Chapter 4
Conditional Marginal and Conditional Joint Reliability Importance

4.1: Introduction

Gao et al. (2007) introduced the idea of Conditional Marginal Reliability Importance (CMRI) and Conditional Joint Reliability Importance (CJRI). These reliability importance measures are calculated under the condition that some different components/groups of components are known to be working or non-working. Components with higher values of CMRI or CJRI are important for the functioning of the system and need careful monitoring or maintenance.

In this chapter, the expressions for the CMRI and CJRI are derived for series-in-parallel and series-parallel systems when the components are independent but not identically distributed. We generally come across such type of configurations in electronic systems, water pumping systems and GSM systems used for communication. Phased Mission (PM) systems, in which a number of different tasks or functions are performed in phases also illustrate such systems. The prime examples of such systems are aircrafts, missiles and rockets. Each phase must be completed so that a mission can be a success. CJRI of PM systems quantitatively measures the interaction between different components within the same phase or among different phases. Schur-convexity (concavity) of the reliability function of a system is used to identify the sign of the joint importance of three components and CJRIs. It is shown that the difference in the reliability functions for more than two statistically independent components and mutually dependent components is measured in terms of covariance, the JRIs and the CJRIs.

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In Section 4.2, we consider series-parallel and series-in-parallel systems and derive expressions for the CMRIs for independent but non-identical components in subsystems. In Section 4.3, expressions for CJRIs are derived for series-in-parallel and series-parallel systems. Section 4.4 explores the effect of Schur-convexity (Schur-concavity) of \( R(p) \) on CJRIs and JRI of three or more components. The difference in the reliability function for three or more statistically independent and dependent components is shown to be measured by their covariance, the JRIs and the CJRIs. In Section 4.5, the idea of CMRI and CJRI is illustrated for a phased type of electronic system and a bridge system. Section 4.6 reports the conclusion.

### 4.2: Conditional Marginal Reliability Importance

In this section, we study the CMRI for a series-in-parallel and a series-parallel system. These systems were described with their configurations in Chapters 2 and 3. We are repeating the figures for the sake of convenience.

#### 4.2.1: CMRI for series-in-parallel system

We consider a series-in-parallel system (shown in Figure 4.1) having \( s \) subsystems connected in parallel and each subsystem with \( n_i \) independent and non-identical components connected in series.

Assume \( A_{ai} \) to be the \( a^{th} \) component in \( i^{th} \) series subsystem and \( p_{ai} \), the probability of functioning of \( a^{th} \) component in \( i^{th} \) subsystem for \( a = 1, 2 \ldots n_i \) and \( i = 1, 2 \ldots s \).

Conditional Marginal Reliability Importance (CMRI) of component \( u \) given \( z_i \), the state of component \( l \) is defined as
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\[ CMRI(A_i \mid z_i) = \frac{\partial R(p_1, p_2, \ldots, p_n, z_i, p_{1i}, \ldots, p_{ni})}{\partial p_i}, i \neq u \]  (4.1)

where \( z_i \) takes value 1 or 0 depending upon whether component \( i \) is functioning or non-functioning.

For independent components

\[ CMRI(A_i \mid z_i) = R(z_i, 1_u, p) - R(z_i, 0_u, p), i \neq u. \]  (4.2)

where \( R(z_i, 1_u, p) \): reliability function with component \( u \) in functioning state and \( z_i \) as state (functioning or non-functioning) of component \( i \).

\( R(z_i, 0_u, p) \): reliability function with component \( u \) in nonfunctioning state and \( z_i \) as (functioning or non-functioning) state of component \( i \).

Conditional MRI of component \( i \) given the states of two components \( u \) and \( v \) has also been defined analogously by Gao et al. (2007).

The reliability of the above system, with independent and non-identical components is

\[ R(p) = \prod_{i=1}^{n} (1 - \prod_{a=1}^{n_i} p_{ia}) \]  (4.3)

Let the state of component \( i \) in \( j^{th} \) subsystem be known. Component \( u \) may or may not lie in the \( j^{th} \) subsystem. Then using equation (4.2), the CMRI of component \( u \) given the state of component \( i \) in \( j^{th} \) subsystem is presented below for two possible cases.

**Case I:** Let components \( i \) and \( u \) belong to the same \( j^{th} \) series subsystem. Then

\[ CMRI(A_{iu} \mid z_{j, i} = 0) = \left\{ \prod_{i=1}^{s} \left(1 - \prod_{a=1}^{n_i} p_{ia} \right) \right\}; \]

\[ CMRI(A_{iu} \mid z_{j, i} = 1) = \left\{ \prod_{i=1}^{s} \left(1 - \prod_{a=1}^{n_i} p_{ia} \right) \right\} \left( \prod_{a=1}^{n_i} p_{ia} \right); \]

**Case II:** Let component \( i \) belong to \( j^{th} \) series subsystem and component \( u \) belong to subsystem \( h \neq j \), then

\[ CMRI(A_{hu} \mid z_{j, i} = 0) = \left\{ \prod_{i=1}^{s} \left(1 - \prod_{a=1}^{n_i} p_{ia} \right) \right\} \left( \prod_{a=1}^{n_i} p_{ia} \right); \]
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\[ CMRI (A_{hu} | z_{jl} = 1) = \left( \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{n_j} P_{ia} \right) \right) \left( \prod_{a=1}^{n_j} P_{ja} \right) \left( \prod_{a=1}^{n_{h \neq u}} P_{ha} \right). \]

Hence \( CMRI (A_{hu} | z_{jl} = 1) \leq CMRI (A_{hu} | z_{jl} = 0). \)

**Remark 4.1:** If a component \( l \) in \( j^{th} \) series subsystem is non-functioning, then component \( u \) in a different series subsystem has a greater effect on the system reliability as compared to the case when component \( l \) is functioning.

