NIL SUBSETS OF ENDMORPHISM RINGS OF MODULES WITH CHAIN CONDITIONS
0. INTRODUCTION$^3$

It is well known that if $R^\times M$ is a finite length module, then $E = \text{End}_R^\times (M)$ is semiprimary (see [AF, Corollary 29.3]) with $J^k = 0$ (see [S]), where $J$ is the Jacobson radical of $E$ and $k$ is the homogeneous length of $M$. Also nil multiplicatively closed subsets of a semiprimary ring are nilpotent of bounded index (see [R, Theorem 2.6.31]). Some more classes of modules with semiprimary endomorphism rings are: (a) Artinian semi-projective module (see [H, Proposition 2.4]); (b) semi-injective module $R^\times M$ with $\mathcal{L}_M(S) = \{l_M(T) : T \subseteq S\}$ satisfying ascending chain condition (acc), where $S = \text{End}_R^\times (M)$ (see [W, 31.12]); (c) Artinian module with finite homogeneous length (see [Sc, Theorem 5]).

Goldie and Small [GOS] (for proof see [Fil, Theorem 2.1]), improving upon an earlier result of Procesi and Small [PS], proved that nil subrings (=subrings without identity) of endomorphism ring of a Noetherian module are nilpotent. Fisher [Fil, Theorem 1.5] proved an analogous result for Artinian modules and also proved that in the case of Noetherian modules the indices of nilpotency of nil subrings is bounded. In this paper Fisher raised the following

**Question** ([Fil, Question (2), page 78]). *If $M$ is an Artinian $R$ module, then are the indices of nilpotency of nil subrings of $\text{End}(M)$ bounded?*

He answered this question in affirmative for finitely generated modules (unpublished).

Gupta [G] showed that if $M$ is an Artinian or Noetherian module, then any nil multiplicatively closed subset in $\text{End}(M)$ is nilpotent. Further he

$^3$The work of this chapter has been accepted for publication [GK1].
showed that if $M$ is an Artinian module of finite homogeneous length $k$, then the indices of nilpotency of nil multiplicatively closed subsets in $\text{End}(M)$ are bounded by $k$. Also he asked: Suppose $M$ is Noetherian, then are the indices of nilpotency of nil multiplicatively closed subsets bounded? In other words does the Fisher’s result also hold for nil multiplicatively closed subsets in $\text{End}(M)$?

The question of Fisher was answered by Wu and Golan [QG, Proposition 6]. They proved, more generally, that if $\tau$ is a torsion theory on $R\text{M}$, the category of all left $R$-modules, and $M$ is $\tau$-torsionfree, $\tau$-Noetherian ($\tau$-torsionfree, $\tau$-Artinian) module in $R\text{M}$, then any nil subrng of $\text{End}(M)$ is nilpotent and the indices of nilpotency of nil subrngs of $\text{End}(M)$ are bounded. In section 3 we prove the following two results:

If $M$ is a $\tau$-torsionfree, $\tau$-Artinian ($\tau$-torsionfree, $\tau$-Noetherian) module, then nil multiplicatively closed subsets of $\text{End}(M)$ are nilpotent and the indices of nilpotency are bounded.

This, in particular, answers the question [G, 1.5, page 1479] and also the dual question for Artinian modules.

Let $\tau$ be any torsion theory. In section 2 we recall some basic torsion theoretic facts, in particular, the concepts of $\tau$-Noetherian, $\tau$-Artinian, $\tau$-torsionfree modules; $\tau$-Goldie dimension ($\tau - G.\text{dim}(M)$, $\tau$-dual Goldie dimension ($\tau$-dual $G.\text{dim}(M)$)) and the lattice of $\tau$-pure submodules ($\mathcal{P}_\tau(M)$) of a module $M$ and observe the following:

1. Let $\tau$ be a torsion theory and $M$ any left $R$-module. Then
   (i). $\tau - G.\text{dim}(M) = G.\text{dim}\mathcal{P}_\tau(M)$. 
Goldie and Dual Goldie Dimensions

(ii). \( \tau - \text{dual } G.\dim(M) = \text{dual } G.\dim_{\tau}(M) \).

2. A \( \tau \)-Noetherian module has finite \( \tau \)-Goldie dimension.

3. A \( \tau \)-Artinian module has finite \( \tau \)-dual Goldie dimension.

4. \( \tau \)-Goldie dimension of a \( \tau \)-Artinian module is finite.

The results 2 and 3 above strengthen [QG, Proposition 2] where these results are proved for \( \tau \)-torsionfree modules.

In section 1, we recall the concepts of Goldie and dual-Goldie dimensions and give some examples.

1. GOLDIE AND DUAL GOLDIE DIMENSIONS

Almost all the results of this section are well known and for notations, terminology and the proofs of most of the results, we refer the reader to [FA], [GP], and [NO].

Throughout \( L \) will denote a modular lattice with 0 and 1 and \( a \wedge b \) \( (a \vee b) \) will denote the infimum (supremum) of two elements \( a \) and \( b \) of \( L \).

Definition 1.1. An element \( a \in L \) is said to be large in \( L \) \((a \leq L)\) if for any \( x \neq 0 \in L \), \( a \wedge x \neq 0 \). We call \( L \) uniform if each of its nonzero elements is large.

Analogously, an element \( a \in L \) is said to be small in \( L \) \((a \ll L)\) if for all \( 1 \neq x \in L \), \( a \vee x \neq 1 \). \( L \) is said to be a co-uniform lattice if \( a \ll L \) for each \( 1 \neq a \in L \).
Definition 1.2. A finite subset \( \{a_i : i \in I\} \) of \( L \setminus \{0\} \) is said to be \textit{join independent} or simply \textit{independent} if
\[
a_i \land (\lor_{j \neq i} a_j) = 0 \quad \forall i \in I.
\]
An arbitrary subset \( A \) of \( L \setminus \{0\} \) is called \textit{independent} if each of its finite subsets is independent. Analogously, we say that a finite subset \( \{a_i : i \in I\} \) of \( L \setminus \{1\} \) is \textit{meet-independent} or \textit{co-independent} if
\[
a_i \lor (\land_{j \neq i} a_j) = 1 \quad \forall i \in I.
\]
An arbitrary subset \( A \) of \( L \setminus \{1\} \) is called \textit{co-independent} if each of its finite subsets is co-independent.

