Chapter II

ENDOMORPHISM RINGS OF HARADA MODULES
AND
FULL LINEAR RINGS
0. INTRODUCTION

Let $V$ be a left vector space over a division ring $D$. Then $E = \text{End}_D(V)$ is called a left full linear ring (see [F1, page 95]). F. Kasch, in his unpublished lecture notes [K1], has proved the following:

**Theorem.** If $M$ is a Harada module and $S = \text{End}(M)$ then $S/J(S)$ is isomorphic to a direct product of left full linear rings.

A module $M$ is said to be discrete if it satisfies the following two conditions:

$(D_1)$. For any submodule $N$ of $M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M$;

$(D_2)$. If $N$ is a submodule of $M$ such that $M/N$ is isomorphic to a direct summand of $M$ then $N$ is a direct summand of $M$.

It is well known that discrete modules are Harada modules (see [MM, Theorem 4.15 and Corollary 5.5]). However, converse is not true as any module which has a finite LE decomposition is a Harada module (see [AF, Corollary 12.7]), but may not be a discrete module. For example, $(\mathbb{Z}_p^\infty)^{(n)}$, for any natural number $n$, is a Harada module which is not discrete as $\mathbb{Z}_p^\infty$ does not satisfy $D_2$. Recently, Ali and Zelmanowitz [ALZ] proved that if $M$ is a discrete module and $S = \text{End}(M)$, then $S/J(S)$ is a left continuous ring. Later Zelmanowitz [Z], improving upon this result, proved the Kasch’s Theorem for the particular case of discrete modules using the results from monograph of Mohamed and Müller [MM] and from Kasch [K1].

The work of this chapter has been accepted for publication [GI2].
In this chapter we give a simple proof of Kasch's Theorem using a well-known property of Harada modules given in Corollary 1.7.

In section 1 we recall some definitions and some known results about Harada modules. In section 2 we prove our main result.

1. DEFINITIONS AND NOTATIONS

For notations and terminology, we refer the reader to [AF] and [MM].

Recall that a ring is called local if its Jacobson radical is a maximal left ideal.

Definition 1.1. A module $M$ is said to be an LE (local endomorphism ring) module if it has local ring of endomorphisms. A decomposition $M = \bigoplus_A M_a$ is said to be an LE decomposition if each $M_a$ is an LE module.

Definition 1.2. A decomposition $M = \bigoplus_A M_a$ is said to complement direct summands if for any direct summand $K$ of $M$, there exists a subset $B$ of $A$ such that

$$M = K \oplus (\bigoplus_B M_b).$$

Definition 1.3. A module $M$ is called a Harada module if it has an LE decomposition that complements direct summands.

It follows from [AF, Corollary 12.5] that any LE decomposition of a Harada module complements direct summands. Also from [AF, Lemma 12.3] it follows that a direct summand of a Harada module is a Harada module.

Definition 1.4. A family $\{M_a : \alpha \in A\}$ of modules is called locally semi-$T$-
nilpotent if for any subfamily \( \{M_{\alpha_i} : i = 1, 2, \ldots \} \), with distinct \( \alpha_i \), and any family of non-isomorphisms \( f_i : M_{\alpha_i} \to M_{\alpha_{i+1}} \), and for every \( x \in M_{\alpha_1} \), there exists a natural number \( n \), depending upon \( x \), such that \( f_n \cdots f_2 f_1(x) = 0 \) (see [MM, Definition 2.23]). A decomposition \( M = \bigoplus A M_\alpha \) of a module \( M \) will be called locally semi-T-nilpotent if \( \{M_\alpha : \alpha \in A\} \) is a locally semi-T-nilpotent family.

**Lemma 1.5** ([MM, Theorem 2.25]). An LE decomposition \( M = \bigoplus A M_\alpha \) of a module \( M \) complements direct summands if and only if \( \{M_\alpha : \alpha \in A\} \) is a locally semi-T-nilpotent family. \( \square \)

**Lemma 1.6** (Kanbara [Kan, Corollary 1]). Let \( M = \bigoplus A M_\alpha \) be an LE decomposition of a module \( M \), \( S = \text{End}(M) \) and \( T = \{f \in S : l_a f \pi_b \text{ is a non-isomorphism for each } a, b \text{ in } A\} \), where \( l_a : M_\alpha \to M = \bigoplus A M_\alpha \) and \( \pi_a : M = \bigoplus A M_\alpha \to M_\alpha \) are natural injections and projections respectively. Then \( \{M_\alpha\}_{\alpha \in A} \) is a locally semi-T-nilpotent family if and only if \( J(S) = T \). \( \square \)

**Corollary 1.7.** Let \( M = \bigoplus A M_\alpha \) be an LE decomposition of a Harada module \( M \), \( S = \text{End}(M) \) and \( T \) as in Lemma 1.6. Then \( J(S) = T \).

**Proof.** By Lemma 1.5 \( \{M_\alpha : \alpha \in A\} \) is a locally semi-T-nilpotent family and thus by Lemma 1.6 \( J(S) = T \). \( \square \)

**Definition 1.8.** Let \( M = \bigoplus A M_\alpha \) be a decomposition of an arbitrary module. For fixed \( a_1 \in A \), we set \( A_1 = \{a \in A : M_\alpha \cong M_{a_1}\} \). Then \( H_1 = \bigoplus A_1 M_\alpha \) will
be called a *homogeneous component* of $M$ with respect to the decomposition $M = \bigoplus_A M_a$.