### 4.2.2: CMRI for Series-Parallel System

In this subsection, we study a system which is a combination of series and parallel subsystems with independent but non-identical components.

Let \( r = r_p + r_s \) be the total number of subsystems with \( r_s \) series subsystems and \( r_p \) parallel subsystems.

and \( n_i(m_j) \) : number of components in \( i^{th} \) series \( (j^{th} \) parallel) subsystem.

For \( a = 1, 2 \ldots n_l, i = 1, 2 \ldots r_s, b = 1, 2 \ldots m_j, j = 1, 2 \ldots r_p, \)

- \( A_{sis} \): the \( a^{th} \) component in \( i^{th} \) series subsystem;
- \( A_{pjb} \): the \( b^{th} \) component in \( j^{th} \) parallel subsystem;
- \( p_{sis} \): probability of functioning of \( a^{th} \) component in \( i^{th} \) series subsystem;
- \( p_{pjb} \): probability of functioning of \( b^{th} \) component in \( j^{th} \) parallel subsystem.

**CMRI \( (A_{jsj} \ | \ z_{jl}) \)**: Conditional Marginal Reliability Importance of component \( u \) in \( j^{th} \) series subsystem given \( z_{si} \), the state of component \( l \) in \( j^{th} \) series subsystem;

**CMRI \( (A_{pju} \ | \ z_{jl}) \)**: Conditional Marginal Reliability Importance of component \( u \) in \( g^{th} \) parallel subsystem given \( z_{jl} \), the state of component \( l \) in \( j^{th} \) series subsystem;

The following figure illustrates the series-parallel system under study. This system was also depicted in Figure 3.1 and is being repeated for convenience.

![Figure 4.2: A series-parallel system](image-url)
The reliability of the system configured in above is same as given by (3.1).

\[ R(p) = \left( \prod_{i=1}^{r_j} \left( \prod_{a=1}^{r_i} p_{sia} \right) \right) \left( \prod_{j=1}^{m_j} (1 - \prod_{b=1}^{m_j} (1 - p_{pjb})) \right) \]

We derive the CMRI of component \( u \) given the state of component \( l \) for this system in five possible cases.

**Case I:** Both components \( u \) and \( l \) are in \( j^{th} \) series subsystem.

\[
CMRI \left( A_{gu} | z_{ql} = 1 \right) = \left\{ \prod_{i=1}^{r_i} \left( \prod_{a=1}^{r_j} p_{sia} \right) \right\} \left\{ \prod_{n=1}^{r_n} \left( 1 - \prod_{b=1}^{r_n} (1 - p_{nph}) \right) \right\} \left\{ \prod_{a=1}^{n_a} p_{sga} \right\}
\]

\[
\geq CMRI \left( A_{gu} | z_{ql} = 0 \right) = 0.
\]

**Case II:** Component \( u \) is in \( g^{th} \) series subsystem and component \( l \) is in \( j^{th} (j \neq g) \) series subsystem.

\[
CMRI \left( A_{gu} | z_{ql} = 1 \right) = \left\{ \prod_{i=1}^{r_i} \left( \prod_{a=1}^{r_j} p_{sia} \right) \right\} \left\{ \prod_{n=1}^{r_n} \left( 1 - \prod_{b=1}^{r_n} (1 - p_{nph}) \right) \right\} \left\{ \prod_{a=1}^{n_a} p_{sga} \right\}\left\{ \prod_{a=1}^{n_a} p_{sga} \right\}
\]

\[
\geq CMRI \left( A_{gu} | z_{ql} = 0 \right) = 0.
\]

**Case III:** Component \( u \) is in \( g^{th} \) parallel subsystem and component \( l \) is in \( j^{th} \) series subsystem, then

\[
CMRI \left( A_{pgu} | z_{ql} = 1 \right) = \left\{ \prod_{i=1}^{r_i} \left( \prod_{a=1}^{r_j} p_{sia} \right) \right\} \left\{ \prod_{n=1}^{r_n} \left( 1 - \prod_{b=1}^{r_n} (1 - p_{nph}) \right) \right\}
\times
\]

\[
\left\{ \prod_{a=1}^{n_a} p_{sga} \right\} \left\{ \prod_{a=1}^{n_a} p_{sga} \right\}
\]

Thus \( CMRI \left( A_{pgu} | z_{ql} = 1 \right) \geq CMRI \left( A_{pgu} | z_{ql} = 0 \right) = 0 \).
Case IV: Both components \( u \) and \( l \) are in \( j \)th parallel subsystem.

