Chapter 2
Multicomponent Joint Reliability Importance for Series-in-Parallel and Parallel-in-Series Systems

2.1: Introduction

In real life scenario, the design engineer always aims to make highly reliable design/products. But in doing so, there are some constraints on time, cost, space and availability of high end reliable components etc. So, he has to make certain compromises to find a tradeoff between these constraints and the reliability of products. One major study, which contributes to the reliability enhancement, is analysis of reliability importance measures of components/subsystems. These measures help in identifying those components or pairs of components which are more important for the functioning of the system. The system reliability can be improved by focusing on the reliability improvement of such components. For example, the functioning of a series system can be improved by increasing the reliability of the weakest component and the functioning of a parallel system can be improved by increasing the reliability of the strongest component. Hence the need is to quantify the reliability importance of various components of a complex coherent system. Various importance measures have been discussed in Chapter 1. These measures help in determining which components are more important for the system reliability improvement or are more critical for the system failure.

In this chapter, we focus on joint reliability importance (JRI) of two or more components for series-in-parallel and parallel-in-series systems.

MRI of any component measures the rate of change in system reliability, when the component’s reliability changes. It fails to capture the effect of the change of reliability of two or more components on the system reliability. In such situations, the joint reliability importance (JRI) of two or more components plays an important role. JRI quantifies the rate of change in system reliability as a consequence of a change in reliability of these components.

We consider two multi-component coherent systems - a series-in-parallel (series subsystems arranged in parallel) and a parallel-in-series (parallel subsystems arranged in
series) system. It is assumed that all the components in the subsystems are independent but not identically distributed. The expressions for the joint reliability importance of two or more components have been derived for the two systems. The results are extended to include subsystems where some of components are replicated. The findings are illustrated by considering a Bridge structure as a series-in-parallel system wherein some of the components are repeated in different subsystems. Numerical results have been provided for a series-in-parallel system with unreplicated components. Joint reliability importance for various combinations of components for both the illustrations are given through tables and figures.

The organization of the chapter is as follows. Section 2.2 includes some new results for JRI. Section 2.3 considers the series-in-parallel system studied earlier by Chang and Jan (2006). Expressions for JRI of 2, 3... m components have been derived when the components are independent but not identically distributed. The case when different subsystems have replicated components, is also dealt with. In Section 2.4, analogous results for a parallel-in-series system are discussed. Applications are given in Section 2.5. Conclusions are reported in Section 2.6.

2.2: Few relationships for JRI

Consider a coherent system consisting of n components where each component has two states, functioning or non-functioning. For i=1, 2 ... n, let \( A_i \) denote the \( i^{th} \) component with reliability \( p_i \) (probability that the component functions). \( R(p) \) denotes the system reliability for \( p = (p_1, p_2 \ldots p_n) \). The JRI of two distinct components i and j was defined in (1.3).

JRI for two distinct independent components is given by (1.4) and that for \( m \) independent components is given by (1.7).

Gao et al. (2007) gave some relationships for JRIs in terms of MRI defined in (1.1). We provide few more relationships in this direction. In terms of marginal reliability importance measures,

\[
\text{JRI}(A_i, A_j) = \text{MRI}(A_i, J_j) - \text{MRI}(A_i, 0_j) = \text{MRI}(A_j, J_i) - \text{MRI}(A_j, 0_i).
\] (2.1)
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where MRI (Ai, 1j) is the marginal reliability importance of ith component when jth component is functioning and

MRI (Ai, 0j) is the marginal reliability importance of ith component when jth component is non-functioning.

Let (s*A) denote the event that component A is functioning in the system and (s-A) indicate a situation where A is not functioning in the system. Using (2.1),

\[ JRI(A_i, A_j) = MRI_{s*A_j}(A_i) - MRI_{s-A_j}(A_i) - MRI_{s-A_i}(A_j) - MRI_{s*A_i}(A_j) . \]

JRI of three components can be expressed in terms of MRIs of different components as follows

\[ JRI(A_i, A_2, A_3) = MRI_{s*A_j}(A_i, A_2) - MRI_{s-A_j}(A_i, A_2) \]

\[ = [MRI_{s*A_2 A_3}(A_i) - MRI\{s\cdot A_2 A_3\} (A_i)] - [MRI_{s-A_2 A_3}(A_i) - MRI\{s-A_2 A_3\} (A_i)] \]

\[ = [MRI_{s*A_1 A_2}(A_2) - MRI\{s\cdot A_1 A_2\} (A_2)] - [MRI_{s-A_1 A_2}(A_2) - MRI\{s-A_1 A_2\} (A_2)] \]

\[ = [MRI\{A_2\} when A_1 and A_3 function] \]
\[ - [MRI\{A_2\} when A_1 does not function but A_3 functions] \]
\[ - [MRI\{A_2\} when A_3 does not function but A_1 functions] \]
\[ - [MRI\{A_2\} when A_1 and A_3 do not function]. \]