Lemma 1.3. Let \( L \) be a lattice, then

(1). If \( A \) is an independent subset of \( L \setminus \{0\} \) and \( x \) is a non zero element of \( L \) such that for all finite \( B \subseteq A \), \( x \land (\lor B) = 0 \), then \( A \cup \{x\} \) is also independent.

(2). If \( A \) is a co-independent subset of \( L \setminus \{1\} \) and \( x \neq 1 \) is an element of \( L \) such that for every finite \( B \subseteq A \), \( x \lor (\land B) = 1 \), then \( A \cup \{x\} \) is also co-independent.

Proof (1). See [GP, B, page 48].

(2). Follows by dualizing (1). \( \square \)

Corollary 1.4. Let \( a_1, a_2, \ldots, a_n \) be elements of \( L \setminus \{1\} \). Then the following are equivalent:

(1). \( \{a_1, a_2, \ldots, a_n\} \) is a co-independent subset of \( L \);
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(2). \( (a_1 \land \ldots \land a_i) \lor a_{i+1} = 1 \) for all \( i = 1, \ldots, m - 1 \);

(3). \( a_i \lor (a_{i+1} \land \ldots \land a_n) = 1 \) for all \( i = 1, \ldots, m - 1 \).

Proof. Clear from Lemma 1.3(2). \( \square \)

Theorem 1.5 (Grzeszczuk and Puczylowski [GP, Theorem 5]). The following conditions on \( L \) are equivalent:

(1). \( L \) does not contain an infinite independent subset.

(2). \( L \) contains a finite independent subset \( \{a_1, \ldots, a_n\} \) with \([0, a_i]\) a uniform lattice for every \( i \) and

\[ \forall_{i=1}^n a_i \subseteq L; \]

(3). There exists a natural number \( m \) such that the cardinality of any independent subset of \( L \) is \( \leq m \);

(4). If \( a_1 \leq a_2 \leq a_3 \leq \ldots \) is a chain in \( L \setminus \{0\} \), then there exists an \( i \geq 1 \) such that \( a_i \subseteq [0, a_j] \) for every \( j \geq i \).

Moreover, if these equivalent conditions are satisfied, then any independent subset of \( L \) has cardinality \( \leq n \), \( n \) as in (2) above; so \( n = \sup \{|A| : A \text{ independent subset of } L\} \). \( \square \)

Definition 1.6. If \( L \) satisfies the equivalent conditions of the Theorem 1.5, we say that Goldie dimension of \( L \) (\( G. \ dim(L) \)) is finite and is equal to \( n \). By condition (4) of Theorem 1.5 it is clear that a sufficient condition for \( L \) to have finite Goldie dimension is that \( L \) satisfies the acc. If \( L \) does not satisfy the equivalent conditions of Theorem 1.5, we say that the Goldie dimension of \( L \) is infinite, that is, \( G. \ dim(L) = \infty \). Note that if \( G. \ dim(L) = n < \infty \),
then the supremum of the cardinalities of independent subsets of $L$ is $n$. In particular, $G.\dim(L) = 1$ if and only if $L$ is uniform. The Goldie dimension of $S(M)$, the lattice of submodules of a module, is called Goldie dimension of $M$ ($G.\dim(M)$). If $G.\dim(M) < \infty$, then for every submodule $N$ of $M$, $G.\dim(N) \leq G.\dim(M)$; but $G.\dim(M/N)$ need not be finite (see Examples 1.11(c) below). Note that $G.\dim(M) = 1$ if and only if $M$ is a uniform module i.e., $S(M)$ is a uniform lattice.

**Corollary 1.7.** If $M$ is a Noetherian or an Artinian module, then $G.\dim(M)$ is finite.

**Proof.** Follows from Theorem 1.5 (1). \hfill \Box

Dualizing Theorem 1.5 we get

**Theorem 1.8** (Grzeszczuk and Puczylowski [GP, Theorem 9, page 50]). The following conditions on $L$ are equivalent:

1. $L$ does not contain an infinite co-independent subset;
2. $L$ contains a finite co-independent subset $\{a_1, ..., a_n\}$ with $[a_i, 1]$ a co-uniform lattice for every $i$ and
   \[ \bigwedge_{i=1}^{n} a_i \ll L; \]
3. There exists a natural number $m$ such that the cardinality of any co-independent subset of $L$ is $\leq m$;
4. If $a_1 \geq a_2 \geq a_3 \geq ...$ is a chain in $L \backslash \{1\}$, then there exists a $j \geq 1$ such that $a_j \ll [a_k, 1]$ for every $k \geq j$.

Moreover, if these equivalent conditions are satisfied, then any co-independent
subset of $L$ has cardinality $\leq n$, $n$ as in (2) above; so $n = \sup\{|A| : A$ co-independent subset of $L\}$.

**Definition 1.9.** If $L$ satisfies the equivalent conditions of the Theorem 1.8, we say that dual Goldie dimension of $L$ ($dual - G. \ dim(L)$) is finite and is equal to $n$. By condition (4) of Theorem 1.8 it follows that a sufficient condition for a lattice $L$ to have finite dual Goldie dimension is that $L$ satisfies descending chain condition (dcc). If $L$ does not satisfy the equivalent conditions of Theorem 1.8, we say that the dual Goldie dimension of $L$ is infinite, that is $dual - G. \ dim(L) = \infty$. Note that if $dual - G. \ dim(L) = n < \infty$, then the supremum of the cardinalities of co-independent subsets of $L$ is $n$.

In particular, $dual - G. dim(L) = 1$ if and only if $L$ is co-uniform. The dual Goldie dimension of $S(M)$, the lattice of submodules of a module, is called dual Goldie dimension of $M$ ($dual - G. dim(M)$). If $dual - G. dim(M) < \infty$, then for any submodule $N$ of $M$, $dual - G. dim(M/N) \leq dual - G. dim(M)$; but $dual - G. dim(N)$ need not be finite (see Examples 1.11 (d) below). Note that $dual - G. dim(M) = 1$ if and only if $M$ is a co-uniform module i.e., $S(M)$ is a co-uniform lattice.

