Chapter III

PROJECTIVE COVERS
AND
LIFTING IDEMPOTENTS
0. INTRODUCTION

Let $U$ and $M$ be two left $R$-modules. $U$ is said to be $M$-projective in case for each $R$-epimorphism $g : M \rightarrow N$ and each $R$-homomorphism $\gamma : U \rightarrow N$ there is an $R$-homomorphism $\delta : U \rightarrow M$ such that $\delta g = \gamma$. If $U$ is $M$-projective for every left $R$-module $M$ then $U$ is said to be projective. Projective left $R$-modules are just direct summands of free left $R$-modules. A submodule $N$ of a module $M$ is said to be superfluous or small ($N \ll M$) if for any submodule $K$ of $M$, $N + K = M$ implies that $K = M$.

Dualizing the concept of an injective hull of a module, Bass [B] defined a projective cover of a module $M$ to be an epimorphism $p : P \rightarrow M$, such that $P$ is a projective module and $\ker p \ll P$. Thus modules having projective covers are, up to isomorphism, of the form $P/K$, where $P$ is a projective module and $K$ its superfluous submodule. Unlike injective hulls, projective covers of modules seldom exist. For instance, over a semiprimitive ring $R$ (i.e., $J(R) = 0$) a module has a projective cover if and only if it is already projective (see [La, 24.11(5)]). Let $I$ be an ideal of ring $R$ and $f$ be an idempotent of $R/I$. We say that $f$ lifts modulo $I$ in $R$ if there exists an idempotent $e$ in $R$ such that $e = e + I = f$.

The following well known result gives a relationship between the concepts of projective covers and lifting of idempotents.

**Proposition 0.1** ([AF, Proposition 27.4]). Let $R$ be a ring and $I$ be a two sided ideal of a ring $R$ with $I \ll_R R$ (or equivalently $I \subseteq J(R)$). Then the following are equivalent:
(a). Idempotents lift modulo \( I \);

(b). Every direct summand of a left \( R \)-module \( R/I \) has a projective cover. □

From Proposition 0.1 it follows that if \( I \subseteq J(R) \) is an ideal of a ring \( R \) such that idempotents do not lift modulo \( I \) in \( R \) then, although the left \( R \)-module \( R/I \) has a projective cover, there exists a direct summand of left \( R \)-module \( R/I \) which does not have a projective cover. More specifically, Let \( Q \) be the field of rational numbers and \( R = \{a/b \in Q : (6, b) = 1\} \). Then \( R \) is a ring with \( R/J(R) \cong R/2R \oplus R/3R \), but neither \( R/2R \) nor \( R/3R \) have projective covers. So the natural question arises that what exactly are the modules with projective covers whose every direct summand also has a projective cover.

One can easily prove the following

**Lemma 0.2.** Let \( p : P \to M \), with \( \ker p = K \), be a projective cover of \( M \). If \( E = \text{End}_R(P); S = \{\alpha \in E : K \alpha \subseteq K\} \) and \( T = \{\alpha \in S : Pf \subseteq K\} \), then \( \theta : S/T \to \text{End}_R(M) \) defined as \( p(x)\theta(\bar{\alpha}) = p((x)\alpha) \forall x \in P, \alpha \in S \), is a ring isomorphism. □