Analogously,

\[ JRI(A_i, A_2, A_3) = [MRI\{s\cdot A_2 A_3\} (A_2) - MRI\{s\cdot A_2\cdot A_3\} (A_2)] \]

\[ - [MRI\{s-A_2 A_3\} (A_2) - MRI\{s-A_2\cdot A_3\} (A_2)]. \]

\[ JRI(A_i, A_2, A_3) = [MRI\{s\cdot A_1 A_3\} (A_2) - MRI\{s\cdot A_1\cdot A_3\} (A_2)] \]

\[ - [MRI\{s-A_1 A_3\} (A_2) - MRI\{s-A_1\cdot A_3\} (A_2)]. \]
For m components, the above relationships can be generalized as follows:

\[ \text{JRI}(A_1, A_2, A_3, \ldots, A_m) \]

\[ = \text{JRI}_{s \rightarrow 0}^*(A_1, A_2, \ldots, A_{m-1}) - \text{JRI}_{s \rightarrow 0}^*(A_1, A_2, \ldots, A_{m-1}) \]

\[ = [\text{MRI}^*_{s \rightarrow 0}^*(A_1) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast \ldots \ast A_{m-1})](A_1)] - \]

\[ [\text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast \ldots \ast A_{m-1})](A_1) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast \ldots \ast A_{m-1})](A_1)] \]

\[ = [\text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m})](A_2) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m})](A_2)] - \]

\[ [\text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m})](A_2) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m})](A_2)] \]

\[ \vdots \]

\[ = [\text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-2})](A_{m-1}) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-2})](A_{m-1})] - \]

\[ [\text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-2})](A_{m-1}) - \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-2})](A_{m-1})] \]

**Remark 2.1:**

(i) If \( \text{JRI}^*_{s \rightarrow 0}^*[(A_1, A_2, \ldots, A_{m-1})] < 0 \), then

\[ \text{MRI}^*_{s \rightarrow 0}^*[(A_1, A_2, \ldots, A_{m-1})] < \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-1})](A_1) \]

which implies that component \( A_1 \) is more important when \( A_2, A_3, \ldots, A_{m-1} \) have failed and \( A_m \) is working than when \( A_2, A_3, \ldots, A_m \) are all working.

(ii) If \( \text{JRI}^*_{s \rightarrow 0}^*[(A_1, A_2, \ldots, A_{m-1})] < 0 \), then

\[ \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-1})](A_1) < \text{MRI}^*_{s \rightarrow 0}^*[(s \rightarrow A_m)^*(A_2 \ast A_3 \ast \ldots \ast A_{m-1})](A_1) \]

which implies that component \( A_1 \) is more important when \( A_2, A_3, \ldots, A_{m-1}, A_m \) have failed than when \( A_2, A_3, \ldots, A_{m-1} \) are all working and \( A_m \) has failed.

### 2.3: Multicomponent Joint Reliability Importance for Series-in-Parallel System

In this section, we study a series-in-parallel system depicted in Figure 2.1 and give expressions for the JRI of two or more components when components are independent but non-identical. Such systems are used for example when connecting speakers in an amplifier.

We assume that there are \( r \) series subsystems connected in parallel and \( n_i \) is the number of independent and non-identical components in each series subsystem.
For \( a=1, \ldots, n_i \) and \( i=1, 2, \ldots, r \), let

\( S_i \) denote the \( i \)th series subsystem;

\( A_{ia} \) denote the \( a \)th component in \( i \)th series subsystem and

\( p_{ia} \) be the corresponding probability of functioning.

Figure 2.1 depicts such a system.

![Diagram of a series-in-parallel system](image)

**Figure 2.1: Series-in-parallel system**

Chang and Jan (2006) considered this system with independent and identically distributed components and obtained the expression for joint reliability importance (JRI) of two components. We extend the results to \( m (\geq 2) \) independent but non-identical components.

**Theorem 2.2:** The reliability of a series-in-parallel system with independent, non-identical and unreplicated components in different subsystems, is

\[
R(p) = 1 - \prod_{i=1}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right).
\]  

(2.2)

**Proof** – The reliability of \( i \)th series subsystem is \( \prod_{a=1}^{n_i} p_{ia} \).

Since all the series subsystems are in a parallel network, the result follows.

**2.3.1: JRI of two and three components**

For deriving JRI of two components, there are two possibilities.

- Both components are in same series subsystem;
- they are in different series subsystems.
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**Theorem 2.3:** JRI of two components \( u \) and \( v \) in series-in-parallel system is

(i) \[ JRI(A_{ju}, A_{jv}) = \left( \prod_{i \neq j}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left\{ \prod_{a=x}^{n_j} P_{ja} \right\} \]

for \( u \) and \( v \) in the same \( j \)th series subsystem;

(ii) \[ JRI(A_{ju}, A_{kv}) = \left( \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left\{ \prod_{a=x}^{n_j} P_{ja} \right\} \left\{ \prod_{a=y}^{n_k} P_{ka} \right\} \]

for \( u \) in \( j \)th series subsystem and \( v \) in \( k \)th series subsystem and \( j \neq k = 1, 2, ..., r \).

**Proof** – (i) Using (1.4), we get

\[ JRI(A_{ju}, A_{jv}) = R(1_{ju}, 1_{jv}, p) + R(0_{ju}, 0_{jv}, p) - R(1_{ju}, 0_{jv}, p) - R(0_{ju}, 1_{jv}, p). \] (2.3)

Using Theorem 2.2, the reliabilities are

\[ R(1_{ju}, 1_{jv}, p) = 1 - \left( \prod_{i \neq j}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left( \prod_{a=x}^{n_j} P_{ja} \right) \]

and \[ R(1_{ju}, 0_{jv}, p) = R(0_{ju}, 1_{jv}, p) = R(0_{ju}, 0_{jv}, p) = 1 - \left( \prod_{i \neq j}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left( \prod_{a=y}^{n_k} P_{ka} \right). \]

Using (2.3),

\[ JRI(A_{ju}, A_{jv}) = \left( \prod_{i \neq j}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left\{ \prod_{a=x}^{n_j} P_{ja} \right\}. \]

