Mock - theta functions and \((n + t)\) - colour partitions

5.1 Introduction

Mock - theta functions have been studied in different directions. For example, Ramanujan gave, without proof, the following identities satisfied by the functions in Group - A of mock theta functions of order 5.
He further proved that the Group - B of fifth order mock - theta functions also satisfies similar relations given by,

(i) \( \psi_1(q) - q^{-1} \phi_1(-q) = 2F_1(q) \)

(ii) \( f_1(-q) + 2q F_1(q^2) = q^{-1} \phi_1(-q^2) + \psi_1(-q) \)

\[ = 2q^{-1} \phi_1(-q^2) + f_1(q) \]

\[ = \theta_4(0, q)[(q^2; q^5)_{\infty}(q^3; q^5)_{\infty}]^{-1}, \]

(iii) \( \psi_1(q) - q F_1(q^2) = \frac{(q^4; q^4)_{\infty}}{(q^2; q^4)_{\infty}}[(q^4; q^{20})_{\infty}(q^{16}; q^{20})_{\infty}]^{-1}, \) \hspace{1cm} (5.1.2)
where $\theta_4(z, q)$ is the Jacobi’s theta function, defined by

$$\theta_4(z, q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1}\cos 2z + q^{4n-2}).$$

Watson [50] proved all the identities given in (5.1.1) and (5.1.2) by the methods of rearrangement of series. He also made use of the basic hypergeometric series for finding new representations of third order mock theta functions given by (1.4.1) - (1.4.4) and (1.4.18) - (1.4.20).

Watson [49] obtained the following new alternative definitions of mock - theta functions of order three:

$$f(q) \prod_{n=1}^{\infty} (1 - q^n) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n (3n+1)/2}{1 + q^n}, \tag{5.1.3}$$

$$\phi(q) \prod_{n=1}^{\infty} (1 - q^n) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n)q^{n(3n+1)/2}}{1 + q^{2n}}, \tag{5.1.4}$$

$$\chi(q) \prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n)q^{n(3n+1)/2}}{1 - q^n + q^{2n}}, \tag{5.1.5}$$

$$\psi(q) \prod_{n=1}^{\infty} (1 - q^{4n}) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{6n(n+1)+1}}{1 - q^{4n+1}}, \tag{5.1.6}$$
Mock - theta functions and \((n + t)\) - colour partitions

\[
\omega(q) \prod_{n=1}^{\infty} (1 - q^{2n}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)} (1 + q^{2n+1})}{1 - q^{2n+1}}, \tag{5.1.7}
\]

\[
\nu(q) \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n+1)/2} (1 - q^{2n+1})}{1 + q^{2n+1}}, \tag{5.1.8}
\]

and

\[
\rho(q) \prod_{n=1}^{\infty} (1 - q^{2n}) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n(n+1)} (1 - q^{4n+2})}{1 + q^{2n+1} + q^{4n+2}}. \tag{5.1.9}
\]

Fine [33] has shown that all the third order mock theta functions can be expressed in terms of a \(2\phi_1\) - series as follows:

\[
f(q) = 2\phi_1(q, 0; -q; q; -1) - 2 - [\phi_1(q, 0; -q; q; -q)]
\]

\[
\phi(q) = (1 - i)\phi_1(q, 0; -iq; q; i) = (1 + i) - i\phi_1(q, 0; -iq; q; iq)
\]

\[
\psi(q) = q\phi_1(q^2, -q^2; 0; q^2; q)
\]

\[
\chi(q) = (1 + \omega^2)2\phi_1(q, 0; -\omega q; q; -\omega^2) = (1 + \omega) - \omega\phi_1(q, 0; -\omega q; q; -\omega^2 q)
\]

\[
\omega(q) = (1 - q)^{-1}\phi_1(q^2, 0; q^2, q^2(q); q)
\]

\[
\nu(q) = 2\phi_1(q^2, q, 0; q^2; -q)
\]

\[
\rho(q) = (1 + \omega q)^{-1}\phi_1(q^2, 0; \omega q^2; q^2; q\omega^{-1}),
\]

where \(\omega\) is a primitive cube root of unity.

