2.1 Introduction

In an attempt to unify the Göllnitz - Gordon partition functions appearing in Theorems 1.3.1, 1.3.2 and the Göllnitz partition function of Theorem 1.3.3, Agarwal in [2] defined a more generalized partition function $A_k(\nu)$ as the number of partitions of $\nu$ in which each part $\geq k$, where $k$ is a positive integer, the parts differ by atleast
2, consecutive odd integers are not allowed if $k$ is even and consecutive even integers are not allowed if $k$ is odd. Obviously, $A_1(\nu)$ and $A_3(\nu)$ are Göllnitz - Gordon functions of Theorems 1.3.1, 1.3.2 and $A_2(\nu)$ is Göllnitz function appearing in Theorem 1.3.3. Agarwal [2] also proved the following:

\[ \sum_{n=0}^{\infty} A_k(\nu)q^n = \sum_{n=0}^{\infty} \left( \frac{(-q;q^2)_n q^{n(n+k-1)}}{(q^2; q^2)_n} \right). \] (2.1.1)

Our objective here is to interpret the right hand side of (2.1.1) as a generating function of a restricted $n$ - coloured partition function. This results in an infinite family of combinatorial identities. In some particular cases we get even 3-way identities. We discuss three such cases and obtain two new combinatorial versions of each of the Theorems 1.3.1, 1.3.2 and 1.3.3.

### 2.2 The Main Result

The following result is the main result of this chapter:

**Theorem 2.2.1.** Given a positive integer $k$, let $B_k(\nu)$ denote the number of $n$-colour partitions of $\nu$ such that parts are greater than or equal to $k$, parts used are of the type $(2l - 1)_1$ and $(2l)_2$ if $k$ is odd, $(2l - 1)_2$ and $(2l)_1$ if $k$ is even. The weighted difference between any two parts is non-negative and even. Then

\[ A_k(\nu) = B_k(\nu), \] (2.2.1)

for all $\nu$, where $A_k(\nu)$ is as defined above.

**Remark.** For each value of $k$, this theorem yields a two-way combinatorial identity. For some particular values of $k$ when the series on the right hand side of (2.1.1) corresponds to a product series, we get even three-way combinatorial identities. Our
2.3 Proof of the Theorem 2.2.1

We shall prove that

\[ \sum_{\nu=0}^{\infty} B_k(\nu)q^\nu = \sum_{n=0}^{\infty} (-q; q^2)_nq^{n(n+k-1)/(q^2; q^2)_n}. \]  

(2.3.1)

Let \( B_k(m, \nu) \) denote the number of partitions enumerated by \( B_k(\nu) \) into exactly \( m \) parts. We shall first prove the identity,

\[ B_k(m, \nu) = B_k(m-1, \nu - k - 2m + 2) + B_k(m-1, \nu - k - 4m + 3) + B_k(m, \nu - 2m). \]  

(2.3.2)

We give the proof of (2.2.1) for odd \( k \) as the proof for even \( k \) is similar.

To prove (2.2.1) for odd \( k \) we split the partitions enumerated by \( B_k(m, \nu) \) into three classes:

(i) those that have least part equal to \( k_1 \),

(ii) those that have least part equal to \( (k + 1)_2 \), and

(iii) those that have least part greater than or equal to \( (k + 2)_1 \).
We now transform the partitions in class (i) by deleting the least part $k_1$ and then subtracting 2 from all remaining parts ignoring the subscripts. This produces a partition of $\nu - k - 2(m - 1)$ into exactly $m - 1$ parts, each of which $\geq k_1$ (since originally the second smallest part was $\geq (k + 2)_1$).

Obviously this transformation does not disturb the weighted difference condition between the parts and so the transformed partition is of the type enumerated by $B_k(m - 1, \nu - k - 2m + 2)$.

Next, we transform the partitions in class (ii) by deleting the least part $(k + 1)_2$ and then subtracting 4 from all the remaining parts ignoring the subscripts. This produces a partition of $\nu - (k + 1) - 4(m - 1) = \nu - k - 4m + 3$ into $m - 1$ parts, each of which is $\geq k_1$ (since originally the second smallest part was $\geq (k + 4)_1$). Note that originally $(k + 2)_1$ and $(k + 3)_2$ could not be the smallest part because of the weighted difference condition. Furthermore, since the weighted difference condition between the parts is not disturbed. We see that the transformed partition is of the type enumerated by $B_k(m - 1, \nu - k - 4m + 3)$.

Finally, we transform the partitions in class (iii) by subtracting 2 from each part ignoring the subscripts. This produces a partition $\nu - 2m$ into $m$ parts, each $\geq k_1$, as in the first two cases, here too, the weighted difference condition between the parts is not disturbed. We see that the transformed partition is of the type $B_k(m, \nu - 2m)$.