(ii) Similarly for \( u \) in \( j \)th series subsystem and \( v \) in \( k \)th series subsystem, we have

\[ R(1_{ju}, 1_{kv}, p) = 1 - \left( \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left( \prod_{a=x}^{n_j} P_{ja} \right) \left( \prod_{a=y}^{n_k} P_{ka} \right); \]

\[ R(1_{ju}, 0_{kv}, p) = 1 - \left( \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left( \prod_{a=x}^{n_j} P_{ja} \right) \left( \prod_{a=y}^{n_k} P_{ka} \right); \]

\[ R(0_{ju}, 1_{kv}, p) = 1 - \left( \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} P_{ia} \right) \right) \right) \left( \prod_{a=x}^{n_j} P_{ja} \right) \left( \prod_{a=y}^{n_k} P_{ka} \right); \]

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\[ R(0_{ju}, 0_{kv}, p) = 1 - \left\{ \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\}. \]

Using (2.3), we get

\[ JRI(A_{ju}, A_{kv}) = -\left\{ \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\} \{\prod_{a=1}^{n_j} p_{ja}\} \{\prod_{a=1}^{n_k} p_{ka}\}. \]

**Remark 2.4:**

(i) When components \( u \) and \( v \) are in the same series subsystem, JRI is non-negative and hence components are reliability complements. This means that the system reliability improves if the reliability of both the components is improved.

(ii) JRI of \( u \) and \( v \) in different subsystems is non-positive implying that they are reliability substitutes. This means that component \( v \) becomes more important in case component \( u \) fails.

We extend Theorem 2.3 to JRI of three components. There exist three possibilities viz.

(i) All three components lie in the same subsystem;

(ii) Two components are in the same subsystem and third is in a different subsystem;

(iii) All three lie in different subsystems.

**Theorem 2.5:** The JRI of three components in series-in-parallel system is

(i) \( JRI(A_{ju}, A_{jv}, A_{jw}) = \left\{ \prod_{i \neq j}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\} \{\prod_{a=1}^{n_j} p_{ja}\} \{\prod_{a=1}^{n_k} p_{ka}\} \]

for \( u, v \) and \( w \) in the same subsystem;

(ii) \( JRI(A_{ju}, A_{jv}, A_{kw}) = -\left\{ \prod_{i \neq j, k}^{r} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\} \{\prod_{a=1}^{n_j} p_{ja}\} \{\prod_{a=1}^{n_k} p_{ka}\} \]

for \( u \) and \( v \) in same subsystem and \( w \) in a different subsystem;

(iii) \( JRI(A_{hu}, A_{jv}, A_{kw}) = \{\prod_{i \neq j, h, k} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \} \{\prod_{a=1}^{n_h} p_{ha}\} \{\prod_{a=1}^{n_j} p_{ja}\} \{\prod_{a=1}^{n_k} p_{ka}\} \]

for \( u, v \) and \( w \) in different subsystems.
Proof: (i) Using (1.7), we get

\[ \text{JRI}(A_{ju}, A_{jv}, A_{jw}) = R(1_{ju}, 1_{jv}, 1_{jw}; p) - R(1_{ju}, 1_{jv}, 0_{jw}; p) + R(1_{ju}, 0_{jv}, 0_{jw}; p) \]

\[ + R(0_{ju}, 1_{jv}, 0_{jw}; p) + R(0_{ju}, 0_{jv}, 1_{jw}; p) - R(1_{ju}, 0_{jv}, 0_{jw}; p) - R(0_{ju}, 1_{jv}, 1_{jw}; p) \]

\[ = R(1_{ju}, 1_{jv}, 1_{jw}; p) \]

\[ = 1 \left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \left( 1 - \left( \prod_{a=1}^{n_j} p_{ja} \right) \right) \left( 1 - \left( \prod_{a=1}^{n_k} p_{ka} \right) \right) \right\}. \]

The other terms in (2.4) are all equal to 1 – \left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\}.

Hence (i) follows.

(ii) When components u and v are in the same subsystem and w is in a different subsystem, the reliabilities can be written as

\[ R(1_{ju}, 1_{jv}, 1_{kw}; p) = \]

\[ = 1 \left\{ \prod_{i \neq j, k} \left( 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \left( 1 - \left( \prod_{a=1}^{n_j} p_{ja} \right) \right) \left( 1 - \left( \prod_{a=1}^{n_k} p_{ka} \right) \right) \right\}.

Using (1.7) and the above reliability expressions, we get the expression for JRI.
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(iii) When components u, v and w are in different subsystems, the reliabilities can be written as

\[ R(1_{hu}, 1_{jv}, 1_{kw}, P) = \cdots \]

\[ 1 - \left\{ \prod_{i \neq j, k, h} \left[ 1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right] \right\} \times \]

\[ \left\{ \left(1 - \left( \prod_{a=i}^{n_h} p_{ha} \right) \right) \left(1 - \left( \prod_{a=j}^{n_v} p_{ja} \right) \right) \left(1 - \left( \prod_{a=k}^{n_w} p_{ka} \right) \right) \right\}; \]

\[ R(0_{ju}, 0_{jv}, 0_{kw}, P) = 1 - \left\{ \prod_{i \neq h, j, k} \left(1 - \left( \prod_{a=1}^{n_i} p_{ia} \right) \right) \right\}; \]

Using (1.7) and the above reliability expressions, we get the expression for JRI.
2.3.2: JRI of m components

To find JRI of m components, we consider $P(m)$, the partition function that gives the number of partitions of integer $m$ into non-negative integers with sum as $m$. $P(0)$ is defined as 1 in partition theory.

The following lemma known as Euler’s recurrence formula (Andrews (1976)) states a recurrence relation for the partition function $P(m)$.

