CHAPTER I

SOCLE OF PRODUCT OF MODULES
AND
SEMILOCAL RINGS
0. INTRODUCTION

It is well known that the socle of a direct sum of modules is equal to the direct sum of their socles (see [AF, Proposition 9.19]). But the socle of a direct product of simple modules may not be equal to the direct product of their socles. In other words, a direct product of simple modules is, in general, not a semisimple module. For example, the socle of the $\mathbb{Z}$-module $\prod_p \mathbb{Z}/(p)$ is $\bigoplus_p \mathbb{Z}/(p)$, where $P$ denotes the set of prime natural numbers (see [AF, Exercise 9.1.2]). Even a product of copies of one simple module need not be semisimple (see Example 1.2 below). However there are situations where an infinite product of copies of one simple module is semisimple. For example, the product $(\mathbb{Z}/(p))^A$ of the $\mathbb{Z}$-module $\mathbb{Z}/(p)$, where $p$ is a fixed prime and $A$ any infinite set, is semisimple (see [AF, Exercise 9.1.2]).

In section 1 we give an explicit description of the socle of product of copies of one simple module (Theorem 1.4). As a consequence we prove that if $A$ is an infinite set and $S$ is a simple left module, then $S^A$ is semisimple if and only if $R/I_R(S)$ is a simple Artinian ring (Corollary 1.7).

It is well known that a ring $R$ is semilocal if and only if every product of simple left $R$-modules is semisimple (see [AF, Proposition 15.17]). Thus for any non semilocal ring $R$ there exists a set of simple left $R$-modules whose product is not semisimple. In section 2 we explicitly write certain products of simple $R$-modules over a non-semilocal ring $R$ which are not semisimple (Theorem 2.5). This sharpens [AF, Proposition 15.17]. We also give a necessary and sufficient condition for a ring to be semilocal (Proposition 2.3). Moreover,

\footnote{The work of this chapter has been accepted for publication [GK].}
we show that the direct product of infinite family of mutually non-isomorphic simple modules is never semisimple (Lemma 2.4).

1. SOCLE OF PRODUCTS OF SIMPLE MODULES

Recall that a non-zero module is simple if it has no non-trivial submodules. A module is said to be semisimple if it has a semisimple decomposition i.e., it can be written as a direct sum of its simple submodules. A ring $R$ is said to be semisimple if $R$ is a semisimple module. For properties and different characterizations of semisimple modules and rings see [AF].

Socle of a module is defined to be the sum of all its simple submodules. It turns out to be the intersection of all its essential submodules (see [AF, Proposition 9.7]). It follows that a module is semisimple if and only if it coincides with its socle. A semisimple module is said to be homogeneous if it is isomorphic to a direct sum of copies of one simple module. For example, every vector space is a homogeneous module. Let $T$ be any simple module. A module $M$ is said to be $T$-homogeneous if $M \cong T^A$ for some set $A$.

By $[V : D]$ we denote the dimension of a vector space $V$ over a division ring $D$.

Lemma 1.1 ([AF, Exercise 10.8]). Let $M$ be a non-zero homogeneous semisimple module with $S = \text{End}_R(M)$. Then
\[ \text{Soc}(sS) = \text{Soc}(Ss) = \{ f \in S : \text{Im } f \text{ is finitely generated} \}. \]

The following example shows that a product of copies of one simple module may not be semisimple.
Example 1.2. Let $V_D$ be an infinite dimensional vector space and $E = \text{End}(V_D)$. Obviously, $V$ is a simple left $E$-module. Suppose $(v_\alpha)_{\alpha \in A}$ be a basis of the vector space $V_D$. We show that $E \cong V^A$. Let $\theta : E \rightarrow V^A$ be defined as $\theta(\sigma) = (((\sigma(v_\alpha)))_{\alpha \in A}$. It is easily checked that $\theta$ is an $E$-homomorphism. Also for any $(x_\alpha)_{\alpha \in A} \in V^A$, we define $\sigma \in E$ as $\sigma(v_\alpha) = x_\alpha$ for each $\alpha \in A$. Then $\theta(\sigma) = (x_\alpha)_{\alpha \in A}$ showing that $\theta$ is onto. Also for any $\sigma \in E$, $\sigma(v_\alpha) = 0$ for each $\alpha \in A$ implies that $\sigma = 0$. Thus $\theta$ is an isomorphism and $\text{Soc}(V^A) = \theta(\text{Soc}(E))$.