\[
\text{CMRI} \left( A_{pjli} \mid z_{pj} = 0 \right) = \left[ \prod_{l=1}^{r_j} \prod_{n_{li} = 1}^{m_{li}} P_{sia} \right] \left[ \prod_{l=1}^{r_j} \prod_{n_{hi} = 1}^{m_{hi}} \left( 1 - \prod_{b=1}^{h_{su}} \left( 1 - P_{phb} \right) \right) \right] \left[ \prod_{l=1}^{r_j} \prod_{n_{hu} = 1}^{m_{hu}} \left( 1 - P_{phb} \right) \right].
\]

Hence \( \text{CMRI} \left( A_{pjli} \mid z_{pj} = 1 \right) = 0 \leq \text{CMRI} \left( A_{pjli} \mid z_{pj} = 0 \right) \)

Case V: Component \( u \) is in \( g \)th parallel subsystem and component \( l \) is in \( j \)th \((j \neq g)\) parallel subsystem.

\[
\text{CMRI} \left( A_{pgui} \mid z_{pj} = 0 \right) = \left[ \prod_{l=1}^{r_g} \prod_{n_{gi} = 1}^{m_{gi}} P_{sia} \right] \left[ \prod_{l=1}^{r_g} \prod_{n_{hi} = 1}^{m_{hi}} \left( 1 - \prod_{b=1}^{h_{su}} \left( 1 - P_{phb} \right) \right) \right] \left[ \prod_{l=1}^{r_g} \prod_{n_{hu} = 1}^{m_{hu}} \left( 1 - P_{phb} \right) \right];
\]

\[
\text{CMRI} \left( A_{pgui} \mid z_{pj} = 1 \right) = \left[ \prod_{l=1}^{r_g} \prod_{n_{gi} = 1}^{m_{gi}} P_{sia} \right] \left[ \prod_{l=1}^{r_g} \prod_{n_{hi} = 1}^{m_{hi}} \left( 1 - \prod_{b=1}^{h_{su}} \left( 1 - P_{phb} \right) \right) \right] \left[ \prod_{l=1}^{r_g} \prod_{n_{hu} = 1}^{m_{hu}} \left( 1 - P_{phb} \right) \right].
\]

Hence \( \text{CMRI} \left( A_{pgui} \mid z_{pj} = 1 \right) \geq \text{CMRI} \left( A_{pgui} \mid z_{pj} = 0 \right) \)

Remark 4.2: Hence, for a series-parallel system,

(i) if component \( l \) in a series subsystem is non-functioning, CMRI is zero except when both components are in parallel subsystems;

(ii) if a component in a parallel subsystem is known to be functioning, then change in the reliability of another component in the same subsystem does not affect the system reliability.

(iii) when components \( l \) and \( u \) are in the same parallel subsystem and component \( l \) is known to be non-functioning, then the effect of change in the reliability of component \( u \) on the system reliability is more as compared to the case when component \( l \) is functioning. The inequality is reversed in all other cases.
4.3: Conditional Joint Reliability Importance

In this section, we study the CJRI for series-in-parallel and series-parallel systems. When the state of component \( l \) is given, Gao et al. (2007) defined the conditional JRI of components \( u \) and \( v \) as

\[
CJRI(A_u, A_v | z_l) = \frac{\partial^2 R(p_1, ..., z_l, ..., p_n)}{\partial p_u \partial p_v}, l \neq u, v.
\]  
(4.4)

When components are independent,

\[
CJRI(A_u, A_v | z_l) = R(z_l, 0_u, 1_v, p) - R(z_l, 1_u, 0_v, p) - R(z_l, 1_u, 1_v, p) + R(z_l, 0_u, 0_v, p)
\]  
(4.5)

where

- \( R(z_l, 1_u, 1_v, p) \): Reliability function with \( z_l \) as the state of component \( l \) (functioning or non-functioning) and components \( u \) and \( v \) in functioning state;
- \( R(z_l, 1_u, 0_v, p) \): Reliability function with \( z_l \) as the state of component \( l \) (functioning or non-functioning) and component \( u \) in functioning state and component \( v \) in non-functioning state;
- \( R(z_l, 0_u, 1_v, p) \): Reliability function with \( z_l \) as the state of component \( l \) (functioning or non-functioning) and component \( u \) in non-functioning state and component \( v \) in functioning state;
- \( R(z_l, 0_u, 0_v, p) \): Reliability function with \( z_l \) as the state of component \( l \) (functioning or non-functioning) and components \( u \) and \( v \) in non-functioning state.

They also extended the idea of CJRI to the case when state of two components is known.

4.3.1: CJRI for series-in-parallel system

Given the state of component \( l \) in \( j \)th series subsystem, the CJRI of components \( u \) and \( v \) can be found by using equation (4.4). Four different cases arise.

Case I: Let components \( l, u \) and \( v \) belong to the same \( j \)th series subsystem, then

\[
CJRI(A_{j\nu}, A_{jv} | z_{j\nu} = 1) = \left\{ \prod_{i=1}^{t_j} \left( 1 - \prod_{q=1}^{i} P_{q_{j\nu}} \right) \right\} \left\{ \prod_{q=1}^{j} P_{q_{j\nu}} \right\}
\]  
\( \geq 0 = CJRI(A_{j\nu}, A_{jv} | z_{j\nu} = 0) \).
Case II: Let component \( l \) belong to \( j^{th} \) series subsystem and components \( u \) and \( v \) belong to \( h^{th} \) series subsystem, where \( h \neq j \), then

\[
CJRl\left( A_{hu}, A_{hv} \mid z_{j^l} = 1 \right) = - \left\{ \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{t_a} p_a \right) \left( 1 - \prod_{a=1}^{t_a} p_{ja} \right) \prod_{a=1}^{l_a} p_{ha} \right\} ^{1}\left[ \prod_{a=1}^{l_a} p_{ha, v} \right];
\]

\[
CJRl\left( A_{hu}, A_{hv} \mid z_{j^l} = 0 \right) = - \left\{ \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{t_a} p_a \right) \prod_{a=1}^{l_a} p_{ha} \right\} ^{1};
\]

Hence \( CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 1 \right) \geq CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 0 \right) \).