**Lemma 2.6:** For $m < 0$, $P(m) = 0$ and for $m > 0$

$$
(P(m) - P(m - 1) - P(m - 2) + P(m - 5) + P(m - 7) + \ldots + (-1)^k P(m - \frac{k}{2}(3k - 1)) + (-1)^k P(m - \frac{k}{2}(3k + 1)) + \ldots) = 0.
$$

Using the above lemma, Table 2.1 lists values of $P(m)$ for $m = 1$ through 16.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$P(m)$</th>
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<th>$P(m)$</th>
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<tr>
<td>1</td>
<td>1</td>
<td>9</td>
<td>30</td>
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<tr>
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<td>2</td>
<td>10</td>
<td>42</td>
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<td>3</td>
<td>3</td>
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<td>56</td>
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<td>15</td>
<td>15</td>
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</tr>
<tr>
<td>8</td>
<td>22</td>
<td>16</td>
<td>231</td>
</tr>
</tbody>
</table>

**Table 2.1:** values of partition function

The following theorem gives an expression for the JRI of m components.

**Theorem 2.7:** Without loss of generality, we assume that m components are chosen from the first $q$ subsystems where $m = \sum_{c=1}^{q} m_c$ and $m_c$ is the number of components from $c^{th}$ subsystem, $0 \leq m_c \leq n_c, c = 1, 2, \ldots, q$. Then JRI of m components is

$$
(-1)^{q+1} \left\{ \prod_{i=1}^{r} \left(1 - \left(\prod_{a=1}^{n_c} P_{ia}\right)\right) \right\} \left\{ \prod_{c=1}^{q} \left(\prod_{a=1}^{n_c} P_{ca}\right) \right\}.
$$
Remark 2.8:

(i) JRI of m components belongs to [-1, 1].

(ii) In Theorem 2.7,
- take $p_i a = p_i$, when components lying in the same subsystem are identical for every $a=1, 2, ..., n_i$;
- take $p_i a = p$, when all components are identical for every $a=1, 2, ..., n_i$ and $i=1, 2, ..., r$.

(iii) The result of Chang and Jan (2006) for JRI of two components of the same system with independent and identically distributed components, is a special case of the result given in Theorem 2.7.

2.3.3: Series-in-Parallel system with replicated components

In practice, it may be possible to express a complex system with independent components as a series-in-parallel system wherein some of the subsystems have few common components. An example of such a system is the Bridge structure which can be represented as a series-in-parallel system shown in Figure 2.2. All the components are replicated in different subsystems in this system.

![Figure 2.2: Bridge structure as a series-in-parallel system with replicated components](image)

Let $C_{12...r}$ denote the set of components that are common in $S_1, S_2$ and $S_r$. The reliability of the system when components are repeated in the different subsystems is given by
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\[ R(p) = \sum_{i=1}^{r} \left[ \prod_{a=1}^{n_i} P_{ia} \right] - \sum_{i \neq j} \left[ \prod_{a=1}^{n_i} P_{ia} \right] \left[ \prod_{b=1}^{n_j} P_{jb} \right] + \]

\[ \sum_{i>j<k=1}^{r} \left[ \prod_{a=1}^{n_i} P_{ia} \right] \left[ \prod_{b=1}^{n_j} P_{jb} \right] \left[ \prod_{c=1}^{n_k} P_{kc} \right] + \]

\[ ... + (-1)^{r} \left[ \prod_{a=1}^{n_1} P_{ia} \right] \left[ \prod_{b=1}^{n_2} P_{jb} \right] \left[ \prod_{l=1}^{n_r} P_{rl} \right] \]

(2.5)

Theorem 2.9 gives expressions of JRIs for two and three components for systems with replicated components. In this theorem, we write u for Au, v for Av and w for Aw for simplicity of notation.

**Theorem 2.9:** For a system with components u and v replicated in different subsystems,

\[ JR1(u, v) = \left\{ \begin{array}{c}
\sum_{(u,v) \in S_g}^{r} \left[ \prod_{a \neq u,v}^{g} P_{ia} \right] - \sum_{i \neq j}^{r} \left[ \prod_{a \neq u,v}^{i} P_{ia} \right] \left[ \prod_{b \neq u,v}^{j} P_{jb} \right] \\
+ \sum_{i < j < k}^{r} \left[ \prod_{a \neq u,v}^{i} P_{ia} \right] \left[ \prod_{b \neq u,v}^{j} P_{jb} \right] \left[ \prod_{c \neq u,v}^{k} P_{kc} \right] \\
+ \sum_{i \neq j}^{r} \left[ \prod_{a \neq u,v}^{i} P_{ia} \right] \left[ \prod_{b \neq u,v}^{j} P_{jb} \right] \left[ \prod_{c \neq u,v}^{k} P_{kc} \right] \end{array} \right\} \]

(2.5)
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\[ (-1)^{r-1} \left[ \prod_{a=1}^{n_1} p_{ia} \right] \left[ \prod_{b=1}^{n_2} p_{2b} \right] \cdots \left[ \prod_{i=1}^{n_i} p_{ia} \right] \cdots \left[ \prod_{k=1}^{n_k} p_{jk} \right] \cdots \left[ \prod_{l=1}^{n_l} p_{rl} \right] I_{S_{ij}} \]

where \( I_{S_i} \): indicator function of the event that \( S_i \) is the first set to which \( (u, v) \) belongs,

\( I_{S_{ij}} \): indicator function of the event that \( u \in S_i \) and \( v \in S_j \) for the first time.