**Corollary 1.10.** If $M$ is an Artinian module then $dual - G. dim(M)$ is finite.

**Proof.** Follows from Theorem 1.8 (4) because if $K \leq N \neq M$ are two submodules of a module $M$, then $N \ll [K, M]$ (i.e., $N$ as an element of the lattice $S(M)$ is small in the sub-lattice $\{X \leq M : K \leq X \leq M\}$) if and only if $N/K \ll M/K$. \qed
Examples 1.11. (a). \( \text{G.dim}(\mathbb{Z}) = 1 \) as \( \mathbb{Z} \) is uniform. (See Definition 1.6.)
\( \text{dual} - \text{G.dim}(\mathbb{Z}) = \infty \) because \( \{p\mathbb{Z}\}_{p \in \mathbb{P}} \), where \( \mathbb{P} \) is the set of all prime natural numbers, is an infinite co-independent family of submodules of \( \mathbb{Z} \). (See Theorem 1.8 (1.).)

(b). Let \( F \) be a field such that there exists a monomorphism \( \sigma : F \to F \) with \( \sigma(F) \neq F \). Let
\[
R = F_\sigma[[x]] = \{ \sum_{i=0}^{\infty} a_i x^i : a_i \in F, \ \forall \ i \}.
\]
Define ‘+’ coefficientwise and ‘.’ using the rule: \( xa = \sigma(a)x \). It is easy to check that \( R \) is a local domain with
\[
J(R) = \{ \sum_{i=1}^{\infty} a_i x^i : a_i \in F, \ \forall \ i \}.
\]
As \( R \) is a local ring, \( R_R \) is co-uniform. Hence \( \text{dual} - \text{G.dim}(R_R) = 1 \). (See Definition 1.9.) Pick \( a \in F \setminus \sigma(F) \). It is easy to see that \( xR \cap axR = 0 \). So if \( I = axR \), then \( \{I, xI, x^2I, \ldots\} \) is an independent family of non-zero submodules of \( R_R \). Hence \( \text{G.dim}(R_R) = \infty \). (See Theorem 1.5 (1.).)

(c). \( \text{G.dim}(\mathbb{Q}/\mathbb{Z}) = 1 \) as \( \mathbb{Q}/\mathbb{Z} \) is uniform. By [AF, Exercise 6.8] \( \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p^\infty \). So \( \text{G.dim}(\mathbb{Q}/\mathbb{Z}) = \infty \).

(d). Let \( V_F \) be a countable dimensional vector space over a field \( F \). Let \( R = F + V \). Define ‘+’ on \( R \) componentwise and ‘.’ by \( (\alpha, v)(\beta, w) = (\alpha\beta, v\beta + w\alpha) \). Clearly, \( R \) is a commutative local ring with \( J(R) = V \). So \( \text{dual} - \text{G.dim}(R_R) = 1 \). Any subspace of \( V_F \) is an \( R \)-submodule of \( RV \). Let \( \{e_1, e_2, \ldots\} \) be a basis of \( V \). Define \( W_i = \oplus_{j \neq i} e_j F, \ i \in \mathbb{N} \). Then \( \{W_i\}_{i \in \mathbb{N}} \) is an infinite co-independent family of submodules of \( RV \). So \( \text{dual} - \text{G.dim}(RV) = \infty \).
2. SOME TORSION THEORY

For terms and notations we refer the reader to [AF] and [GO].

Let $U$ and $M$ be two left $R$-modules. Recall that $U$ is $M$-injective in case for each $R$-monomorphism $g : N \to M$ and each $R$-homomorphism $\gamma : N \to U$ there is an $R$-homomorphism $\delta : M \to U$ such that $g\delta = \gamma$. If $U$ is $M$-injective for every left $R$-module $M$ then $U$ is said to be injective. If $U$ is $U$-injective (i.e. $U$ is self-injective) then $U$ is called quasi-injective.

A module $C$ is said to cogenerate $M$ if $M$ embeds in a product of copies of $C$. A cogenerator in $\mathcal{M}$, the category of all left $R$-modules, is a module which cogenerates every left $R$-module.

**Definition 2.1.** Two injective $R$-modules $E$ and $E'$ are said to be equivalent if each cogenerates the other. This defines an equivalence relation on the class of injective left $R$-modules. An equivalence class $\tau$, with respect to this relation, is called a torsion theory on $\mathcal{M}$. The equivalence class of all injective cogenerators is called the trivial torsion theory and is denoted by $\tau$. Throughout, we shall assume that $\tau$ is a fixed torsion theory on $\mathcal{M}$

**Definition 2.2.** Let $E \in \tau$. A module $M$ is called $\tau$-torsion if $\text{Hom}(M, E) = 0$; the class of all $\tau$-torsion modules is denoted by $\mathcal{T}_\tau$. A module $N$ is said to be $\tau$-torsionfree if $N$ is cogenerated by $E$. The class of all $\tau$-torsion free modules is denoted by $\mathcal{F}_\tau$.

**Proposition 2.3** ([GO, Proposition 1.7 and Proposition 1.10]). (1). $\mathcal{T}_\tau$ is closed under taking submodules, homomorphic images, direct sums and extensions.
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(2). \( T \) is closed under taking submodules, injective hulls, direct products and isomorphic copies.

Proposition 2.4 ([GO, Proposition 3.11]) If \( M \) is a left \( R \)-module, then \( M \) has a unique maximal \( \tau \)-torsion submodule denoted by \( T_\tau(M) \).

Proposition 2.5. A non zero module cannot be both \( \tau \)-torsion as well as \( \tau \)-torsionfree.

Proof. Follows from [GO, Proposition 1.11].

Definition 2.6. A submodule \( N \leq M \) is said to be \( \tau \)-pure (resp. \( \tau \)-dense) if \( M/N \) is \( \tau \)-torsionfree (\( \tau \)-torsion).

Proposition 2.7. The class of \( \tau \)-pure (resp. \( \tau \)-dense) submodules is closed under taking arbitrary intersections (finite intersections).

Proof. Follows from Proposition 2.3 (2) and (1) respectively.

Proposition 2.8 (1). Let \( \alpha : N \to M \) be a module homomorphism. If \( W \) is a \( \tau \)-dense submodule of \( M \), then \( \alpha^{-1}(W) \) is a \( \tau \)-dense submodule of \( N \). In particular, if \( W, N \leq M \) with \( W \) \( \tau \)-dense in \( M \), then \( W \cap N \) is a \( \tau \)-dense submodule of \( N \).

(2). If \( L_1 \leq L, N \) are submodules of \( M \) and \( L_1 \) is a \( \tau \)-dense submodule of \( L \), then \( L_1 + N \) is a \( \tau \)-dense submodule of \( L + N \).