In the notations of Lemma 0.2 we note that \( T \) is a left ideal of \( S \). Suppose that \( q : Q \to M \) is also a projective cover of \( M \) with \( K_1 = \ker q \). Then there exists an isomorphism \( \gamma : P \to Q \) such that \( \gamma q = p \) (see [AF, Lemma 17.17]). Let \( E_1 = \text{End}_R(Q); S_1 = \{\alpha \in E_1 : K_1 \alpha \subseteq K_1\} \) and \( T_1 = \{\alpha \in S_1 : Qf \subseteq K_1\} \). One easily checks that the map \( \delta : S \to S_1 \) defined as \( \delta(\sigma) = \gamma^{-1}\sigma\gamma \), where \( \sigma \) is in \( S \), is an isomorphism with \( \delta(T) = T_1 \). Thus \( S \) is unique upto isomorphism.
Let $M$ be as in Lemma 0.2 and $N \subseteq M$. Fix an idempotent $e$ in $\text{End}_R(M)$ such that $N = Me$. There exists an idempotent $\bar{g}$ in $S/T$ such that $\theta(\bar{g}) = e$. In section 1 we prove that $N$ has a projective cover if and only if $\bar{g}$ lifts to an idempotent in $S$. This improves upon [AF, Proposition 27.4]. We also prove that every direct summand of the left $R$-module $M$ has a projective cover if and only if every direct summand of the left $S$-module $S/T$ has a projective cover.

Let $L$ be a left ideal of a ring $R$. Following Nicholson [N], we say that idempotents lift modulo $L$ in $R$ if for given $r \in R$ with $r - r^2 \in L$, there exists an idempotent $e$ in $R$ such that $e - r \in L$. Nicholson [N], defined suitable rings and characterised them as rings in which idempotents lift modulo every left (equivalently right) ideal. In section 2 we study the rings in which idempotents lift modulo every left ideal contained in the Jacobson radical. We prove that these rings coincide with the rings in which idempotents lift modulo the Jacobson radical. Using the results proved in section 1 we prove that if idempotents lift modulo the Jacobson radical of a ring $R$ then if a cyclic left $R$-module has a projective cover then so does its every direct summand. It is proved that direct summands of finitely generated left $R$-modules, with projective covers, have projective covers if and only if idempotents lift modulo $M_n(J)$ in $M_n(R)$ for every natural number $n$.

In section 3 we give some sufficient conditions for direct summands of left $R$-modules, with projective covers, to have projective covers. In particular, we prove that if idempotents lift modulo the Jacobson radical of the endomorphism ring of each free left $R$-module then a left $R$-module has a projective
cover then so does its every direct summand.

A submodule $K$ of a left $R$-module $M$ is called supplement submodule if for some submodule $N$ of $M$, $N + K = M$ but $N + K_1 \neq M$ for every proper submodule $K_1$ of $K$. Let $K$ be a supplement submodule of a projective module $P$. In section 4 we prove that $P/K$ has a projective cover if and only if $K$ is a direct summand of $P$.

1. PROJECTIVE COVERS OF DIRECT SUMMANDS

We first list some basic results which will often be referred to later.

**Lemma 1.1** ([AF, Lemma 5.18]). If $K \ll M$ and $f : M \to N$ is a homomorphism then $Kf \ll N$. In particular, if $K \ll M \leq N$, then $K \ll N$. □

**Lemma 1.2** ([AF, Proposition 17.11]). Let $P$ be a projective left $R$-module with endomorphism ring $E = \text{End}_R(P)$. Let $a \in E$. Then

$$a \in J(E) \iff \text{Im } a \ll P.$$ □

**Lemma 1.3** ([AF, Exercise 17.15(2)]). Let $M$ and $N$ be two modules. If both $M$ and $M \oplus N$ have projective covers, then $N$ also has a projective cover. □

**Lemma 1.4** ([AF, Lemma 17.17]). Suppose $M$ has a projective cover $p : P \to M$. If $Q$ is projective and $q : Q \to M$ is an epimorphism, then $Q$ has a decomposition $Q = P' \oplus P''$ such that

(a). $P' \cong P$.

(b). $P'' \subseteq \text{Ker } q$. 