**Definition 1.9.** A family $(f_\alpha)_{\alpha \in A}$ of $R$-homomorphisms, with domain $R M$, is called *summable* if for each $m \in M$, $m f_\alpha = 0$ for all but finitely many $\alpha \in A$.

## 2. ENDO Morphism Rings of Harada Modules

In this section we prove the main result of this chapter. First we fix the notations for the results to follow.

**Notations 2.1.** Let $M = \bigoplus_A M_a$ be an LE decomposition of a Harada module. Then $M = \bigoplus_I H_i$, where for each $i \in I$, $H_i = \bigoplus_{A_i} M_a$, for some $A_i \subseteq A$, is a homogeneous component of $M$ with respect to the decomposition $M = \bigoplus_A M_a$.

We set $S = \text{End}(M)$ and $S_i = \text{End}(H_i)$, for each $i \in I$. For any $B \subseteq A$, we denote by $l_B : \bigoplus_B M_b \rightarrow \bigoplus_A M_a$ and $\pi_B : \bigoplus_A M_a \rightarrow \bigoplus_B M_b$ the natural injection and projection respectively. We will identify, whenever convenient, $S$ with the ring of all row-summable matrices $(f_{ij})_{i,j \in I}$ (i.e. for each $i \in I$, the family $(f_{ij})_{j \in I}$ is summable), where $f_{ij} : M_i \rightarrow M_j$ is a homomorphism; under the map $f \rightarrow (l_i f_{ij})_{i,j \in I}$.

To prove the *Theorem* we show that:

(a). $S/J(S) \cong \prod_I S_i/J(S_i)$ and

(b). For each $i \in I$, $S_i/J(S_i)$ is isomorphic to a left full linear ring.

**Lemma 2.2.** In the notations of 2.1 above, $\theta_i : S \rightarrow S_i/J(S_i)$ defined as

$$\theta_i(f) = l_A f \pi_A + J(S_i)$$
where maps are acting on the right, is a ring epimorphism with $J(S) \subseteq \text{Ker}(\theta_i)$.

**Proof.** Obviously $\theta_i$ is additive and onto. We now show that for $f, g \in S$, $\theta_i(fg) = \theta_i(f)\theta_i(g)$. Observe that, $l_{A_i} f g \pi_{A_i} = l_{A_i} f \pi_{A_i} + l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} - l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} = l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} + l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_{A_i}$. So it suffices to show that $l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_{A_i} \in J(S_i)$ and for which in view of Corollary 1.7, we show that for $a, b \in A_i$, $l_{A_i} l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_{A_i} \pi_b = l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_b$ is a non-isomorphism. Suppose, on the contrary, there exists $a : M_b \rightarrow M_a$ such that $l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_b \alpha = I_{M_a}$. Thus $l_{A_i} f (1 - \pi_{A_i} l_{A_i}) : M_a \rightarrow \oplus_{A \setminus A_i} M_a$ is a split monomorphism. Hence $M_a$ is isomorphic to summand of $\oplus_{A \setminus A_i} M_a$. Thus for some $c \in A \setminus A_i$, $M_a \cong M_c$. This contradiction shows that $\theta_i$ is a ring epimorphism.

Let $\alpha \in J(S)$. By Corollary 1.7 we have that, for all $a, b \in A$, $l_{A_i} \alpha \pi_b$ is a non-isomorphism. In particular, for all $a_i, b_i \in A_i$, $l_{A_i} l_{A_i} \alpha \pi_{A_i} \pi_b = l_{A_i} \alpha \pi_b$ is a non-isomorphism. Hence, by Corollary 1.7, $l_{A_i} \alpha \pi_{A_i} \in J(S_i)$ and thus $J(S) \subseteq \text{Ker}(\theta_i)$.

**Proof of the Theorem.** We prove the assertions (a) and (b) of 2.1 above.

(a). For $i \in I$, let $\theta_i : S \rightarrow S_i / J(S_i)$ be as in Lemma 2.2 and $\theta = \prod_i \theta_i : S \rightarrow \prod_i S_i / J(S_i)$. In view of Lemma 2.2, $\theta$ is a ring epimorphism and $J(S) \subseteq \text{Ker}(\theta)$. Suppose that $f = (f_{ab})_{a, b \in A} \in \text{Ker}(\theta)$. In order to show that $f \in J(S)$, we show that each $f_{ab}$ is a non-isomorphism (see Corollary 1.7). Suppose on the contrary, $f_{ab} : M_a \rightarrow M_b$ is an isomorphism for some $a, b \in A$. Thus $a, b \in A_i$ for some $i$ in $I$. So $f_{ab} = l_{A_i} f \pi_b = l_{A_i} l_{A_i} f \pi_{A_i} \pi_b$ is an isomorphism.
This in view of Corollary 1.7, violates the fact that $l_{A_i} f \pi_{A_i} \in J(S_i)$.

(b). We need to show that if $M = N^{(A)}$, for some set $A$, is locally semi-T-nilpotent LE decomposition of $M$ with $S = \text{End}(M)$, then $S/J(S)$ is isomorphic to a left full linear ring. Let $L = \text{End}(N)$ and $D = L/J(L)$. As $L$ is a local ring, $D$ is a division ring.

Let $Y$ be the ring of all $A \times A$ row finite matrices with entries from $D$. We define $\phi : S \to Y$ as $\phi(f_{ab})_{a,b \in A} = (f_{ab})_{a,b \in A}$. Then $\phi$ is a ring epimorphism. By Corollary 1.7 $J(S) = T$. Obviously, $\text{Ker}(\phi) = T = J(S)$. Hence

$$S/J(S) \cong Y.$$