Case III: Let component \( l \) lies in \( j^{th} \) series subsystem, component \( u \) in \( h^{th} \) subsystem and component \( v \) belong to \( k^{th} \) series subsystem where \( k \neq j \neq h \), then

\[
CJRl\left( A_{hu}, A_{hv} \mid z_{j^l} = 1 \right) = - \left\{ \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{t_a} p_a \right) \left( 1 - \prod_{a=1}^{t_a} p_{ja} \right) \prod_{a=1}^{l_a} p_{ha} \right\} ^{1};
\]

\[
CJRl\left( A_{hu}, A_{hv} \mid z_{j^l} = 0 \right) = - \left\{ \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{t_a} p_a \right) \prod_{a=1}^{l_a} p_{ha} \right\} ^{1};
\]

Hence \( CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 1 \right) \geq CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 0 \right) \).

Case IV: Let components \( l, u \) belong to \( j^{th} \) series subsystem and component \( v \) belong to \( h^{th} \) series subsystem where \( h \neq j \) then

\[
CJRl\left( A_{ju}, A_{hv} \mid z_{j^l} = 0 \right) = 0;
\]

\[
CJRl\left( A_{ju}, A_{hv} \mid z_{j^l} = 1 \right) = - \left\{ \prod_{i=1}^{n} \left( 1 - \prod_{a=1}^{t_a} p_a \right) \prod_{a=1}^{l_a} p_{ja} \right\} ^{1};
\]

Hence \( CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 1 \right) \leq CJRl \left( A_{hu}, A_{hv} \mid z_{j^l} = 0 \right) = 0. \)
Remark 4.3: It can be concluded that for a series-in-parallel system,
(i) if component \( l \) in a series subsystem is non-functioning, then CJRI of two components with at least one in the same subsystem is zero.
(ii) given the state of one component, the negative CJRI of two components indicates that change in the reliability of these two components has a negative effect on the system reliability and in that case, they are reliability substitutes (Hong et al. (2002)).

4.3.2: CJRI for series-parallel system
Let u, v and \( l \) be three components. Several possibilities arise depending upon whether the components are in the same or different series/parallel subsystems. Few of the cases are illustrated below.

Case I: Component \( l \) is in \( j \)th series subsystem and components \( u \) and \( v \) belong to \( j \)th series subsystem,

\[
CJRI \left( A_{ju}, A_{jv} \mid z_{jl} = 1 \right) = \left\{ \prod_{i \neq j} \left( 1 - \prod_{b=1}^{k} \left( 1 - P_{b,j} \right) \right) \right\} \left\{ \prod_{a=1}^{n} P_{a,j} \right\} .
\]

\[
CJRI \left( A_{ju}, A_{jv} \mid z_{jl} = 0 \right) = 0.
\]

Case II: All three components belong to \( j \)th parallel subsystem

\[
CJRI \left( A_{pu}, A_{jv} \mid z_{jl} = 0 \right) = \left\{ \prod_{i \neq j} \left( 1 - \prod_{b=1}^{k} \left( 1 - P_{b,j} \right) \right) \right\} \left\{ \prod_{b=1}^{k} \left( 1 - P_{b,j} \right) \right\} ;
\]

\[
CJRI \left( A_{pu}, A_{jv} \mid z_{jl} = 1 \right) = 0 \leq CJRI \left( A_{pu}, A_{jv} \mid z_{jl} = 0 \right).
\]

Case III: Components \( u \) and \( v \) are in \( g \)th series subsystem and component \( l \) is in \( j \)th series subsystem for \( j \neq g \).

\[
CJRI \left( A_{sgu}, A_{sgv} \mid z_{jl} = 1 \right) = \left\{ \prod_{i \neq g,j} \left( 1 - \prod_{b=1}^{k} \left( 1 - P_{b,j} \right) \right) \right\} \left\{ \prod_{a=1}^{n} P_{a,g} \right\} ;
\]

\[
CJRI \left( A_{sgu}, A_{sgv} \mid z_{jl} = 0 \right) \geq CJRI \left( A_{sgu}, A_{sgv} \mid z_{jl} = 0 \right) = 0.
\]
Case IV: Components u and v are in $g^{th}$ series subsystem and component $l$ is in $j^{th}$ parallel subsystem.