Claim: $\theta(\text{Soc}(E)) = \{(x_\alpha)_{\alpha \in A} : [\sum_{D} x_\alpha D : D] < \infty\}$.

Let $\sigma \in \text{Soc}(E)$. Then $\theta(\sigma) = ((\sigma(v_\alpha)))_{\alpha \in A}$ and, by Lemma 1.1, $\sigma(V)$ is a finite dimensional subspace of $V_D$. Now as $\sum_{A} \sigma(v_\alpha)D = \sigma(V)$, $[\sum_{A} \sigma(v_\alpha)D : D] < \infty$. Conversely, suppose $(x_\alpha)_{\alpha \in A} \in V^A$ is such that $[\sum_{D} x_\alpha D : D] < \infty$. Define $\sigma \in E$ such that $\sigma(v_\alpha) = x_\alpha$ for each $\alpha \in A$. Then $\sigma(V) = \sum_{\alpha \in A} \sigma(v_\alpha)D = \sum_{\alpha \in A} x_\alpha D$, $\sigma(V)$ is finitely generated. This establishes the claim. Hence

$$\text{Soc}(V^A) = \{(x_\alpha)_{\alpha \in A} : [\sum_{A} x_\alpha D : D] < \infty\}.$$ 

Thus $(v_\alpha)_{\alpha \in A}$ is not in $\text{Soc}(V^A)$ and so $V^A$ is not semisimple left $E$-module.

As endomorphism ring of a simple module is a division ring, Example 1.2 makes one ask the following

Question 1.3. Let $S$ be a simple left $R$-module and $D = \text{End}(S)$. Then is 

$$\text{Soc}(S^A) = \{(x_\alpha)_{\alpha \in A} : [\sum_{A} x_\alpha D : D] < \infty\}?$$
Before we address this question we make some further interesting observations about the ring $E$ of Example 1.2. Define $e_{\alpha} \in E$ as $e_{\alpha}(v_{\beta}) = \delta_{\alpha\beta}v_{\alpha}, \alpha \in A$. It is easy to show that $Soc(E) = \oplus_{\alpha \in A} e_{\alpha}E$. However we observe that $\oplus_{\alpha \in A} Ee_{\alpha}$ is a proper subset of $Soc(E)$. In fact, it is easy to check that $\sigma \in \oplus_{A} Ee_{\alpha}$ if and only if $\sigma(v_{\alpha}) = 0$ for almost all $\alpha$. Pick $v \neq 0 \in V$. Define $\sigma \in E$ as $\sigma(v_{\alpha}) = v$ for all $\alpha \in A$. Obviously, $\sigma$ is a finite rank linear transformation so it is in $Soc(E)$, but not in $\oplus_{A} Ee_{\alpha}$.

It is easy to check that $Ee \cong \prod_{A} Ee_{\alpha}$ under the map $\psi : 1 \rightarrow (e_{\alpha})_{\alpha \in A}$. But interestingly, the map $\phi : 1 \rightarrow (e_{\alpha})_{\alpha \in A}$ from $E_{E}$ to $\prod_{A} e_{\alpha}E$, which is easily seen to be 1-1, is not onto. In fact, one checks easily that

$$Im(\phi) = \{(e_{\alpha}\sigma_{\alpha})_{\alpha \in A} \in \prod_{A} e_{\alpha}E : e_{\alpha}\sigma_{\alpha}(v) = 0 \text{ for almost all } \alpha \in A\}.$$ 

Pick $v \neq 0 \in V$. Choose for each $\alpha \in A$ an element $\sigma_{\alpha} \in E$ such that $\sigma_{\alpha}(v) = v_{\alpha}$. Clearly the element $(e_{\alpha}\sigma_{\alpha})_{\alpha \in A}$ is not in $Im(\phi)$. We do not know whether $Ee \cong \prod_{A} e_{\alpha}E$.

The following Theorem affirms the Question 1.3.