(3). Suppose \( X \leq Y \leq M \). If \( X \) is \( \tau \)-dense in \( Y \) and \( Y \) is \( \tau \)-dense in \( M \), then \( X \) is \( \tau \)-dense in \( M \).
Proof. (1). See [GO, Proposition 4.1 (3)].

(2) Let \( i : L \rightarrow L + N \) and \( \nu : L + N \rightarrow (L + N)/(L_1 + N) \) be the canonical maps. The map \( i\nu : L \rightarrow (L + N)/(L_1 + N) \) is an epimorphism and \( L_1 \leq \text{Ker}(i\nu) \). So by Factor Theorem (see [AF, Proposition 3.6 (1)]) there exists an epimorphism \( L/L_1 \rightarrow (L + N)/(L_1 + N) \). Thus by Proposition 2.3 (1) \( L_1 + N \) is a \( \tau \)-dense submodule of \( M \).

(3) The sequence
\[
0 \rightarrow Y/X \rightarrow M/X \rightarrow M/Y \rightarrow 0
\]
is exact. Since \( Y/X \) and \( M/Y \) belong to \( T_\tau \) so by Proposition 2.3 (1) \( M/X \in T_\tau \).

\[ \square \]

Definition 2.9. Let \( N \leq M \). \( N' \), called the \( \tau \)-purification of \( N \), is defined to be the intersection of all \( \tau \)-pure submodules of \( M \) containing \( N \).

Proposition 2.10 ([GO, Proposition 6.4]). For any submodule \( N \) of \( M \),
\[
N'/N = T_\tau(M/N).
\]

Proposition 2.11 Let \( N \) and \( W \) be submodules of \( M \), then

(1). \( (N')' = N' \).

(2). \( (N : m)' = (N' : m) \), where \( m \in M \).

(3). If \( N \leq W \leq M \), then \( N' \leq W' \).

(4). \( (N \cap W)' = N' \cap W' \).

(5). \( (N + W)' = (N' + W')' \).

Proof. For (1) to (4) see [GO, Proposition 6.5].
Now (5) easily follows from (1) and (3).

Corollary 2.12. Let $M$ be a $\tau$-torsionfree and $N,W \leq M$ with $W$ $\tau$-pure. Then $N \cap W = 0 \Rightarrow N' \cap W = 0$.

Proof. Follows from Proposition 2.11 (4).

2.13. The lattice of $\tau$-pure submodules

Let $\mathcal{P}_\tau(M)$ denote the set of $\tau$-pure submodules of a module $M$. $\mathcal{P}_\tau(M)$ is a partially ordered set with respect to set inclusion and $M$ is its largest element. Since $\mathcal{P}_\tau(M)$ is closed under taking arbitrary intersections (see Proposition 2.7), by [AF, 0.6, page 4], $\mathcal{P}_\tau(M)$ is a complete lattice; its smallest element is $T_\tau(M)$. So for any subset $\{N_i\}_i \in I$ of $\mathcal{P}_\tau(M)$,

$$\bigwedge_i N_i = \cap_i N_i \text{ and } \bigvee_i N_i = \left( \sum_i N_i \right)'.$$  

Proposition 2.14 ([GO, Proposition 6.11]). $\mathcal{P}_\tau(M)$ is a complete modular lattice with ' 0', the smallest element, $T_\tau(M)$ and ' 1', the largest element, $M$.

Definition 2.15. A submodule $N \leq M$ is said to be $\tau$-large, $(N \unlhd_\tau M)$, if $N \cap W \in T_\tau \Rightarrow W \in T_\tau$. Dually, $N$ is said to be $\tau$-small in $M$, $(N \lhd_\tau M)$ if $N + W$ $\tau$-dense in $M$, $W \leq M$, implies $W$ $\tau$-dense in $M$. It is immediate that intersection of two $\tau$-large submodules of $M$ is $\tau$-large in $M$ and sum of two $\tau$-small submodules of $M$ is $\tau$-small in $M$.

Proposition 2.16. Let $M$ be a module and $S = \text{End}(M)$, then

$$H(\tau) = \{ \alpha \in S : \text{Im}(\alpha) \lhd_\tau M \} \text{ and } N(\tau) = \{ \alpha \in S : \text{Ker}(\alpha) \unlhd_\tau M \}$$
are 2-ideals of $S$.

Proof. See [QG, page 2598].

Definition 2.17. A module $M$ is said to be $\tau$-Artinian (resp. $\tau$-Noetherian) if $\mathcal{P}_\tau(M)$ is an Artinian (Noetherian) lattice. If $M$ is $\tau$-Artinian (resp. $\tau$-Noetherian) the lattice $\mathcal{P}_\tau(M)$ has finite dual Goldie dimension (Goldie dimension). This follows from Definition 1.9 (Definition 1.6).

Definition 2.18. A non-empty family $\mathcal{A}$ of submodules of $M$ is called $\tau$-independent if

(1) $N \in \mathcal{A} \Rightarrow N \notin \mathcal{T}_\tau$

(2) For any finitely many $M_1, \ldots, M_n \in \mathcal{A}$

$$M_i \cap \left( \sum_{j \neq i} M_j \right) \in \mathcal{T}_\tau \ \forall \ i = 1, \ldots, n.$$

Definition 2.19. A module $M$ is said to have finite $\tau$-Goldie dimension $n$ ($\tau - G.\dim(M) = n$), $n$ a non negative integer, if there exists a $\tau$-independent family of $n$ submodules of $M$ and no $\tau$-independent family of more than $n$ submodules of $M$. If no such $n$ exists, we say that $\tau$-Goldie dimension of $M$ is infinite. Clearly, $\xi - G.\dim(M) = G.\dim(M)$. Dually a module $M$ is said to have finite $\tau$-dual Goldie dimension $n$ ($\tau - \text{dual } G.\dim(M) = n$), $n$ a non negative integer, if there exists a $\tau$-co-independent family of $n$ submodules of $M$ and no $\tau$-co-independent family of more than $n$ submodules of $M$. If no such $n$ exists, then we say that $\tau$-dual Goldie dimension of $M$ is infinite. Clearly, $\xi - \text{dual } G.\dim(M) = \text{dual } G.\dim(M)$.