$$CJRI \left(A_{ugu}, A_{ugv} \mid z_{pg}, 0 \right) = \left\{ \begin{array}{l}
\prod_{a=1}^{r} \left( \prod_{v=a}^{n_a} P_{sa} \right) \prod_{b=1}^{r} \left( \prod_{h=1}^{m_h} \left( 1 - \prod_{b=1}^{1} \left( 1 - P_{pgh} \right) \right) \right.
\end{array} \right.$$ 

$$\left\lfloor \prod_{a=1}^{n_a} P_{sa} \right\rfloor \left( 1 - \prod_{b=1}^{1} \left( 1 - P_{pgh} \right) \right);$$

$$CJRI \left(A_{ugu}, A_{ugv} \mid z_{pg}, 0 \right) \geq CJRI \left(A_{ugu}, A_{ugv} \mid z_{pg}, 1 \right)$$

Case V: Components u and v are in $g^{th}$ parallel subsystem, component $l$ is in $j^{th}$ parallel subsystem ($j \neq g$).

$$CJRI \left(A_{pgu}, A_{pgv} \mid z_{pg}, 0 \right) = \left\{ \begin{array}{l}
\prod_{a=1}^{r} \left( \prod_{v=a}^{n_a} P_{sa} \right) \prod_{b=1}^{r} \left( \prod_{h=1}^{m_h} \left( 1 - \prod_{b=1}^{1} \left( 1 - P_{pgh} \right) \right) \right. \\
\left. \prod_{a=1}^{n_a} P_{sa} \right\rfloor \left( 1 - \prod_{b=1}^{1} \left( 1 - P_{pgh} \right) \right);$$

$$CJRI \left(A_{pgu}, A_{pgv} \mid z_{pg}, 0 \right) \geq CJRI \left(A_{pgu}, A_{pgv} \mid z_{pg}, 1 \right)$$

The results can be interpreted as earlier.
4.4: Some general results for JRI and CJRI

To identify the sign of the JRI of two components without finding its value, Hong et al. (2002) used Schur-convexity (concavity) of the reliability function. They also measured the difference between reliability functions for pairwise dependent components and statistically independent components.

In this sub-section, we study the properties of JRI and CJRIs for three or more components using the Schur-convexity (concavity) of the reliability function. These are useful to identify the sign of the joint and conditional joint reliability importance rather than its magnitude. The difference in the reliability functions for three statistically independent and mutually dependent components is shown to be measured in terms of covariance, the JRIs and the CJRIs. The above results have been generalized for n components.

The following theorem establishes the symmetry of CJRI for a Schur-convex (concave) reliability function.

**Theorem 4.4:** If the reliability function $R(p)$ is Schur convex (concave), then
\[
\text{CJRI} (A_i, A_j | z = 0) = \text{CJRI} (A_j, A_k | z = 0) = \text{CJRI} (A_k, A_i | z = 0).
\]

**Proof:** Using Schur’s condition (Marshall and Olkin (1979), Thm 3.A.4), a necessary and sufficient condition for $R(p)$ to be Schur-convex (concave) is symmetry of $R(p)$.

Hence
\[
R(1_i, 1_j, 0_k, p) = R(0_i, 1_j, 1_k, p) = R(1_i, 0_j, 1_k, p); \\
R(1_i, 0_j, 0_k, p) = R(0_i, 1_j, 0_k, p) = R(0_i, 0_j, 1_k, p).
\]

Using (4.5), we get the desired result.

The following result identifies the signs of JRI and CJRI’s when $R(p)$ is Schur-Convex (Schur-Concave). It is observed that rate of change of system reliability as a function of the reliability of the components is negative.

Theorem 4.5: If the reliability function $R(p)$ is Schur-convex (Schur-concave), then

(i) $JRI(A_i, A_j, A_k) \leq 0 \ (\geq 0)$,

(ii) $CJRI(A_i, A_j | z \neq 0) \leq 0 \ (\geq 0)$,

(iii) $CJRI(A_i, A_k | z \neq 0) \leq 0 \ (\geq 0)$ and

(iv) $CJRI(A_k, A_i | z \neq 0) \leq 0 \ (\geq 0)$.

Proof: Since $R(p)$ is Schur convex, hence

$$ (p_i - p_j) \left( \frac{\partial R(p)}{\partial p_i} \frac{\partial R(p)}{\partial p_j} \right) \geq 0 \ \text{for every } 1 \leq i, j \leq n. \ (Marshall \ and \ Olkin \ (1979)), $$

Decomposing $R(p)$ by pivoting on $p_i, p_j$ and $p_k$, we get

$$ \frac{\partial R(p)}{\partial p_i} = p_j p_k R(1,1,1_k,p) + p_j (1-p_k) R(1,1,0_k,p) - p_j p_k R(0,1,1_k,p) +$$

$$ + (1-p_j) p_k R(1,0,0_k,p) + (1-p_j) (1-p_k) R(1,0,0_k,p) - p_j (1-p_k) R(0,1,0_k,p) -$$

$$ - (1-p_j) p_k R(0,0,1_k,p) - (1-p_j) (1-p_k) R(0,0,0_k,p).$$

Similarly, we get

$$ \frac{\partial R(p)}{\partial p_j} = p_i p_k R(1,1,1_k,p) + p_i (1-p_k) R(1,1,0_k,p) + (1-p_i) p_k R(0,1,1_k,p)$$

$$ + p_k (1-p_j) R(1,0,0_k,p) + (1-p_i) (1-p_k) R(0,1,0_k,p) -$$

$$ - (1-p_i) p_k R(0,0,1_k,p) - (1-p_i) (1-p_k) R(0,0,0_k,p).$$

Using the symmetry condition implied by Schur-convexity (concavity), we get

$$ \frac{\partial R(p)}{\partial p_k} = R(1,1,1_k,p) p_j p_k + R(1,1,0_k,p) [p_j + p_k - 3 p_j p_k]$$