(b) When \( u, v \) and \( w \) are replicated in different subsystems, then

\[ JRI(u, v, w) = \]

\[ \sum_{(u,v,w) \in S_g} \left\{ \prod_{a=u,v,w} p_{ga} \right\} - \sum_{(u,v,w) \in (S_i \cap S_j) \text{ only}} \left\{ \prod_{a=u,v,w} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \]

\[ - \sum_{(u,v) \in S_1 \text{ only}} \left\{ \prod_{a=u,v} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \]

\[ + \sum_{(u,v,w) \in S_i \cap S_j \cap S_k} \left\{ \prod_{a=u,v,w} p_{ia} \right\} \left\{ \prod_{b=1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]

\[ + \sum_{(u,v) \in S_1 \cap S_j \cap S_k} \left\{ \prod_{a=u,v} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]

\[ + \sum_{(u,v,w) \in S_i \cap S_j \cap S_k} \left\{ \prod_{a=u,v,w} p_{ia} \right\} \left\{ \prod_{b=1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]

\[ + \sum_{(u,v) \in S_1 \cap S_j \cap S_k} \left\{ \prod_{a=u,v} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]

\[ + \sum_{(u,v,w) \in S_i \cap S_j \cap S_k} \left\{ \prod_{a=u,v,w} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]

\[ + \sum_{(u,v) \in S_1 \cap S_j \cap S_k} \left\{ \prod_{a=u,v} p_{ia} \right\} \left\{ \sum_{j \neq 1}^{n_j} p_{jb} \right\} \left\{ \prod_{c=1}^{n_k} p_{kc} \right\} \]
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\[ + \left\{ \sum_{(u,v) \in S_i \text{ only}} \left[ \prod_{a \neq u,v} p_{ia} \right] \right\} \left\{ \sum_{j<k=1} \left[ \prod_{b \in C_{ij}} p_{jb} \right] \left[ \prod_{c \in C_{ik}} p_{kc} \right] \right\} \]

\[ + \left\{ \sum_{i<j<k=1} \left[ \prod_{a \neq u,v} p_{ia} \right] \left[ \prod_{b \neq v} p_{jb} \right] \left[ \prod_{c \neq w} p_{kc} \right] \right\} \]

\[ + ... + \]

\[ + (-1)^{r-1} \left[ \prod_{a=1}^{n_1} p_{1a} \right] \left[ \prod_{b=1}^{n_2} p_{2b} \right] ... \left[ \prod_{g=1}^{n_k} p_{rg} \right] \left[ \prod_{l=1}^{r_i} p_{ri} \right] l_{Si} \]

\[ + (-1)^{r-1} \left[ \prod_{a=1}^{n_1} p_{1a} \right] \left[ \prod_{b=1}^{n_2} p_{2b} \right] ... \left[ \prod_{g=1}^{n_k} p_{rg} \right] \left[ \prod_{l=1}^{r_i} p_{ri} \right] l_{Sij} \]

\[ + (-1)^{r-1} \left[ \prod_{a=1}^{n_1} p_{1a} \right] \left[ \prod_{b=1}^{n_2} p_{2b} \right] ... \left[ \prod_{g=1}^{n_k} p_{rg} \right] \left[ \prod_{l=1}^{r_i} p_{ri} \right] l_{Sijk} \]

where \( l_{Si} \): indicator function of the event that \( S_i \) is the first set to which \((u, v, w)\) belongs;

\( l_{Sij} \): indicator function of the event that \( u, v \in S_i \) and \( w \in S_j \) for the first time;

\( l_{Sijk} \): indicator function of the event that \( u \in S_i, v \in S_j \) and \( w \in S_k \) for the first time.

**Proof:** It follows by using the definition of JRI.

Next, we find the JRI's of the components of the system depicted in Figure 2.2 by using Theorem 2.9. The reliability function of this system can be written as

\[ R(p) = p_1 p_4 + p_2 p_5 + p_2 p_3 p_4 + p_1 p_3 p_5 - p_1 p_2 p_3 p_4 + p_1 p_2 p_4 p_5 - p_1 p_3 p_4 p_5 - p_1 p_3 p_4 p_5 + 2 p_1 p_2 p_3 p_4 p_5. \]

where \( p_i \) denotes the reliability of the \( i \)-th component.
The JRI’s are given as
\[ JRI(1,2) = -p_4 p_5 - p_3 p_4 - p_3 p_5 + 2p_3 p_4 p_5; \]
\[ JRI(1,3) = p_5 - p_2 p_5 - p_2 p_4 - p_4 p_5 + 2p_2 p_4 p_5; \]
\[ JRI(1,4) = 1 - p_2 p_5 - p_2 p_3 - p_3 p_5 + 2p_2 p_3 p_5; \]
\[ JRI(1,2,3) = -p_5 - p_4 + 2p_4 p_5; \]
\[ JRI(3,4,5) = -p_1 - p_2 + 2p_1 p_2; \]
\[ JRI(1,3,5) = 1 - p_2 - p_4 + 2p_2 p_4; \]
\[ JRI(2,3,4) = 1 - p_1 - p_5 + 2p_1 p_5. \]

Using the definition of JRI, it is easy to see that
\[ JRI(1,2,3,4) = -1 + 2p_5, \]
\[ JRI(1,2,3,4,5) = 1. \]

2.4: Parallel-in-Series System

We consider a system with \( p \) parallel subsystems connected in series and \( n_i \) as the number of independent and non-identical components in each parallel subsystem. For \( a=1,\ldots,n_i \) and \( i=1,2,\ldots,p \), let
- \( P_i \) denote the \( i^{th} \) parallel subsystem,
- \( A_{ia} \) be the \( a^{th} \) component in \( P_i \) and
- \( p_{ia} \) be the probability of functioning of \( a^{th} \) component.

Such a system is illustrated in Figure 2.3.