Andrews ([24],[25]) gave a set of very elegant basic hypergeometric transformations which yielded as special cases the mock - theta functions relations.
5.1 Introduction

Besides these above mentioned relations of mock-theta functions, number theoretic interpretations of some of the mock-theta functions are also found in the literature. For example, Fine [33] gave two number theoretic interpretations of mock-theta functions $\psi(q)$ of order three as follows:

**Theorem 5.1.1** $\psi(q) = \sum_{n=1}^{\infty} \beta(n)q^n$, where $\beta(n)$ is the number of partition of $n$ of the form,

$$n = n_1.1 + n_3.3 + n_5.5 + \cdots + n_{2k-1}(2k - 1), \quad n_j > 0.$$ 

**Theorem 5.1.2** $\psi(q) = \sum_{n=1}^{\infty} \gamma(n)q^n$, where $\gamma(n)$ is the number of partition of $n$ of the form,

$$n = n_0 + 2n_1 + 2n_2 + \cdots + 2n_r, \quad (n_0 > n_0 > n_0 > \cdots > n_r).$$

Andrews [27] gave number theoretic interpretations of $f(q)$, and $F_0(q)$ as follows:

**Theorem 5.1.3** Let $f(q)$ be the first mock theta function of order three given by (1.4.1). Then

$$f(q) = \sum_{n=0}^{\infty} D(gn, p; n)q^n$$

where $D(gn, p; n)$ is the total number of partitions $\pi$ of $n$ with $gn(\pi)$ even minus the number with $gn(\pi)$ odd.

Here $gn(\pi)$ denotes the largest part plus the number of parts.

**Theorem 5.1.4** Let $F_0(q)$ be mock-theta function of order five given by (1.4.8). Then $F_0(q)$ is the generating function for the partitions into odd parts wherein every
odd number up to and including the largest part appears at least twice.

and used hook differences to give number theoretic interpretation of $f_0(q)$.

**Theorem 5.1.5** Let $f_0(q)$ be mock-theta function of order five given by (1.4.5). Then $f_0(q)$ is interpreted as the excess of the number of partitions of $n$ which have hook differences 0 or 1 on the diagonal 0 and largest part plus number of parts even over the same type of partitions of $n$ with largest part plus number of parts odd.

where the hook differences and diagonal is defined as follows:

**Definition 5.1.1.** Let $\pi$ be a partition whose Ferrers graph has a node in the $i^{th}$ row and $j^{th}$ column; we call this node the $(i,j)^{th}$ node (Some authors represents the Ferrers graph by square nodes rather than dots). We define the hook differences at the $(i,j)^{th}$ node to be the number of nodes in the $i^{th}$ row of $\pi$ minus the number of nodes in the $j^{th}$ column of $\pi$.

**Definition 5.1.2.** The $(i,j)^{th}$ node of $\pi$ lies on the diagonal $c$ if $i - j = c$.

Example. The Ferrers graph of the partition $4+4+3+2+2$ of 15 is
and the hook difference of each node is given as below:

<table>
<thead>
<tr>
<th>Hook Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1  -1  1  2</td>
</tr>
<tr>
<td>-1  -1  1  2</td>
</tr>
<tr>
<td>-2  -2  0</td>
</tr>
<tr>
<td>-3  -3</td>
</tr>
<tr>
<td>-3  -3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0  -1  -2  -3</td>
</tr>
<tr>
<td>1  0  -1  -2</td>
</tr>
<tr>
<td>2  1  0</td>
</tr>
<tr>
<td>3  2</td>
</tr>
<tr>
<td>3  2</td>
</tr>
</tbody>
</table>