**Theorem 1.4.** Let $S$ be a simple left $R$-module and $D = \text{End}^r(S)$. Let $A$ be a non-empty set. Then

$$Soc(S^A) = \{(x_{\alpha})_{\alpha \in A} : [\sum_{\alpha} x_{\alpha}D : D] < \infty\}.$$ 

**Proof.** Claim(*): If $m = (x_{\alpha})_{\alpha \in A}$, then $T = Rm$ is a simple $R$-module if and only if $[\sum_{\alpha} x_{\alpha}D : D] = 1$.

Suppose $T = Rm$ is simple. Let $x_{\alpha}$ be a nonzero component of $m = (x_{\alpha})_{\alpha \in A}$.
Let \( \beta \in A, \beta \neq \alpha \). We show that the elements \( x_\alpha, x_\beta \) are \( D \)-linearly dependent.

For any \( \alpha \in A \) let \( \pi_\alpha : S^A \to S \) be the natural projection. Assume that \( x_\beta \neq 0 \). Since \( T \) and \( S \) are simple \( R \)-modules so \( (\pi_\alpha|T) : T \to S \) is an isomorphism. Similarly, \( (\pi_\beta|T) : T \to S \) is also an isomorphism. Set \( \theta = (\pi_\alpha|T)^{-1}(\pi_\beta|T) \), where the maps act on the right. Then \( \theta \in D \) and \( (x_\alpha)\theta = x_\beta \). Hence \( x_\alpha, x_\beta \) are \( D \)-linearly dependent and so

\[ [\sum_A x_\alpha D : D] = 1. \]

Conversely, suppose \( m = (x_\alpha)_{\alpha \in A} \in S^A \) is such that \( [\sum_A x_\alpha D : D] = 1 \). Let \( 0 \neq x \in \sum_A x_\alpha D \). Then \( xd_\alpha = x_\alpha \) for a unique \( d_\alpha \in D \). Since \( x \neq 0 \), \( S = Rx \). We show that \( S = Rx \cong Rm \). Let \( \phi : Rx \to Rm \) be defined as \( \phi(rx) = (rx_\alpha)_{\alpha \in A}, r \in R \). For any \( r_1 \) and \( r_2 \) in \( R \),

\[ r_1x = r_2x \Rightarrow (r_1x)d_\alpha = (r_2x)d_\alpha \Rightarrow r_1(xd_\alpha) = r_2(xd_\alpha) \Rightarrow r_1x_\alpha = r_2x_\alpha, \]

for each \( \alpha \in A \). This shows that \( \phi \) is well defined. It is easily seen that \( \phi \) is an epimorphism. Lastly, for any \( r_1 \) and \( r_2 \) in \( R \),

\[ \phi(r_1x) = \phi(r_2x) \Rightarrow r_1x_\alpha = r_2x_\alpha \Rightarrow r_1xd_\alpha = r_2xd_\alpha \quad \forall \ \alpha \in A. \]

Now as \( d_\alpha \neq 0 \) for at least one \( \alpha \in A \), it follows that \( r_1x = r_2x \) showing that \( \phi \) is an isomorphism. Thus \( Rm \) is a simple left \( R \)-module. This establishes the claim \((*)\).

Now let \( T \) be a minimal submodule of \( S^A \). Let \( 0 \neq m = (x_\alpha)_{\alpha \in A} \in T \), then \( T = Rm \). So by \((*)\) \( [\sum_A x_\alpha D : D] = 1 \). Thus
Now as $\text{Soc}(S^A)$ is the sum of simple left $R$–modules, if $s = \sum_{\alpha \in A} y_\alpha \in \text{Soc}(S^A)$, then $s$ belongs to a finite sum of simple submodules of $S^A$, which shows that $[\sum_{\alpha \in A} y_\alpha D : D] < \infty$. Thus

$$\text{Soc}(S^A) \subseteq \{(x_\alpha)_{\alpha \in A} : [\sum_{\alpha \in A} x_\alpha D : D] < \infty\}.$$ 