Proposition 2.20. Let $N_1, \ldots, N_n$ be submodules of a module $M$. Then:
(1). \( \{N_1, ..., N_n\} \) are \( \tau \)-co-independent submodules of \( M \) if and only if \( \{N'_1, ..., N'_n\} \) are co-independent in the lattice \( \mathcal{P}_\tau(M) \). In particular,
\[
\tau - \text{dual } G.\dim(M) = \text{dual } - G.\dim(\mathcal{P}_\tau(M)).
\]

(2). \( \{N_1, ..., N_n\} \) are \( \tau \)-independent submodules of \( M \) if and only if \( \{N'_1, ..., N'_n\} \) are independent in the lattice \( \mathcal{P}_\tau(M) \). In particular,
\[
\tau - G.\dim(M) = G.\dim(\mathcal{P}_\tau(M)).
\]

**Proof** (1). Note that \( N \) is \( \tau \)-dense in \( M \) iff \( N' = M \). Now \( N_i + (\cap_{j \neq i} N_j) \) is \( \tau \)-dense in \( M \)
\[
\text{iff } (N_i + (\cap_{j \neq i} N_j))' = M
\]
\[
\text{iff } N_i' \lor (\cap_{j \neq i} N_j') = M \quad (\text{see Proposition 2.11 and 2.13}).
\]

(2). Note that \( N \) is \( \tau \)-torsion iff \( N' = T_\tau(M) \). Now \( N_i \cap (\sum_{j \neq i} N_j) \) is \( \tau \)-torsion
\[
\text{iff } (N_i \cap (\sum_{j \neq i} N_j))' = T_\tau(M)
\]
\[
\text{iff } N_i' \land (\lor_{j \neq i} N_j') = T_\tau(M) \quad (\text{see Proposition 2.11 and 2.13}).
\]

**Corollary 2.21.** (1). If \( M \) is \( \tau \)-Artinian, then
\[
\tau - \text{dual } G.\dim(M) = G.\dim(\mathcal{P}_\tau(M)) < \infty.
\]

(2). If \( M \) is \( \tau \)-Noetherian, then
\[
\tau - G.\dim(M) = G.\dim(\mathcal{P}_\tau(M)) < \infty.
\]

**Proof** (1). As \( M \) is \( \tau \)-Artinian, the dual Goldie dimension of \( \mathcal{P}_\tau(M) < \infty \)
(see 2.17). Now the result follows from Proposition 2.20(1).
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(2). Since $M$ is $\tau$-Noetherian so by 2.17 Goldie dimension of $\mathcal{P}_r(M) < \infty$. Now Proposition 2.20(2) completes the proof. □

We observed in Corollary 1.7 that an Artinian module has finite Goldie dimension. This observation raises the following question:

**Question 2.22.** If $M$ is a $\tau$-Artinian, then is it true that $\tau - G.dim(M) < \infty$?

In the Corollary 2.25 below we prove that Question 2.22 has affirmative answer. The proof was suggested by Professor M. L. Teply in a personal communication. First we prove

**Proposition 2.23.** For any module $M$

$$G.dim(\mathcal{P}_r(M)) = \tau - G.dim(M) = G.dim(M/\mathcal{T}_r(M)).$$

**Proof.** In view of Proposition 2.20 we only have to show that

$$G.dim(\mathcal{T}_r(M)) = G.dim(M/\mathcal{T}_r(M)).$$

For any submodule $N$ of $M$ we have

$N$ is $\tau$-torsion if and only if $N' = \mathcal{T}_r(M)$. (*)

Now suppose that $N_1, \ldots, N_k$ are independent in $\mathcal{P}_r(M)$. Thus, by Proposition 2.11 and 2.13,

$$N_1 \land (\bigvee_{j=2}^{k} N_j) = \mathcal{T}_r(M) \Rightarrow N_1 \cap \left( \sum_{j=2}^{k} N_j \right)' = \mathcal{T}_r(M)$$

$$\Rightarrow (N_1 \cap \left( \sum_{j=2}^{k} N_j \right))' = \mathcal{T}_r(M) \Rightarrow N_1 \cap \left( \sum_{j=2}^{k} N_j \right) = \mathcal{T}_r(M).$$

This in view of (*) gives that $N_1/\mathcal{T}_r(M), \ldots, N_k/\mathcal{T}_r(M)$ are independent in $M/\mathcal{T}_r(M)$. 

Conversely, suppose that $N_1/T_\tau(M), ..., N_k/T_\tau(M)$ are independent in $M/T_\tau(M)$. Thus, by Proposition 2.11 and 2.13,

$$N_1 \cap \left( \sum_{j=2}^{k} N_j \right) = T_\tau(M) \Rightarrow (N_1 \cap \left( \sum_{j=2}^{k} N_j \right))' = (T_\tau(M))' = T_\tau(M)$$

$$\Rightarrow (N_1)' \cap \left( \sum_{j=2}^{k} N_j \right)' = T_\tau(M) \Rightarrow (N_1)' \land (\forall_{j=2}^{k} N_j)' = T_\tau(M).$$

This in view of (*) gives that $N_1', ..., N_k'$ are independent in $T\tau(M)$. □

**Lemma 2.24.** For a $\tau$-Artinian module $M$

$$G\text{.dim}(M/T_\tau(M)) < \infty.$$ 

**Proof.** Suppose, on the contrary, \{ $N_i/T_\tau(M) : i = 1, 2, ...$ \} is an infinite independent set of submodules of $M/T_\tau(M)$. Then for each $i$, $T_\tau(M) \subset N_i$ properly. For any finite subset $F$ of natural numbers and $i \in F$, we have

$$N_i \cap \left( \sum_{j \in F \setminus \{i\}} N_j \right) = T_\tau(M).$$

Thus for arbitrary subset $K$ of natural numbers and $i \in K$ also

$$N_i \cap \left( \sum_{j \in K \setminus \{i\}} N_j \right) = T_\tau(M) \Rightarrow (N_i \cap \left( \sum_{j \in K \setminus \{i\}} N_j \right))' = T_\tau(M)$$

$$\Rightarrow (N_i)' \cap \left( \sum_{j \in K \setminus \{i\}} N_j \right)' = T_\tau(M) \Rightarrow (N_i)' \land (\sum_{j \in K \setminus \{i\}} N_j)' = T_\tau(M).$$

Thus $(\sum_{i=1}^{\infty} N_i)' \supset (\sum_{i=2}^{\infty} N_i)' \supset ...$ is a proper chain in $P_\tau(M)$. This contradiction completes the proof. □

**Corollary 2.25.** If $M$ is a $\tau$-Artinian module then $\tau - G\text{.dim}(M) < \infty$. 

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Proof. Follows from Lemma 2.24 and Proposition 2.23.