Very recently, Agarwal in [5] gave the $n$-colour number theoretic interpretations of four mock theta functions $\psi(q)$, $F_0(q)$, $\phi_0(q)$, $\phi_1(q)$ given by,

**Theorem 5.1.6.** For $\nu \geq 1$, let $A_1(\nu)$ denote the number of $n$-colour partitions of $\nu$ such that even parts appear with even subscripts and odd with odd. For some $k$, 

$k_k$ is a part, and the weighted difference of any two consecutive parts is 0. Then
\[ \sum_{\nu=1}^{\infty} A_1(\nu)q^\nu = \psi(q). \] (5.1.10)

**Theorem 5.1.7.** For $\nu \geq 0$, let $A_2(\nu)$ denote the number of $n$-colour partitions of $\nu$ such that even parts appear with even subscripts and odd with odd greater than 1, for some $k$, $k_k$ is a part, and the weighted difference of any two consecutive parts is 0. Then
\[ \sum_{\nu=0}^{\infty} A_2(\nu)q^\nu = F_0(q). \] (5.1.11)

**Theorem 5.1.8.** For $\nu \geq 0$, let $A_3(\nu)$ denote the number of $n$-colour partitions of $\nu$ such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k - 1)_1$ or $(2k)_2$, the minimum part is $1_1$ or $2_2$ and the weighted difference of any two consecutive part is 0. Then
\[ \sum_{\nu=0}^{\infty} A_3(\nu)q^\nu = \phi_0(q). \] (5.1.12)

**Theorem 5.1.9.** For $\nu \geq 1$, let $A_4(\nu)$ denote the number of $n$-colour partitions of $\nu$ such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is $1_1$ and the weighted difference of any two consecutive part is 0. Then
\[ \sum_{\nu=0}^{\infty} A_4(\nu)q^\nu = \phi_1(q). \] (5.1.13)

In this chapter we consider one more mock theta function of order 5 defined by (1.4.13), viz.,
\[ F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_n}_n. \] (5.1.14)
5.2 The Main Result

Theorem 5.2.1 For $\nu \geq 0$, let $B(\nu)$ denote the number of partitions of $\nu$ with "$n + 2$ copies of $n$" in which even parts appear with even subscripts and odd with odd greater than 1. For some $i$, $i_{i+2}$ is a part and the weighted difference of any two consecutive parts is zero. Then

$$
\sum_{\nu=0}^{\infty} B(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^{n+1})} = F_1(q). \tag{5.2.1}
$$

Example. $B(7) = 3$, since the relevant $(n + 2)$-colour partitions are

$$
7_9, \quad 7_5 + 0_2, \quad 6_2 + 1_3.
$$

5.3 Proof of the Theorem 5.2.1

Let $A_2(m, \nu)$ denote the number of partitions of $\nu$ enumerated by $A_2(\nu)$ into $m$ parts. Let for $|q| < 1$ and $|z| < |q|^{-1}$, $f(z, q)$ is defined by

$$
f(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_2(m, \nu) z^m q^{\nu}. \tag{5.3.1}
$$

We shall first prove that

$$
A_2(m, \nu) = A_2(m - 1, \nu - 4m + 2) + A_2(m, \nu - 2m + 1). \tag{5.3.2}
$$
To prove (5.3.2), we split the partitions enumerated by \( A_2(m, \nu) \) into two classes, viz.,

(i) those which contain 2 as a part, and

(ii) those which contain \( k^k (k > 2) \) as a part.

We now transform the partitions in class (i) by deleting the part 2 and then subtracting 4 from all the remaining parts without disturbing the subscripts. The transformed partition will be of the type enumerated by

\[ A_2(m - 1, \nu - 4m + 2). \]

Conversely, if we have a partition enumerated by \( A_2(m - 1, \nu - 4m + 2) \), we add 4 to each part and then add 2 as a part. The resulting partition will be a partition of class (i). This shows that the partitions in class (i) are in one-to-one correspondence with the partitions enumerated by \( A_2(m - 1, \nu - 4m + 2) \).