Conversely, let $x = (x_\alpha)_{\alpha \in A}$ be such that $[\sum_{\alpha \in A} x_\alpha D : D] = n$. Suppose $e_1, e_2, \ldots, e_n$ is a basis of the $D$–space $\sum_{\alpha \in A} x_\alpha D$. Then $x_\alpha = \sum_{i=1}^n e_i d_{i\alpha}$ for unique $d_{i\alpha} \in D$. For fixed $i$, there exists at least one $\alpha$ such that $d_{i\alpha} \neq 0$ because otherwise $[\sum_{\alpha \in A} x_\alpha D : D] < n$. For $1 \leq i \leq n$, let $m_i = (e_i d_{i\alpha})_{\alpha \in A}$. Then clearly $x = \sum_{i=1}^n m_i$. Now as each $m_i = (e_i d_{i\alpha})_{\alpha \in A} \in \{(x_\alpha)_{\alpha \in A} : [\sum_{\alpha \in A} x_\alpha D : D] = 1\}$, by (*) each $Rm_i$ is simple. Thus $x \in \text{Soc}(S^A)$.

A ring $R$ is said to be left primitive if there exists a simple and faithful left $R$–module.
Theorem 1.6. (Jacobson Density Theorem [AF, Theorem 14.4]). Let R be a left primitive ring with faithful and simple left R-module S and $D = \text{End}_R(S)$. Then R is isomorphic to a dense subring of linear transformations of $S^D$.

Corollary 1.7. Let S be a simple R-module and A an infinite set. Then $S^A$ is semisimple if and only if $R/I_R(S)$ is a simple Artinian ring.

Proof. Suppose $S^A$ is semisimple. Now $R/I_R(S)$ is a left primitive ring with S a simple and faithful left $R/I_R(S)$-module. Let $D = \text{End}_{R/I_R(S)}(S)$. In view of Theorem 1.6 it suffices to show that $[S : D] < \infty$. Suppose not. Then there exists a $D$-linearly independent subset $\{s_i : i \in \mathbb{N}\}$ in S, where N denotes the set of natural numbers. Let $\{\alpha_i : i \in \mathbb{N}\}$ be a countable set of mutually distinct elements of A. Define $x \in S^A$ as $x_{\alpha_i} = s_i$, if $i \in \mathbb{N}$ and $x_{\alpha} = 0$, if $\alpha \neq \alpha_i$ for every i. By Theorem 1.4 it is clear that $x \notin \text{Soc}(S^A)$. A contradiction.

Conversely, if $R/I_R(S)$ is simple Artinian, then $(I_R(S))S^A = 0$ implies that $S^A$ is a semisimple left $R/I_R(S)$-module. Thus $S^A$ is a semisimple left R-module.

Remark 1.8. Professor Mark Teply communicated the following direct argument of the Corollary 1.7 in the case when $A = S$. Let $e = (s)_{s \in S}$. Define $\gamma : R \rightarrow S^S$ such that $\gamma(1) = e$. Clearly Ker($\gamma$) = $I_R(S)$. So $R/I_R(S)$ is semisimple left primitive ring and hence a simple Artinian ring.

It would be of interest to see a direct proof of Corollary 1.7.
2. SIMPLE MODULES OVER SEMILOCAL RINGS

In Theorem 2.5 below a sharpened version of [AF, Proposition 15.17] is given.

Definition 2.1. A set $I_1, \ldots, I_n$ of ideals of a ring $R$ is said to be co-maximal if $I_i + I_j = R$ whenever $i \neq j$.

Lemma 2.2 (Chinese Remainder Theorem [AF, Exercise 7.13]). If $I_1, \ldots, I_n$ is a set of co-maximal ideals of ring $R$, then the natural map

$$
\phi : R \rightarrow R/I_1 \times \ldots \times R/I_n
$$

is a ring epimorphism with kernel $I_1 \cap \ldots \cap I_n$. □

A ring $R$ is said to be semilocal if $R/J(R)$ is a semisimple ring.

Proposition 2.3. A ring $R$ is semilocal if and only if $R$ has finitely many nonisomorphic simple left $R$-modules and $R/I_R(S)$ is simple Artinian for every simple left $R$-module $S$.