![Figure 2.3: Parallel-in-series system](image)
Theorem 2.10: (a) The reliability of a parallel-in-series system with independent, non-identical and unreplicated components in different subsystems is

\[ R(p) = \prod_{i=1}^{p} \left( 1 - \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \]  

(2.6)

(b) The JRI of two components \( u \) and \( v \) in a parallel-in-series system is

(i) \( JRI(A_{ju}, A_{jv}) = -\left\{ p \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ \prod_{a \neq u, v}^{n_j} (1 - p_{ja}) \right\} \)

for \( u \) and \( v \) in the same \( j \)th parallel subsystem;

(ii) \( JRI(A_{ju}, A_{kv}) = \)

\[ \left\{ p \prod_{i \neq j, k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ \prod_{a \neq u, v}^{n_j} (1 - p_{ja}) \right\} \left\{ \prod_{a \neq u, v}^{n_k} (1 - p_{ka}) \right\} \]

for \( u \) in \( j \)th parallel subsystem and \( v \) in \( k \)th parallel subsystem and \( j \neq k = 1, 2, \ldots, p \).

(c) The JRI of three components for a parallel-in-series system is

(i) \( JRI(A_{ju}, A_{jv}, A_{jw}) = \left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ \prod_{a \neq u, v, w}^{n_j} (1 - p_{ja}) \right\} \)

for \( u, v \) and \( w \) in the same \( j \)th parallel subsystem;

(ii) \( JRI(A_{hu}, A_{jv}, A_{kw}) = \)

\[ -\left\{ p \prod_{i \neq h, j, k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ \prod_{a \neq u, v}^{n_j} (1 - p_{ja}) \right\} \left\{ \prod_{a \neq u, v}^{n_k} (1 - p_{ka}) \right\} \]

for \( u \) and \( v \) in \( j \)th parallel subsystem and \( w \) in \( k \)th parallel subsystem and \( j \neq k = 1, 2, \ldots, p \).

(iii) \( JRI(A_{hu}, A_{jv}, A_{kw}) = \left\{ \prod_{i \neq h, j, k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \times \)

\[ \left\{ \prod_{a \neq u}^{n_h} (1 - p_{ha}) \right\} \left\{ \prod_{a \neq v}^{n_j} (1 - p_{ja}) \right\} \left\{ \prod_{a \neq w}^{n_k} (1 - p_{ka}) \right\}. \]

for \( u \) in \( h \)th parallel subsystem, \( v \) in \( j \)th and \( w \) in \( k \)th parallel subsystem and \( h \neq j \neq k = 1, 2, \ldots, p \).
Proof- (a) The reliability of \(i^{th}\) parallel subsystem is \(1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right)\). Since all the parallel subsystems are in a series network, hence the result follows.

(b) (i) Using (2.6), the reliabilities are

\[
R(1_{ju}, 1_{jv}, p) = R(1_{ju}, 0_{jv}, p) = R(0_{ju}, 1_{jv}, p) = \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right)
\]

\[
R(0_{ju}, 0_{jv}, p) = \left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left( 1 - \left( \prod_{a=1}^{n_j} (1 - p_{ja}) \right) \right)
\]

Using (1.4), we get

\[
JRI(A_{ju}, A_{jv}) =
\]

\[
R(1_{ju}, 1_{jv}, p) + R(0_{ju}, 0_{jv}, p) - R(1_{ju}, 0_{jv}, p) - R(0_{ju}, 1_{jv}, p).
\]

Hence

\[
JRI(A_{ju}, A_{jv}) =
\]

\[- \left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left( 1 - \left( \prod_{a=1}^{n_j} (1 - p_{ja}) \right) \right).\]

(ii) For \(u\) in \(j^{th}\) parallel subsystem and \(v\) in \(k^{th}\) parallel subsystem, we have

\[
R(1_{ju}, 1_{kv}, p) = \prod_{i \neq j, k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right);
\]

\[
R(1_{ju}, 0_{kv}, p) =
\]

\[
\left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left( 1 - \left( \prod_{a=1}^{n_k} (1 - p_{ka}) \right) \right);
\]

\[
R(0_{ju}, 1_{kv}, p) =
\]

\[
\left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left( 1 - \prod_{a \neq u,v} (1 - p_{ja}) \right);
\]

\[
R(0_{ju}, 0_{kv}, p) =
\]

\[
\left\{ \prod_{i \neq j} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left( 1 - \prod_{a \neq u,v} (1 - p_{ja}) \right) \left( 1 - \left( \prod_{a=1}^{n} (1 - p_{ka}) \right) \right).
\]
Using (1.4), we get

\[ JRI(A_{ju}, A_{kv}) = \]
\[ \prod_{i \neq j,k} \left( 1 - (\Pi_{a=1}^{n_i}(1 - p_{ia})) \right) \left( 1 - \Pi_{a=1}^{n_j}(1 - p_{ja}) \right) \left( 1 - \Pi_{a=1}^{n_k}(1 - p_{ka}) \right). \]

(c) (i) \( R(0_{ju}, 0_{jv}, 0_{jw}, p) \)
\[ = \left\{ \prod_{i \neq j,k} \left( 1 - (\Pi_{a=1}^{n_i}(1 - p_{ia})) \right) \right\} \left\{ 1 - \left( \prod_{a=1}^{n_j} (1 - p_{ja}) \right) \right\} \left\{ 1 - \left( \prod_{a=1}^{n_k} (1 - p_{ka}) \right) \right\}. \]

For other possible choices, the reliabilities are equal to \( \left\{ \prod_{i \neq j} \left( 1 - \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right\}. \)

Hence (i) follows.