3. NIL MULTIPLICATIVELY CLOSED SETS

In this section we prove our main results.

Proposition 3.1. Let $M$ be a $\tau$-torsion free $\tau$-Artinian ($\tau$-Noetherian) module, $S = \text{End}(M)$. Then $S$ satisfies acc on right (left) annihilators.

Proof. For $\tau$-Artinian case see [QG, Proposition 3] and similar proof works when $M$ is given to be $\tau$-Noetherian.

A sequence $x_1, x_2, ...$ of elements of a ring $R$ is said to be left vanishing (resp. right vanishing) if there exists a natural number $n$ such that $x_1x_2...x_n = 0$ ($x_nx_{n-1}...x_1 = 0$). A subset $X$ of $R$ is said to be left $T$-nilpotent (right $T$-nilpotent) if every sequence of elements of $X$ is left vanishing (right vanishing).

Clearly every left or right $T$-nilpotent subset of a ring is nil.

The following result was proved by Fisher [Fi, page 1215] for subrings. On the same lines we prove

Lemma 3.2. Let $S$ be a ring satisfying acc on right annihilators (resp. acc on left annihilators), then any right $T$-nilpotent (left $T$-nilpotent) subset $X$ of $S$ is nilpotent.

Proof. Suppose $S$ satisfies acc on right annihilators and $X \subseteq S$ is not nilpotent. Then we have $r_S(X) \subseteq r_S(X^2) \subseteq ...$. So by hypothesis there exists a natural number $n$ such that $r_S(X^n) = r_S(X^{n+1})$. Since $X$ is not nilpotent so there exists $x_1 \in X$ such that $X^n x_1 \neq 0$. This gives that $X^{n+1} x_1 \neq 0$ and so
there exists $x_2 \in X$ such that $X^n x_2 x_1 \neq 0$ which implies that $X^{n+1} x_2 x_1 \neq 0$.

On proceeding by induction we get a non right vanishing sequence $(x_1, x_2, ...)$. Similar argument works when $S$ satisfies acc on left annihilators.

**Lemma 3.3.** Let $M$ be a module and $\alpha, \beta, \gamma \in S = \text{End}(M)$. Then,

1. $\text{Ker}(\alpha) = \text{Ker}(\alpha \beta) \Rightarrow M \alpha \cap \text{Ker}(\beta) = 0$.
2. $\text{Ker}(\alpha) = \text{Ker}(\beta) \Rightarrow \text{Ker}(\gamma \alpha) = \text{Ker}(\gamma \beta)$.

**Proposition 3.4.** Let $M$ be a $\tau$-torsionfree $\tau$-Artinian ($\tau$-torsionfree $\tau$-Noetherian) module having $\text{End}(M) = S$. Then any nil multiplicatively closed subset $X$ of $S$ is nilpotent.

**Proof.** The argument of [QG, Proposition 4] works when $M$ is $\tau$-Artinian.

Now suppose that $M$ is $\tau$-Noetherian $\tau$-torsionfree and $X \subseteq S$ be nil multiplicatively closed. In view of Proposition 3.1 and Lemma 3.2 it suffices to show that $X$ is left $T$-nilpotent. Suppose not. Let $C$ be the set of first terms of all non left vanishing sequences in $X$. Since $M$ is $\tau$-torsionfree, $\text{Ker}(\alpha)$ is $\tau$-pure for all $\alpha \in S$. Let $s_1 \in C$ be such that $\text{Ker}(s_1)$ is maximal in $\{\text{Ker}(s) : s \in C\}$. Clearly $s_1 C \cap C \neq \emptyset$. Let $s_2 \in C$ be such that $\text{Ker}(s_2)$ is maximal in $\{\text{Ker}(s) : s \in C, s_1 s \in C\}$. By induction we can choose a sequence $s_1, s_2, ...$ in $C$ such that, for each $n$, $\text{Ker}(s_{n+1})$ is maximal in $\{\text{Ker}(s) : s \in C, s_1 s_2 ... s_n s \in C\}$.

Thus, in particular, the sequence $s_1, s_2, ...$ is non left vanishing.

**Claim 1:** $\text{Ker}(s_m ... s_n) = \text{Ker}(s_m)$ for all $n \geq m$.

Clearly $s_m ... s_n \in C$, $s_1 ... s_{m-1} s_m ... s_n \in C$ and $\text{Ker}(s_m ... s_n) \supseteq \text{Ker}(s_m)$. So by the maximality of $\text{Ker}(s_m)$ we get the desired claim.
Let $p_n = s_1 \ldots s_n$. By Claim 1 $\ker(p_n) = \ker(p_{n+1}) = \ker(s_1)$. Thus by Lemma 3.3(1)

$$M_{p_n} \cap \ker(s_{n+1}) = 0 \quad (1)$$

**Claim 2:** $p_n s_m = 0$ for every $m \leq n$

Suppose $p_n s_m \neq 0$. By Claim 1 and Lemma 3.3 (2) we get that $\ker(p_n s_m \ldots s_k) = \ker(p_n) \neq M$. Therefore $p_n s_m \ldots s_k \neq 0$ for all $k \geq m$. In other words

$$s_1 \ldots s_{m-1}(s_m \ldots s_n s_m) s_{m+1} \ldots s_k \neq 0$$

for every $k \geq m$, showing that $s_m \ldots s_n s_m \in C$ and $s_1 \ldots s_{m-1} s_m \ldots s_n s_m \in C$. Maximality of $\ker(s_m)$ thus gives that $\ker(s_m \ldots s_n s_m) = \ker(s_m)$. Let $\sigma = s_m \ldots s_n$. By Lemma 3.3 (2) $\ker(s_m) = \ker(\sigma s_m) = \ker(\sigma^2 s_m) = \ldots = \ker(\sigma^r s_m)$ for every $r$. Since $X$ is nil so $\sigma^r = 0$ for some $r$. Thus $s_m = 0$, a contradiction. Hence Claim 2 is established. So

$$M_{p_n} \subseteq \ker(s_m) \forall m \leq n \quad (2)$$

**Claim 3:** $(\sum_{k=1}^n M_{p_k}) \cap (\cap_{j=2}^{n+1} \ker(s_j)) = 0 \quad (3)$

By (1) this claim is true for $n = 1$. Assume that

$$(\sum_{k=1}^n M_{p_k}) \cap (\cap_{j=2}^{n+1} \ker(s_j)) = 0 \quad (4)$$

By (1)

$$M_{p_{n+1}} \cap \ker(s_{n+2}) = 0 \quad (5)$$

Note that by (2) $M_{p_{n+1}} \subseteq \cap_{j=2}^{n+1} \ker(s_j)$. Now

$$M_{p_{n+1}} = M_{p_{n+1}} + ((\sum_{k=1}^n M_{p_k}) \cap (\cap_{j=2}^{n+1} \ker(s_j)))$$

$$= (\sum_{k=1}^{n+1} M_{p_k}) \cap (\cap_{j=2}^{n+1} \ker(s_j)). \quad (\text{by modular law}) \quad (6)$$