Thus the number of partitions in class (i) is \( A_2(m - 1, \nu - 4m + 2) \). Next we transform the partitions in class (ii) by first replacing the part \( k^k \) by \( (k - 1)^k \), and then subtracting 2 from each of remaining parts. The transformed partition is a partition enumerated by \( A_2(m, \nu - 2m + 1) \). Since this transformation is also reversible, we see that the number of partitions in class (ii) is \( A_2(m, \nu - 2m + 1) \). This completes the proof of (5.3.2).

On substituting for \( A_2(m, \nu) \) from (5.3.2) in (5.3.1) and then simplifying, we get

\[ f(z, q) = zq^2 f(zq^4, q) + q^{-1} f(zq^2, q). \]  

(5.3.3)

Setting

\[ f(z, q) = \sum_{n=0}^{\infty} a_n(q) z^n, \]

(5.3.4)

in (5.3.3) and then comparing the coefficients of \( z^n \) in the resulting expression, we
5.3 Proof of the Theorem 5.2.1

obtain

\[ \alpha_n(q) = \frac{q^{4n-2}}{1 - q^{2n-1}} \alpha_{n-1}(q). \]  \hfill (5.3.5)

Iterating (5.3.5) \(n\)-times and noting that \( \alpha_0(q) = 1 \), we are led to

\[ \alpha_n(q) = \frac{q^{2n^2}}{(q; q^2)_n}, \]  \hfill (5.3.6)

and so

\[ f(z, q) = \sum_{n=0}^{\infty} \frac{q^{2n^2} z^n}{(q; q^2)_n}. \]  \hfill (5.3.7)

Now suppose that \( B(m, \nu) \) \((m > 0)\) denotes the number of partitions enumerated by \( B(\nu) \) into \( m \) parts. Clearly if we subtract 2 from each nonempty part of a partition enumerated by \( A_2(m, \nu) \) we get a partition enumerated by \( B(m, \nu - 2m) \) and since the transformation is reversible, we see that

\[ A_2(m, \nu) = B(m, \nu - 2m), \quad m > 0, \nu \geq 2m. \]  \hfill (5.3.8)

Now for \(|q| < 1\) and \(|z| < |q|^{-1}\), let

\[ \sum_{\nu=0}^{\infty} \sum_{m=1}^{\infty} B(m, \nu) z^m q^{\nu} = g(z, q) = \sum_{n=1}^{\infty} \beta_n(q) z^n. \]  \hfill (5.3.9)

Then

\[ \sum_{n=1}^{\infty} \alpha_n(q) z^n = \sum_{\nu=2}^{\infty} \sum_{m=1}^{\infty} A_2(m, \nu) z^m q^{\nu}. \]
\[
B(m, \nu - 2m)z^m q^\nu
\]
\[
g(zq^2, q)
\]
\[
= \sum_{n=1}^{\infty} \beta_n(q)(zq^2)^n. \quad (5.3.10)
\]

On comparing the coefficients of \(z^n (n > 0)\) in the extremes of (5.3.10), we get

\[
\beta_n(q) = \frac{q^{2n^2 - 2n}}{(q; q^2)_n}, \quad n > 0. \quad (5.3.11)
\]

Therefore,

\[
g(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n^2 - 2n}}{(q; q^2)_n} z^n. \quad (5.3.12)
\]

Now

\[
\sum_{\nu=0}^{\infty} B(\nu)q^\nu = \sum_{\nu=0}^{\infty} \left( \sum_{m=1}^{\infty} B(m, \nu) \right)q^\nu
\]

\[
= g(1, q)
\]

\[
= \sum_{n=1}^{\infty} \frac{q^{2n^2 - 2n}}{(q; q^2)_n}
\]
This completes the proof of the Theorem 5.2.1.