Proof. Suppose $R$ is semilocal. Since $R/J(R)$ has only finitely many simple left modules, there are only finitely many simple left $R$-modules. If $S$ is a simple left $R$-module, then $R/I_R(S)$ is a left primitive ring. Also as $R/I_R(S)$ is a finitely generated left $R/J(R)$-module and $R/J$ is left Artinian, $R/I_R(S)$ is an Artinian left $R/J(R)$ module and is thus a left Artinian ring. Now, by Theorem 1.6 and Lemma 1.5, it follows that $[S : D] < \infty$, where $D = End_{R/I_R(S)}(S)$. Now, again by Theorem 1.6, $R/I_R(S)$ is a simple Artinian ring being isomorphic to the endomorphism ring of a finite dimensional vector space.
Conversely, let $S_1, ..., S_n$ be an irredundant set of representatives of isomorphism classes of simple left $R$-modules such that $R/I(S_i)$ is a simple Artinian ring for all $i = 1, ..., n$. Thus each $l_R(S_i)$ is a maximal two-sided ideal of $R$. Also if $i \neq j$, then $l_R(S_i) \neq l_R(S_j)$, because otherwise the simple Artinian ring $R/I(R_i)$ has two nonisomorphic simple left $R$-modules $S_i$ and $S_j$ which is not possible (see [AF, Exercise 13.1.1]). Now as the Jacobson radical of a ring is the intersection of annihilators of all simple left $R$-modules, it follows, by Lemma 2.2, that $R/J(R) \cong \prod_{i=1}^{n} R/I(R_i)$. Hence $R$ is semilocal.

The following Lemma is inspired by the fact that $\prod_p(Z/pZ)$ is not semisimple $Z$-module.

- **Lemma 2.4.** Let $(T_\alpha)_{\alpha \in A}$ be an infinite family of mutually nonisomorphic simple left $R$-modules. For all $\alpha \in A$, let $M_\alpha$ be a $T_\alpha$-homogeneous module, then

$$Soc(\prod_A M_\alpha) = \oplus_A M_\alpha.$$

In particular, $\prod_A T_\alpha$ is not a semisimple module.

**Proof.** Since $\oplus_A M_\alpha$ is a semisimple module so $\oplus_A M_\alpha \subseteq Soc(\prod_A M_\alpha)$. Let $m = (x_\alpha)_{\alpha \in A} \in \prod_A M_\alpha$ such that $x_\alpha \neq 0$ for infinitely many $\alpha$'s.

**Claim:** $Rm$ is not a semisimple module.

Suppose, on the contrary, $Rm = \oplus_{i=1}^{n} S_i$ be a semisimple decomposition of $Rm$. Let $B = \{\alpha \in A : x_\alpha \neq 0\}$. By choice $|B| = \infty$. Let $\pi_\alpha : \prod_A M_\alpha \rightarrow M_\alpha$ denote the canonical projection. Now $(\pi_\alpha|Rm) : Rm \rightarrow M_\alpha$ is nonzero for every $\alpha \in B$. So there exists an epimorphism $Rm \rightarrow T_\alpha$, for every $\alpha \in B$. 


Thus for $a \in B, T_a \cong S_{\sigma(a)}$ for some $\sigma(a)$ in $\{i = 1, \ldots, n\}$ (see [AF, Proposition 9.4]). Since $|B| = \infty$ so there exists $\alpha \neq \beta$ in $B$ such that $\sigma(\alpha) = \sigma(\beta)$. Thus $T_\alpha \cong T_\beta$. This contradiction establishes the claim. \hfill \Box

**Theorem 2.5.** Suppose $R$ is not a semilocal ring. Then

1. If $R$ has infinitely many non-isomorphic simple left modules $\{S_i\}_A$, then $\prod_A S_i$ is not semisimple.

2. If $R$ has finitely many simple left $R$-modules, then there exists a simple left $R$-module $S$ such that $S^A$ is not semisimple for any infinite set $A$.

**Proof.** (1). Follows from Lemma 2.4.

(2). As $R$ is not semilocal, by Proposition 2.3, there exists a simple left $R$-module $S$ such that $R/I_R(S)$ is not a simple Artinian ring. Thus by Corollary 1.7 $S^A$ is not semisimple for any infinite set $A$. \hfill \Box