(ii) The reliabilities can be written as
\[ R(1_{ju}, 1_{jv}, 1_{kw}, p) = R(1_{ju}, 0_{jv}, 1_{kw}, p) = R(0_{ju}, 1_{jv}, 1_{kw}, p) \]
\[ = \left\{ \prod_{i \neq j,k} \left( 1 - \Pi_{a=1}^{n_i}(1 - p_{ia}) \right) \right\}; \]
\[ R(0_{ju}, 0_{jv}, 0_{kw}, p) = \]
\[ \left\{ \prod_{i \neq j,k} \left( 1 - \Pi_{a=1}^{n_i}(1 - p_{ia}) \right) \right\} \left\{ 1 - \Pi_{a=1}^{n_j}(1 - p_{ja}) \right\} \left\{ 1 - \Pi_{a=1}^{n_k}(1 - p_{ka}) \right\}; \]
\[ R(1_{ju}, 1_{jv}, 0_{kw}, p) = R(0_{ju}, 1_{jv}, 0_{kw}, p) \]
\[ = \left\{ \prod_{i \neq j,k} \left( 1 - \Pi_{a=1}^{n_i}(1 - p_{ia}) \right) \right\} \left\{ 1 - \Pi_{a=1}^{n_k}(1 - p_{ka}) \right\}; \]
\[ R(0_{ju}, 0_{jv}, 1_{kw}, p) = \left\{ \prod_{i \neq j,k} \left( 1 - \Pi_{a=1}^{n_i}(1 - p_{ia}) \right) \right\} \left\{ 1 - \left( \prod_{a=1}^{n_j} (1 - p_{ja}) \right) \right\} \left\{ 1 - \left( \prod_{a=1}^{n_k} (1 - p_{ka}) \right) \right\}. \]

Using (1.7) and the above reliability expressions, we get the expression for JRI given by

\[ JRI(A_{ju}, A_{jv}, A_{kw}) = \]
\[ - \left\{ \prod_{i \neq j,k} \left( 1 - (\Pi_{a=1}^{n_i}(1 - p_{ia})) \right) \right\} \left\{ \Pi_{a=1}^{n_j}(1 - p_{ja}) \right\} \left\{ \Pi_{a=1}^{n_k}(1 - p_{ka}) \right\} \]

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(iii) For components u, v and w in different subsystems,

\[ R(1_{hu}, 1_{jv}, 1_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\}; \]

\[ R(0_{hu}, 0_{jv}, 0_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ \prod_{a=x}^{n_h} (1 - p_{ha}) \right\} \times \]

\[ \left\{ 1 - \prod_{a=x}^{n_j} (1 - p_{ja}) \right\} \left\{ 1 - \prod_{a=x}^{n_k} (1 - p_{ka}) \right\}; \]

\[ R(1_{hu}, 0_{jv}, 0_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ 1 - \prod_{a=x}^{n_h} (1 - p_{ha}) \right\}; \]

\[ R(0_{hu}, 1_{jv}, 1_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ 1 - \prod_{a=x}^{n_h} (1 - p_{ha}) \right\}; \]

\[ R(0_{hu}, 1_{jv}, 0_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ 1 - \prod_{a=x}^{n_h} (1 - p_{ha}) \right\} \left\{ 1 - \prod_{a=x}^{n_k} (1 - p_{ka}) \right\}; \]

\[ R(0_{hu}, 0_{jv}, 1_{kw}, p) = \left\{ \prod_{i \neq h,j,k} \left( 1 - \left( \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \right) \right\} \left\{ 1 - \prod_{a=x}^{n_h} (1 - p_{ha}) \right\} \left\{ 1 - \prod_{a=x}^{n_j} (1 - p_{ja}) \right\}. \]

Using (1.7) and the above reliability expressions, we get the expression for JRI.
Remark 2.11: (i) JRI is non-positive when two components are in the same parallel subsystem and hence components are reliability substitutes. This means that if either of them fails, the other becomes more important.

(ii) When components \( u \) and \( v \) are in different subsystems, JRI is non-negative implying that they are reliability complements. This means that component \( v \) becomes more important if component \( u \) is functioning.

2.4.1: JRI of \( m \) components

The JRI of \( m \) components for a parallel-in-series system is given in the following theorem.

Theorem 2.12: Let \( m \) components be chosen from the first \( q \) subsystems where

\[
m = \sum_{c=1}^{q} m_c
\]

and \( m_c \) is the number of components from \( c^{th} \) subsystem, \( 0 \leq m_c \leq n_c \), \( c = 1, 2, \ldots, q \). Then JRI of \( m \) components is

\[
(-1)^{q+m} \prod_{i=1}^{p} \left( 1 - \prod_{a=1}^{n_i} (1 - p_{ia}) \right) \left( \prod_{c=1}^{q} \left( \prod_{d=1}^{m_c} (1 - p_{cd}) \right) \right).
\]

Remark 2.13: For finding the JRI of \( m \) components of a parallel-in-series system using Theorem 2.12,

- put \( p_{ja} = p_j \), when components lying in the same subsystem are identical for \( a=1, 2, \ldots, n_j \);
- put \( p_{ja} = p \), when all components are identical for \( a=1, 2, \ldots, n_j \) and \( j=1, 2, \ldots, p \).

2.5: Applications

In this section, we illustrate the method for finding the JRIs for two series-in-parallel systems. One system has distinct components and the other has replicated components in various subsystems.

Example 2.13: Let 1, 2, 3 and 4 be four clamps, needed to press together two flanges. These flanges are used for fastening of two mechanical parts. It has been experienced that the fastening holds when at least two opposite clamps function. The reliability
block diagram can be put as a series-in-parallel system with different components in two subsystems (Birolini (2007)) and is depicted in Figure 2.5.

![Diagram](image)

**Figure 2.5: Series-in-parallel system with unreplicated components**

For a specified choice of reliabilities of different components, JRIs of two, three and four components for the above system are listed in Table 2.2.