The above transformations establish a bijection between the partitions enumerated by $B_k(m, \nu)$ and those enumerated by

$$B_k(m - 1, \nu - k - 2m + 2) + B_k(m - 1, \nu - k - 4m + 3) + B_k(m, \nu - 2m).$$
Let
\[ g_k(z; q) = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} B_k(m, \nu) z^m q^\nu. \] (2.3.3)

Using (2.3.2) in (2.3.3), we get the following \( q \)-functional equation
\[ g_k(z; q) = zq^k g_k(zq^2; q) + zq^{k+1} g_k(zq^4; q) + g_k(zq^4; q). \] (2.3.4)

Setting
\[ g_k(z, q) = \sum_{n=0}^{\infty} \beta_k(n; q) z^n \]
and then comparing the coefficient of \( z^n \) on each side of (2.3.4), we get
\[ \beta_k(n; q) = q^{2n-2+k} \beta_k(n - 1; q) + q^{2n-3+k} \beta_k(n - 1; q) + q^{2n} \beta_k(n; q) \]

Therefore,
\[ \beta_k(n; q) = (1 + q^{2n-1}) q^{2n-2+k} / (1 - q^{2n}) \beta_k(n - 1; q). \] (2.3.5)

Iterating (2.3.5) \( n \) times and observing that \( \beta_k(0; q) = 1 \), we see that
\[ \beta_k(n; q) = (-q; q^2)_{n} q^{n(n+k-1)} / (q^2; q^2)_n. \]

Therefore,
\[ g_k(z; q) = \sum_{n=0}^{\infty} \beta_k(n; q) z^n = \sum_{n=0}^{\infty} (-q; q^2)_{n} q^{n(n+k-1)} / (q^2; q^2)_n z^n. \]
Now

\[ \sum_{\nu=0}^{\infty} B_k(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} \left[ \sum_{m=0}^{\infty} B_k(m, \nu)|q|^m \right] \frac{q^n}{1-q} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{\nu} n^{n+k-1}}{(q^2; q^2)_n}. \]

This completes the proof of (2.3.1).

### 2.4 Particular cases and new combinatorial identities

For \( k = 1 \), in view of (1.3.1) our Theorem 2.2.1 reduces to:

**Corollary 1.**

\[ A_1(\nu) = B_1(\nu) = D_1(\nu), \]  

(2.4.1)

where \( D_1(\nu) \) is the number of partitions of \( n \) into parts which are congruent to 1, 4, 7 (mod 8).

(2.4.1) gives us following three identities in the usual sense,

\[ A_1(\nu) = B_1(\nu) \quad (2.4.1)_a \]

\[ A_1(\nu) = D_1(\nu) \quad (2.4.1)_b \]

\[ B_1(\nu) = D_1(\nu) \quad (2.4.1)_c \]

The case (2.4.1)_b of (2.4.1) is the first Göllnitz - Gordon identity, that is, Theorem 1.3.1. The other two identities induced by (2.4.1) are new combinatorial versions of...
it.

For \( k = 2 \), in view of (1.3.3) we get the following three way identity:

**Corollary 2.**

\[
A_2(\nu) = B_2(\nu) = D_2(\nu),
\]

(2.4.2)

where \( D_2(\nu) \) is the number of partitions of \( n \) into parts which are congruent to 2, 3, 7 (mod 8).

(2.4.2) gives us following three identities in the usual sense,

\[
\begin{align*}
A_2(\nu) &= B_2(\nu) \quad (2.4.2)_a \\
A_2(\nu) &= D_2(\nu) \quad (2.4.2)_b \\
B_2(\nu) &= D_2(\nu) \quad (2.4.2)_c
\end{align*}
\]

The case \((2.4.2)_b\) of (2.4.2) is the Gollnitz identity, that is, Theorem 1.3.3. The other two identities induced by (2.4.2) are new combinatorial versions of it.

For \( k = 3 \), in view of (1.3.2) we get the following three way identity:

**Corollary 3.**

\[
A_3(\nu) = B_3(\nu) = D_3(\nu),
\]

(2.4.3)

where \( D_3(\nu) \) is the number of partitions of \( n \) into parts which are congruent to 3, 4, 5 (mod 8).

(2.4.2) gives us following three identities in the usual sense,
\[ A_3(\nu) = B_3(\nu) \quad (2.4.3)_a \]
\[ A_3(\nu) = D_3(\nu) \quad (2.4.3)_b \]
\[ B_3(\nu) = D_3(\nu) \quad (2.4.3)_c \]

The case \((2.4.3)_b\) of \((2.4.3)\) is the second Göllnitz - Gordon identity, that is, Theorem 1.3.2. The other two identities induced by \((2.4.3)\) are new combinatorial versions of it.