$$ + R(1,0,0_k,p) (1-2 p_j - 2 p_k + 3 p_j p_k) - (1-p_j) (1-p_k) R(0,0,0_k,p).$$
\[ \frac{\partial R(p)}{\partial p_i} = p_i p_k R(l,1,l,1,k,p) + R(l,1,l,0,k,p)(p_i - 3p_i p_k + p_k) + R(1,0,0,k,p)(1-2p_i - 2p_k + 3p_k p_j)(1-p_i)(1-p_k)R(0,0,0,k,p). \]

Thus using (4.4) and (4.5)

\[ \frac{\partial R(p)}{\partial p_i} = -(p_i - p_j)^2[1 + p_k JRI (A_i, A_j, A_k) + CJRI (A_i, A_j | z_k = 0)] + (1-p_i)(1-p_k)R(0,0,0,k,p). \]

This holds if \( JRI (A_i, A_j, A_k) \leq 0 \) and \( CJRI (A_i, A_j | z_k = 0) \leq 0. \)

Hence the result follows by using the result of Theorem 4.4.

We extend the result of Hong et al. (2002) to JRI of three components and show that difference in the reliability functions for statistically independent and mutually dependent components is measured by their covariance, the JRI and the CJRI. The results are also derived for \( n \) components.

**Theorem 4.6:** Let \( R_{ijk} \) be the system reliability under the assumption of statistical dependence between components \( A_i, A_j \) and \( A_k \) and \( R \) be the system reliability when the components are independent. Then the difference in the reliability functions is given by

\[ R_{ijk} - R = \sigma_{ijk} JRI(A_i, A_j, A_k) + \sigma_{ij} CJRI(A_i, A_j | z_k = 0) + \sigma_{jk} CJRI(A_j, A_k | z_i = 0) + \sigma_{ik} CJRI(A_i, A_k | z_j = 0) \]

where \( \sigma_{ijk} = p_{ijk} - p_i p_j p_k \) and \( \sigma_{ij} \) is the covariance between \( A_i \) and \( A_j \).

\( \sigma_{ik} \) and \( \sigma_{jk} \) are defined analogously.

**Proof:** Under the assumption of statistical dependence between \( A_i, A_j \) and \( A_k \), we write

\[ P[X_i = 1, X_j = 1, X_k = 1] = p_{ijk}; \]
\[ P[X_i = 0, X_j = 0, X_k = 0] = p_{ijk}; \]
\[ P[X_i = 1, X_j = 0, X_k = 0] = p_{ij}; \]
\[ P[X_i = 1, X_j = 0, x_k = 0] = p_{ijk}; \]

and \( P[X_i = 1, X_j = 1] = p_y. \)

\( p_{ijk}, p_{ijk}, p_{ij}, p_{ijk} \) can be defined analogously.
Then the system reliability is
\[
R^{ijk} = p_{ijk} R(1, 1, 1, k; p) + p_{i*j} R(1, 1, 0, k; p) + p_{i*jk} R(0, 1, 1, k; p) + p_{ijk} R(1, 0, 1, k; p) + p_{ijk} R(1, 0, 0, k; p)
\]

It is observed that
\[
\begin{align*}
p_{jk} &= p_{j} - p_{ik} - p_{ik} - p_{ijk} - p_{ijk} \\
p_{ij} &= p_{i} - p_{jk} - p_{ij} + p_{ijk} \\
p_{jk} &= p_{j} - p_{ijk} + p_{ijk} - p_{ijk} \\
p_{ij} &= p_{i} - p_{jk} + p_{ijk} - p_{ijk}
\end{align*}
\]

Writing \( \sigma_{jk} = p_{ijk} - p_{i}p_{j} p_{k} \), \( \sigma_{y} = p_{y} - p_{p_{j}} p_{j} \), \( \sigma_{yj} = p_{yj} - p_{p_{j}} p_{j} \), \( \sigma_{yk} = p_{yk} - p_{p_{k}} p_{k} \)
we get
\[
R^{ijk} - R = \sigma_{ijk} CJR(A_i, A_j, A_k) + \sigma_{ij} CJR(A_i, A_j | z_k = 0) + \sigma_{jk} CJR(A_j, A_k | z_i = 0)
\]

The following theorems generalise the results given by Theorem 4.5 and 4.6 for \( n \) components. For the sake of simplicity of notation, we denote \( A_i \) by \( i \).

**Theorem 4.7**: If the reliability function \( R(p) \) is Schur-convex (concave), then

(i) \( JRI(1, 2, ..., n) \leq 0 \) (\( \geq 0 \));

(ii) \( CJR \) (all the components except \( j \) \( | z_{j} = 0 \)) \( \leq 0 \) (\( \geq 0 \)), for \( j = 1, 2, \ldots, n \);

(iii) \( CJR \) (all the components except \( j_1 \) and \( j_2 \) \( | z_{j_1} = 0, z_{j_2} = 0 \)) \( \leq 0 \) (\( \geq 0 \))

for \( j_1 = 1, 2, \ldots, n-1, j_2 = j_1 + 1, 2, \ldots, n \);

(iv) \( CJR \) (all the components except \( j_1, j_2 \) and \( j_3 \) \( | z_{j_1} = 0, z_{j_2} = 0, z_{j_3} = 0 \)) \( \leq 0 \) (\( \geq 0 \))

for \( j_1 = 1, 2, \ldots, n-2, j_2 = j_1 + 1, 2, \ldots, n-1, j_3 = j_2 + 1, 2, \ldots, n \).