<table>
<thead>
<tr>
<th></th>
<th>p = [.82, .86, .9, .92]</th>
</tr>
</thead>
<tbody>
<tr>
<td>JRI (1, 2)</td>
<td>-0.828</td>
</tr>
<tr>
<td>JRI (1, 3)</td>
<td>0.209</td>
</tr>
<tr>
<td>JRI (1, 4)</td>
<td>-0.774</td>
</tr>
<tr>
<td>JRI (2, 3)</td>
<td>-0.754</td>
</tr>
<tr>
<td>JRI (2, 4)</td>
<td>0.262</td>
</tr>
<tr>
<td>JRI (3, 4)</td>
<td>-0.705</td>
</tr>
<tr>
<td>JRI(1, 2, 3)</td>
<td>-0.920</td>
</tr>
<tr>
<td>JRI(1, 2, 4)</td>
<td>-0.900</td>
</tr>
<tr>
<td>JRI(1, 3, 4)</td>
<td>-0.820</td>
</tr>
<tr>
<td>JRI(2, 3, 4)</td>
<td>-0.820</td>
</tr>
<tr>
<td>JRI(1, 2, 3, 4)</td>
<td>-1</td>
</tr>
</tbody>
</table>

**TABLE 2.2: Values of JRIs**

The values in Table 2.2 illustrate the following:

(a) JRI of two components in the same subsystem is positive implying that they are complements;

(b) JRI of two components in different subsystems is negative implying that they are substitutes.

(c) JRI of three components is always negative. It is due to the fact that one of them lies in a different subsystem;

(d) JRI of all the components attains the lower bound of -1.

Figure 2.6 depicts the changes in the values of JRIs of two components corresponding to a change in the values of p₁.
Figure 2.6: JRIs of 2 components as a function of $p_1$

It can be concluded from Figure 2.6 that

(a) JRI (1, 2), JRI (1, 3) and JRI (1, 4) remain constant with respect to a change in the values of $p_1$;
(b) JRI (2, 3) and JRI (3, 4) decrease with respect to an increase in the values of $p_1$;
(c) JRI (2, 4) decreases with respect to an increase in the values of $p_1$.

The following example considers a bridge structure that can be put in the form of a series-in-parallel system with components replicated in different subsystems.

Example 2.14: For the bridge structure represented as a series-in-parallel system as in Figure 2.2, the JRIs of two components for some values of $p_1$ are listed below:

<table>
<thead>
<tr>
<th></th>
<th>JRI(1, 2)</th>
<th>JRI(2, 4)</th>
<th>JRI(2, 3)</th>
<th>JRI(1, 3)</th>
<th>JRI(2, 5)</th>
<th>JRI(1, 4)</th>
<th>JRI(3, 4)</th>
<th>JRI(1, 5)</th>
<th>JRI(3, 5)</th>
<th>JRI(2, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>0.82</td>
<td>0.86</td>
<td>0.9</td>
<td>0.92</td>
<td>0.88</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JRI(1, 2)</td>
<td>-0.972</td>
<td>-0.038</td>
<td>0.725</td>
<td>0.039</td>
<td>0.069</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JRI(2, 3)</td>
<td>0.143</td>
<td>0.133</td>
<td>0.133</td>
<td>0.082</td>
<td>0.143</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 2.3: JRIs of two components
The values in Table 2.3 illustrate the following:

(a) JRI of two components comprising a single subsystem is positive implying that they are complements;

(b) JRI of two components is negative if they occur in same subsystem and are also replicated in different subsystems;

(c) JRI of two components takes large negative value if they occur only in different subsystems.

Table 2.4 gives the JRI of three components for different combinations of five components.

<table>
<thead>
<tr>
<th>p=[.82, .86, .9, .92, .88]</th>
<th>JRI</th>
<th>JRI</th>
<th>JRI</th>
<th>JRI</th>
<th>JRI</th>
<th>JRI</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3)</td>
<td>-0.181</td>
<td>(1, 2, 5)</td>
<td>-0.164</td>
<td>(3, 1, 4)</td>
<td>-0.226</td>
<td>(4, 2, 3)</td>
</tr>
<tr>
<td>(1, 2, 4)</td>
<td>-0.196</td>
<td>(2, 3, 5)</td>
<td>-0.231</td>
<td>(3, 4, 5)</td>
<td>-0.269</td>
<td>(5, 1, 4)</td>
</tr>
<tr>
<td>(1, 2, 3, 4)</td>
<td>0.760</td>
<td>(1, 2, 3, 5)</td>
<td>0.840</td>
<td>(1, 2, 4, 5)</td>
<td>0.800</td>
<td>(2, 3, 4, 5)</td>
</tr>
</tbody>
</table>

| TABLE 2.4: JRIs of 3 components |

It is seen that JRI (1, 2, 3, 4, 5) =1.

From the values in Table 2.4, it is observed that

(a) JRI of three components lying in the same subsystem is positive;

(b) JRI of three components is negative when one of them lies in a different subsystem;

(c) JRI of four components is always positive;

(d) JRI of all the components attains the upper bound of +1.

The effect of change in JRIs of three components, corresponding to a change in the value of \( p_1 \) is depicted in Figure 2.7.
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Figure 2.7: JRIs of three components as a function of $p_1$ for bridge structure

It is depicted by Figure 2.7 that
(a) JRI (3, 4, 5) remains constant with respect to a change in the values of $p_1$;
(b) JRI (2, 4, 5) and JRI (3, 4, 5) increase with respect to an increase in the values of $p_1$;
(c) JRI (3, 4, 2) increases with respect to $p_1$.

2.6: Conclusions

We generalize the work of Chang and Jan (2006) and Gao et al. (2007). They found the JRI of two components of a series-in-parallel system when components are identical and independently distributed. We derive the JRI of $m \geq 2$ components for series-in-parallel and parallel-in-series systems when components are independent but non-identical. Expressions are obtained for series-in-parallel system when components are replicated or unreplicated in different subsystems. These results can be useful for analyzing the joint effect of reliability of several components on the system reliability and can help in designing systems that function more effectively and have a longer life.