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So by (5) and (6) we get that \((\sum_{k=1}^{n+1} M_{p_k}) \cap (\cap_{j=2}^{n+2} \text{Ker}(s_j)) = 0\). Thus Claim (3) has been proved. This gives, using Proposition 2.11 (5), Corollary 2.12 and 2.13, that
\[
(\forall_{k=1}^{n} M_{p_k})' \cap (\cap_{j=2}^{n+1} \text{Ker}(s_j)) = 0. \tag{7}
\]
Since \(M\) is \(\tau\)-Noetherian so there exists \(n\) such that
\[
(M_{p_{n+1}})' \subseteq \forall_{k=1}^{n} (M_{p_k})' \Rightarrow M_{p_{n+1}} \cap (\cap_{j=2}^{n+1} \text{Ker}(s_j)) = 0 \quad \text{(see (7))}.
\]
But by (2)
\[
M_{p_{n+1}} \subseteq (\cap_{j=2}^{n+1} \text{Ker}(s_j)) = 0
\]
Hence \(M_{p_{n+1}} = 0\). This contradiction proves that \(X\) is left \(T\)-nilpotent. \(\square\)

**Lemma 3.5.** Let \(M\) be an arbitrary module and \(N_1, \ldots, N_n\) be non \(\tau\)-dense submodules of \(M\). Then, \(\{N_1, \ldots, N_n\}\) are \(\tau\)-co-independent if and only if \(N_i + (\cap_{j>i} N_j)\) is \(\tau\)-dense in \(M\) for every \(i = 1, \ldots, n - 1\).

**Proof.** \((\Rightarrow)\) is trivial.

\((\Leftarrow).\) Clearly \(N'_i \neq M\). By Proposition 2.11 and 2.13 \(N'_i \cup (\cap_{j>i} N'_j) = M\).
So, by Corollary 1.4, \(\{N'_1, \ldots, N'_n\}\) are co-independent in \(\mathcal{P}_r(M)\). Hence, by Proposition 2.20 (2), \(\{N_1, \ldots, N_n\}\) are \(\tau\)-co-independent in \(M\). \(\square\)

**Proposition 3.6 (1).** Let \(M\) be a \(\tau\)-torsionfree \(\tau\)-Artinian module with \(\text{End}(M) = S\) and \(\tau\)-dual Goldie dimension \(n\), then
\[(a).\ H = H(\tau)\ is\ a\ nilpotent\ ideal\ of\ S.\]
(b) Any proper descending chain of right annihilator ideals of $S/H$ has at the most $n + 1$ terms.

(2). Let $M$ be a $\tau$-torsionfree $\tau$-Noetherian module with $\text{End}(M) = S$ and $\tau$-Goldie dimension $n$, then

(a) $N = N(\tau)$ is a nilpotent ideal of $S$.

(b). Any proper descending chain of left annihilator ideals of $S/N$ has at the most $n + 1$ terms.

**Proof**

(1) (a). See [QG, Proposition 5].

(b). This part is proved in [QG, Proposition 5(2)]. We present here a streamlined version of the proof given in [QG].

Let $V_1 \subset \ldots \subset V_{m+1}$ be a proper chain of subsets of $S$ such that $(H : V_i) \supset \ldots \supset (H : V_{m+1})$, where $(H : V_i) = (H : V_i)_\tau = \{\alpha \in S : V_i \alpha \subseteq H\}$, is a proper chain. Pick $\gamma_i \in (H : V_i) \setminus (H : V_{i+1})$, $i = 1, \ldots, m$. Thus

$$V_j \gamma_i \subseteq H \forall \ j \leq i$$

(1)

Pick $\alpha_{i+1} \in V_{i+1}$ such that $\alpha_{i+1} \gamma_i \notin H$. Thus there exists a non $\tau$-dense submodule $N_i \leq M$ such that

$$M\alpha_{i+1} \gamma_i + N_i \text{ is } \tau\text{-dense in } M \text{ for } i = 1, \ldots, m$$

(2)

This gives

$$M \gamma_i + N_i \text{ is } \tau\text{-dense in } M \text{ for } i = 1, \ldots, m$$

(3)

and, by Proposition 2.8(1),

$$M\alpha_{i+1} + N_i \gamma_i^- \text{ is } \tau\text{-dense in } M \text{ for } i = 1, \ldots, m$$

(4)

We set $M_1 = N_1 \gamma_1^-$ and $M_i = N_i \gamma_i^- + M\alpha_2 + \ldots + M\alpha_i$ for $i = 2, \ldots, m$. 

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Since $M/M_i = M/N_i \gamma_i \cong (M \gamma_i + N_i)/N_i$, $M_i$ is \( \tau \)-dense in $M$ implies $N_i$ is \( \tau \)-dense in $M \gamma_i + N_i$, which by (3) and Proposition 2.8(3), implies that $N_i$ is \( \tau \)-dense in $M$. A contradiction. Also for $i \geq 2$, $M/M_i \cong (M \gamma_i + N_i + M \alpha_2 \gamma_i + \ldots + M \alpha_i \gamma_i)/(N_i + M \alpha_2 \gamma_i + \ldots + M \alpha_i \gamma_i)$. If $M_i$ is \( \tau \)-dense in $M$, by (3) and Proposition 2.8(3), $N_i + M \alpha_2 \gamma_i + \ldots + M \alpha_i \gamma_i$ is \( \tau \)-dense in $M$. So $N_i$ is \( \tau \)-dense in $M$ by using (1). A contradiction. Thus $M_i$, $i = 1, \ldots, m$, are all non \( \tau \)-dense submodules of $M$. Now observe that $N_i \gamma_i \subseteq M_i$ and $M \alpha_{i+1} \subseteq \cap_{j>i} M_j$. Thus

\[ N_i \gamma_i + M \alpha_{i+1} \subseteq M_i + (\cap_{j>i} M_j). \]

Hence, by (4), $M_i + (\cap_{j>i} M_j)$ is \( \tau \)-dense in $M$. Thus by Lemma 3.5 $M_1, \ldots, M_m$ are \( \tau \)-co-independent.