The \( n \)th inequality is given as
\[
CJR(a, b | z_{j_1} = 0, z_{j_2} = 0, \ldots, z_{a-1} = 0, z_{a+1} = 0, \ldots, z_{b-1} = 0, z_{b+1} = 0, \ldots, z_{j_n} = 0) \leq 0 ( \geq 0),
\]

\( a \) and \( b \) can take values from \( 1, 2, \ldots, n \) but are different from \( j_i \)’s.
Theorem 4.8: Let components 1, 2... , n be mutually dependent and R^d be the system reliability under the assumption of statistical dependence of components. Let R be the reliability function of the system when the components are independent. Then the difference in reliability functions is given by

\[
R^d - R = \sigma_{1,2...n} JRI(1,2,...,n) + \\
+ \sum_{j_1=1}^{n} \sigma_{1,2...j_1-1,j_1+1,...,n} CJRI(1,2,...,j_1-1,j_1+1,...,n|z_{j_1} = 0) \\
+ \sum_{j_1=1}^{n-1} \sum_{j_2=j_1+1}^{n} [(\sigma_{1,2...j_2-1,j_2+1,...,n}) \\
CJRI(1,2,...,j_1-1,j_1+1,...,j_2-1,j_2+1,...,n|z_{j_1} = 0, z_{j_2} = 0)] \\
+ \cdots + \\
\sum_{j_1=1}^{3} \sum_{j_2=j_1+1}^{4} \cdots \sum_{j_n=j_{n-1}+1}^{n} [(\sigma_{a,b}) \\
CJRI(a,b|z_{j_1} = 0, z_{j_2} = 0,..., z_{a-1} = 0, z_{a+1} = 0,..,z_{b-1} = 0, z_{b+1} = 0,...,z_{j_n} = 0)],
\]

j_i's are different from a and b.

\[
\sigma_{1,2...n} = P_{1,2...n} - P_1P_2...P_n \quad \text{and}
\]

\[
\sigma_{ij} \quad \text{is the covariance between } A_i \text{ and } A_j.
\]

The remaining \( \sigma \)'s can be defined analogously.

4.5: Applications

Example 4.9: We consider a series-parallel system with nine components. It is a subsystem of a phased electronic system that is used in defence equipments.

![Figure 4.3: Subsystem of a phased electronic system](image-url)
Table 4.1 gives the CMRIs of some components of phased electronic system shown in Figure 4.3 for same pi’s.

In this table, we have considered five cases-

- both components in same series subsystems,
- both components in the same parallel subsystems,
- both in different series subsystems
- both in different parallel subsystems and
- one in parallel and the other in series subsystem.

Columns 2 and 4 of the table give the CMRIs when the conditioning component is functioning whereas columns 3 and 5 display the CMRIs when the conditioning component is non-functioning.

Table 4.1: Few CMRI’s for system in Figure 4.3

<table>
<thead>
<tr>
<th>Cases considered</th>
<th>Functioning $p_i = 0.9$</th>
<th>Non-functioning $p_i = 0.9$</th>
<th>Functioning $p_i = 0.98$</th>
<th>Non-functioning $p_i = 0.98$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>8, same series subsystems</td>
<td>0.7145</td>
<td>0</td>
<td>0.9404</td>
</tr>
<tr>
<td>8</td>
<td>2, different series subsystems</td>
<td>0.7144</td>
<td>0</td>
<td>0.9404</td>
</tr>
<tr>
<td>5</td>
<td>4, same parallel subsystem</td>
<td>0.0722</td>
<td>0</td>
<td>0.0188</td>
</tr>
<tr>
<td>6</td>
<td>4, different parallel subsystems</td>
<td>0</td>
<td>0.5846</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2, 4 in parallel, 2 in series</td>
<td>0.0656</td>
<td>0.0531</td>
<td>0.0184</td>
</tr>
</tbody>
</table>

Table 4.2 gives the CJRI of two components given the state of another component when all pi’s are same for the system depicted in Figure 4.3. We have considered different cases depending on the location of three components. Columns 2 and 4 give the CJRIs when the conditioning component is functioning whereas columns 3 and 5 display the CJRIs when the conditioning component is non-functioning.
Table 4.2: CJRI of two components for system in Figure 4.3

