6.1 Introduction

Agarwal in [6] translated Theorems (5.1.1) - (5.1.4) for lattice paths as follows:

**Theorem 6.1.1.** For $\nu \geq 1$, let $B_1(\nu)$ denote the number of lattice paths of weight $\nu$ which start from (0,0), have no valley above height 0 and no plain. Then

$$
\sum_{\nu=1}^{\infty} B_1(\nu)q^\nu = \Psi(q).
$$

(6.1.1)
Theorem 6.1.2. For \( \nu \geq 0 \), let \( B_2(\nu) \) denote the number of lattice paths of weight \( \nu \) which start from \((0,0)\), have no valley above height 0, no plain and the height of each peak is \( \geq 2 \). Then
\[
\sum_{\nu=0}^{\infty} B_2(\nu)q^{\nu} = F_0(q).
\] (6.1.2)

Theorem 6.1.3. For \( \nu \geq 0 \), let \( B_3(\nu) \) denote the number of lattice paths of weight \( \nu \) which start from \((0,0)\), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2. Then
\[
\sum_{\nu=0}^{\infty} B_3(\nu)q^{\nu} = \Phi_0(q).
\] (6.1.3)

Theorem 6.1.4. For \( \nu \geq 1 \), let \( B_4(\nu) \) denote the number of lattice paths of weight \( \nu \) which start from \((0,0)\), have no valley above height 0, no plain, the height of each peak of odd weight is 1 while that of even weight is 2 and the weight of the first peak is 1. Then
\[
\sum_{\nu=0}^{\infty} B_4(\nu)q^{\nu} = \Phi_1(q).
\] (6.1.4)

Our objective in this chapter is to interpret \( F_1(q) \) - the fifth order mock - theta function defined by (1.4.13) above as generating function for certain weighted lattice path function. This leads us to a new combinatorial identity. In the following sections we state our main Theorem and prove it by two different methods. The results of this chapter have appeared in [15].
6.2 The Main Result

Theorem 6.2.1. Let $C(\nu)$ denote the number of lattice paths of weight $\nu$ which start at $(0,2)$, have no valley above height 0, no plain, and for which the height of each peak is $\geq 2$. Then

$$\sum_{\nu=0}^{\infty} B(\nu) q^\nu = \sum_{\nu=0}^{\infty} C(\nu) q^\nu = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}} = F_1(q). \quad (6.2.1)$$

where $B(\nu)$ is as defined in Theorem 5.2.1 in Chapter 5.

Remark. Theorem 6.2.1 yields the following new combinatorial identity:

$$B(\nu) = C(\nu), \quad \text{for all } \nu. \quad (6.2.2)$$
Example. $C(7) = 3$, since the relevant lattice paths are:

And as we have already seen in the previous chapter, $B(7)$ is also equal to 3 since the relevant $(n+2)$-colour partitions of 7 are

$$7_9, \quad 7_5 + 0_2, \quad 6_2 + 1_3.$$ 

We give the detail proof of Theorem (6.2.1) in our next section.
6.3 First Proof of the Theorem 6.2.1

In

\[ \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}} \]

the factor \( q^{2n^2+2n} \) generates the lattice paths of \( n+1 \) peaks each of height 2 starting at \((0, 2)\) and terminating at \((4n + 2, 0)\) and with two southeast steps from \((0, 2)\) to \((1, 1)\) and from \((1, 1)\) to \((2, 0)\) at the front of lattice path. For \( n=4 \), the path begins as

Graph A

In the Graph A we consider two successive peaks say \((i+1)\)th and \((i+2)\)th, \(1 \leq i \leq n\), and denote them by \( p_1 \) and \( p_2 \), respectively. We will discuss the case of first peak separately.

Graph B
Clearly
\[ p_1 \equiv (4i, 2), \]
and
\[ p_2 \equiv (4i + 4, 2). \]
The factor \(1/(q; q^2)_{n+1}\) generates non-negative multiples of \((2i - 1), 1 \leq i \leq n + 1\), say,
\[ b_1 \times 1, \; b_2 \times 3, \ldots, b_{n+1} \times (2n + 1). \]
This is encoded by having the \(i^{th}\) peak grow to height \(b_{n-i+2} + 2\). Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. The graph B now changes to graph C or Graph D depending on whether \(b_{n-i} > b_{n-i+1}\) or \(b_{n-i} < b_{n-i+1}\). Note that if \(b_{n-i} = b_{n-i+1}\), then the new graph will look like Graph B.

Graph C
Every lattice path starting at \((0,2)\) with all valleys at height 0, no plain and the height of each peak \(\geq 2\) is generated in this manner. This proves (6.2.1).

6.4 Second Proof (bijective)

We now establish a bijection between the \((n+2)\)-colour partitions enumerated by \(B(\nu)\) and the lattice paths enumerated by \(C(\nu)\). We do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Thus, if we denote the two peaks in Graph C (or Graph D) by \(A_x\) and \(B_y\), respectively, then

\[
A = 4i + 2(b_{n+1} + b_n + \cdots + b_{n-i+2}) + b_{n-i+1}
\]

\[
x = b_{n-i+1} + 2
\]

\[
B = 4i + 4 + 2(b_{n+1} + b_n + \cdots + b_{n-i+1}) + b_{n-i}
\]

\[
y = b_{n-i} + 2.
\]
The weighted difference of these two parts is \(((B_y - A_x)) = B - A - x - y = 0\). Further, if we look at the \((n + 2)\)-colour part \(A_x\), we find that the parity of both \(A\) and \(x\) is determined by \(b_{n-i+1}\). If \(b_{n-i+1}\) is even then both \(A\) and \(x\) are even and if \(b_{n-i+1}\) is odd then both \(A\) and \(x\) are odd and since \(x \geq 2\), we conclude that the even parts appear with even subscripts and odd with odd > 1. The first peak corresponds to the part \((b_{n+1})_{b+n+1+2}\) which is of the form \(k_{b+2}\). Since the second peak corresponds to \((4 + 2b_{n+1} + b_n)_{b+n+2}\), we see that the weighted difference of the smallest and the next smallest part is also equal to 0.

To see the reverse implication we consider two consecutive \((n + 2)\)-colour parts of a partition enumerated by \(B(\nu)\), say, \(C_u\) and \(D_v\). Let \(Q_1 \equiv (C, u)\) and \(Q_2 \equiv (D, v)\) be the corresponding peaks in the associated lattice path.

The length of the plain, if any, between the two peaks is \(D - C - u - v\) which is the weighted difference between the two parts \(C_u\) and \(D_v\) and is therefore equal to zero showing that there are no plains. Also, there can not be a valley above height 0. This can be proved by contradiction.

Suppose there is a valley \(V\) of height \(\gamma\) \((\gamma > 0)\) between the peaks \(Q_1\) and \(Q_2\).
In this case there is a descent of $u - \gamma$ from $Q_1$ to $V$ and an ascent of $v - \gamma$ from $V$ to $Q_2$. This implies that $D = C + (u - \gamma) + (v - \gamma)$, or $D - C - u - v = -2\gamma$. But since the weighted difference is zero, therefore $\gamma=0$. Further since for some $k \geq 0$, $k_{k+2}$ is a part, the path has a peak $(k, k+2)$ showing that the path begins from $(0, 2)$. This completes the bijection between the $(n + 2)$-colour partitions enumerated by $B(\nu)$ and the lattice paths enumerated by $C(\nu)$. 

Graph F