(2). Since $M$ is \( \tau \)-torsionfree, $K$ is large in $M$ iff $K$ is \( \tau \)-large in $M$ for every $K \subseteq M$ and $N_1, \ldots, N_n$ are independent submodules of $M$ if and only if $N_1, \ldots, N_n$ are \( \tau \)-independent.

(2) (a). By Proposition 3.1 and Lemma 3.2 it suffices to show that $N$ is left $T$-nilpotent. Let $s_1, s_2, \ldots$ be a sequence in $N$. Set $p_n = s_1 \ldots s_n$. Since $Ker(p_1) \subseteq Ker(p_2) \subseteq \ldots$ so there exists an $n$ such that $Ker(p_n) = Ker(p_{n+1})$.

So, by Lemma 3.3(1), $M p_n \cap Ker(s_{n+1}) = 0$. But as $Ker(s_{n+1}) \subseteq M$, $M p_n = 0$. 

(b). Let $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_{m+1}$ be a proper chain of subsets in $S$ such that $(N : S_1) \supset (N : S_2) \supset \ldots \supset (N : S_{m+1})$, where $(N : S_i) = (N : S_i \backslash \{ \alpha \in S : \alpha S_i \subseteq N \})$, is also a proper chain. Let $x_i \in (N : S_i) \setminus (N : S_{i+1})$, $i = 1, \ldots, m$. Thus for $1 \leq j < i \leq m$

\[ x_i S_j \subseteq N \quad (1) \]

Pick $s_{i+1} \in S_{i+1}$ such that $x_i s_{i+1} \notin N$. So $Ker(x_i s_{i+1})$ is not large in $M$ and
thus there exists $0 \neq K_i \leq M$ such that for each $i = 1, \ldots, m$

$$Ker(x_is_{i+1}) \cap K_i = 0$$  \hspace{1cm} (2)

This gives us that for each $i = 1, \ldots, m$,

$$Ker(x_i) \cap K_i = 0$$  \hspace{1cm} (3)

and

$$K_ix_i \cap Ker(s_{i+1}) = 0$$  \hspace{1cm} (4)

From (3) it follows that for each $i = 1, ..., m$

$$(x_i|K_i)$$ is a monomorphism  \hspace{1cm} (5)

Using (1) it follows that for all $2 \leq i \leq m$, $K_i \cap Ker(x_is_2) \cap \ldots \cap Ker(x_is_i) \neq 0$.
Thus, using (5), we get $M_1 = K_1x_1$ and $M_i = (K_i \cap Ker(x_is_2) \cap \ldots \cap Ker(x_is_i))x_i$, $i = 2, ..., m$, are all non zero. Further one sees easily that

$M_i \subseteq Ker(s_j)$, $2 \leq j \leq i \leq m$. Thus for each $i = 1, ..., m$, by (4),

$M_i \cap (M_{i+1} + \ldots + M_m) \subseteq K_ix_i \cap Ker(s_{i+1}) = 0$ showing that $M_1, ..., M_n$ are independent. □

**Theorem 3.7** (1). Let $M$ be a $\tau$-torsionfree $\tau$-Artinian module having endomorphism ring $S$. Every nil multiplicatively closed subset of $S$ is nilpotent and has the index of nilpotency at the most $(n+1)k$, where $n$ is the $\tau$-dual Goldie dimension of $M$ and $k$ is the index of nilpotency of the ideal $H = H(\tau)$ of $S$.

(2). Let $M$ be a $\tau$-torsionfree $\tau$-Noetherian module having endomorphism ring $S$. Every nil multiplicatively closed subset of $S$ is nilpotent and has the index of nilpotency at the most $(m+1)l$, where $m$ is the $\tau$-Goldie dimension of $M$. 
and \( l \) is the index of nilpotency of the ideal \( N = N(\tau) \) of \( S \).

**Proof** (1). Let \( L \) be a nil multiplicatively closed subset of \( S \). By Proposition 3.4 \( L \) and thus \( X = (L + H)/H \) is nilpotent. We claim that \( X^{n+1} = 0 \). Suppose not. Let \( m \) be the least natural number such that \( X^m = 0 \). By choice \( m > n + 1 \). So we have a proper chain \( S/H \supset X \supset X^2 \supset \ldots \supset X^m = 0 \). This gives us a chain \( 0 = r_{S/H}(S/H) \subset r_{S/H}(X) \subset \ldots \subset r_{S/H}(X^m) = S/H \). This chain is also proper because \( X^{m-(i+1)} \subset r_{S/H}(X^{i+1}) \) but not in \( r_{S/H}(X^i) \) for all \( i = 1, \ldots, m \). This violates Proposition 3.6 (1)(b). Thus \( L^{(n+1)k} = 0 \).

(2). Follows analogously using Proposition 3.4 and Proposition 3.6 (2)(b). \( \square \)

**Corollary 3.8** (1). Let \( M \) be an Artinian module with endomorphism ring \( S \). Then any nil multiplicatively closed subset of \( S \) is nilpotent and has index of nilpotency at the most \( (n + 1)k \), where \( n \) is the dual Goldie dimension of \( M \) and \( k \) is the index of nilpotency of the ideal \( H = \{ \alpha \in S : M\alpha \ll M \} \).

(2). Let \( M \) be a Noetherian module with endomorphism ring \( S \). Then any nil multiplicatively closed subset of \( S \) is nilpotent and has index of nilpotency at the most \( (m + 1)l \), where \( m \) is the Goldie dimension of \( M \) and \( l \) is the index of nilpotency of the ideal \( N = \{ \alpha \in S : \text{Ker}(\alpha) \trianglelefteq M \} \).

**Proof.** Take \( \tau = \xi \) in Theorem 3.7 above. \( \square \)