<table>
<thead>
<tr>
<th>Considered cases</th>
<th>Functioning $p_i=0.9$</th>
<th>Non-functioning $p_i=0.9$</th>
<th>Functioning $p_i=0.98$</th>
<th>Non-functioning $p_i=0.98$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 2</td>
<td>1, same series subsystem</td>
<td>0.8821</td>
<td>0</td>
<td>0.9792</td>
</tr>
<tr>
<td>8, 9</td>
<td>1, 8 and 9 in one series subsystem and 1 in different series</td>
<td>0.7933</td>
<td>0</td>
<td>0.9596</td>
</tr>
<tr>
<td>8, 3</td>
<td>2, 8 in a series subsystem different from series subsystem containing 3 and 2</td>
<td>0.7933</td>
<td>0</td>
<td>0.9596</td>
</tr>
<tr>
<td>5, 4</td>
<td>2, 4 and 5 in same parallel subsystem and 2 in a series</td>
<td>-0.7217</td>
<td>0</td>
<td>-0.9408</td>
</tr>
<tr>
<td>4, 3</td>
<td>2, 4 in a parallel and 2 and 3 in same series subsystem</td>
<td>0.0802</td>
<td>0</td>
<td>0.0192</td>
</tr>
<tr>
<td>9, 4</td>
<td>8, 8 and 9 in same series subsystem and 4 in a parallel subsystem</td>
<td>0.0721</td>
<td>0</td>
<td>0.0188</td>
</tr>
<tr>
<td>9, 8</td>
<td>4, 8 and 9 in same series subsystem and 4 in parallel</td>
<td>0.7217</td>
<td>0.649539</td>
<td>0.9408</td>
</tr>
<tr>
<td>8, 3</td>
<td>4, 8 and 3 in different series subsystems and 4 in parallel</td>
<td>0.7217</td>
<td>0.649539</td>
<td>0.9408</td>
</tr>
<tr>
<td>8, 5</td>
<td>4, 8 in a series subsystem and 4 and 5 in a parallel subsystem</td>
<td>0</td>
<td>0.649539</td>
<td>0</td>
</tr>
<tr>
<td>8, 6</td>
<td>4, 8 in a series subsystem and 6 and 4 in different parallel</td>
<td>0.0656</td>
<td>0.059049</td>
<td>0.0184</td>
</tr>
<tr>
<td>7, 6</td>
<td>4, 6 and 7 in same parallel subsystem and 4 in different parallel</td>
<td>-0.5905</td>
<td>-0.5314</td>
<td>-0.9039</td>
</tr>
</tbody>
</table>
The results from Tables 4.1 and 4.2 substantiate the conclusions in Sections 4.2 and 4.3. The negative CJRI of two components indicates that they are reliability substitutes of each other.

**Example 4.10:** We consider a bridge structure (Barlow and Proschan (1981)) which can be represented as a series in parallel system where some subsystems have common components.

![Bridge System Diagram](image)

**Figure 4.4: Bridge System**

The reliability function

\[
R(p) = p_1p_4 + p_2p_5 + p_2p_3p_4 + p_1p_3p_5 - p_1p_2p_3p_4 - p_1p_2p_4p_5 \\
- p_1p_2p_3p_5 - p_2p_3p_4p_5 - p_1p_3p_4p_5 + 2p_1p_2p_3p_4p_5.
\]

Figures 4.5 (i) and (ii) display CMRI (4|3) when component 3 is non-functioning / functioning for fixed \( p_5 = 0.9 \).

![CMRI Figures](image)

**Figure 4.5 (i)** – CMRI of 4|3 when component 3 is non-functioning

**Figure 4.5 (ii)** – CMRI of 4|3 when component 3 is functioning

**Figure 4.5**
For non-functioning component 3 and fixed $p_5$, CMRI $(4|3)$ is increasing in $p_1$ and decreasing in $p_2$. When component 3 is functioning, CMRI $(4|3)$ is increasing in $p_1$ and $p_2$.

Figures 4.6 (i) and (ii) graph the CMRI $(5|4)$ for $p_3=.9$.

![Figure 4.6 (i) - CMRI of 5|4 when component 4 is non-functioning](image1)

![Figure 4.6 (ii) - CMRI of 5|4 when component 4 is functioning](image2)

The above figures can be interpreted on similar lines.

Figures 4.7 and 4.8 depict CJRI $(4, 3|2)$ and CJRI $(4, 5|2)$ when component 2 is functioning/non-functioning.

![Figure 4.7 (i) - CJRI of (4,3|2) when component 2 is non-functioning](image3)

![Figure 4.7 (ii) - CJRI of (4,3|2) when component 2 is functioning](image4)
Figures 4.8 (i) and (ii) shows that CJRI (4, 5|2) is a decreasing function of $p_1$ and $p_3$ irrespective of the fact that component 2 is functioning or non-functioning. In both cases, it decreases from 0 to -1. However, CJRI (4, 5|2 functioning) = -1 when
- $p_1=1, p_3=1$;
- $p_1=0, p_3=1$;
- $p_1=1, p_3=0$.
But CJRI (4, 5|2 non-functioning) = -1 only when $p_1=1, p_3=1$.
Other figures can be interpreted similarly.

4.6: Conclusions

Gao et al. (2007) introduced the idea of conditional reliability importance (CRI) when the states of certain components are known to be functioning or non-functioning. It gives a measure of interaction between some components under the condition that few other components are working/non-working. For series-in-parallel and series-parallel systems, the expressions for CMRIs and CJRIs have been derived for independent but non-identical components. The results are illustrated through a phased type of electronic system and a bridge system. Schur-convexity (concavity) of reliability function is used to identify the sign of the joint importance of three or more components and CJRIs. The difference in the reliability functions for $n \geq 3$ statistically independent and mutually dependent components is shown to be measured in terms of covariance, the JRIs and the CJRIs.