(c). \((q|P') : P' \to M\) is a projective cover of \(M\). \(\Box\)

**Lemma 1.5** ([AF, Exercise 27.1]). Let \(I\) be an ideal of \(R\) with \(I \subseteq J(R)\) and \(f\) be a nonzero idempotent of \(R\). If idempotents lift modulo \(I\) in \(R\), then idempotents lift modulo \(ff\) in \(fRf\). \(\Box\)

**Lemma 1.6** ([AF, Lemma 27.2]). Let a left \(R\)-module \(M\) have a decomposition \(M = M_1 \oplus \ldots \oplus M_n\) such that each term \(M_i\) has a projective cover. Then an \(R\)-homomorphism \(p : P \to M\) is a projective cover if and only if \(P\) has a decomposition \(P_1 \oplus \ldots \oplus P_n\) such that for each \(i = 1, \ldots, n\)
\[
(p|P_i) : P_i \to M_i
\]
is a projective cover. \(\Box\)

Authors got the idea of proving the following Theorem during personal communication with Professor Donald S. Passman.

**Theorem 1.7.** Let \(p : P \to M\) be a projective cover of a left \(R\)-module \(M\) with \(K = \text{Ker} \ p\) and \(E, S, T\) and \(\theta\) be as in Lemma 0.2. Let \(M_e\), where \(e \in \text{End}_R(M)\) is an idempotent with \(\theta(\tilde{g}) = e\) for some idempotent \(\tilde{g}\) in \(S/T\), be a direct summand of \(M\). Then the following are equivalent:

(a). There exists a direct summand \(P_1\) of \(P\) such that \(p(P_1) = Me\);

(b). \(Me\) has a projective cover;

(c). There exists a decomposition \(P = P_1 \oplus P_2\) such that
\[
p(P_1) = Me\text{ and }p(P_2) = M(1 - e);
\]

(d). \(\tilde{g}\) lifts modulo \(T\) in \(S\).
In particular, every direct summand of $M$ has a projective cover if and only if idempotents lift modulo $T$ in $S$.

**Proof.** (a) $\Rightarrow$ (b). As $K \ll P$ and $P_1$ is a direct summand of $P$, $\text{Ker}(p|_{P_1}) = K \cap P_1 \ll P_1$ (see Lemma 1.1). Thus $(p|_{P_1}) : P_1 \to Me$ is a projective cover.

(b) $\Rightarrow$ (c). As both $Me$ and $M = Me \oplus M(1 - e)$ have projective covers, $M(1 - e)$ also has a projective cover by Lemma 1.3. Also as $p : P \to M$ is a projective cover, by Lemma 1.6, there exists a decomposition $P = P_1 \oplus P_2$ such that

\[
p(P_1) = Me \text{ and } p(P_2) = M(1 - e);
\]

(c) $\Rightarrow$ (d). Let $f \in E$ be an idempotent such that $P_1 = Pf$ and $P_2 = P(1 - f)$.

\[
p(P_1) = p(Pf) = Me = M\theta(\bar{g}) \quad (1)
\]

and

\[
p(P_2) = p(P(1 - f)) = M(1 - e) = M\theta(\bar{1 - g}) \quad (2)
\]

Now we show that $f \in S$. Let $k \in K$. Then

\[
k = kf + k(1 - f) \Rightarrow 0 = p(k) = p(kf) + p(k(1 - f))
\]

\[
\Rightarrow p(kf) \in Me \cap M(1 - e) = 0 \Rightarrow kf \in \text{Ker } p = K
\]

which in turn implies that $f \in S$. This in view of (1) and (2) gives that,

\[
M\theta(\bar{g}) = p(Pf) = p(P)\theta(\bar{f}) = M\theta(\bar{f}) \quad (3)
\]

and

\[
M\theta(\bar{1 - g}) = p(P(1 - f)) = p(P)\theta(\bar{(1 - f)}) = M\theta(\bar{(1 - f)}). \quad (4)
\]
Now by using (3) and (4) we get
\[ \theta(f) + \theta((1-f)) = 1 \Rightarrow \theta(\bar{g})(\theta(f) + \theta((1-f))) = \theta(\bar{g}) \Rightarrow \theta(\bar{g})\theta(f) = \theta(\bar{g}). \]

Also
\[ \theta(g) + \theta((1-g)) = 1 \Rightarrow (\theta(g) + \theta((1-g)))\theta(f) = \theta(f) \Rightarrow \theta(g)\theta(f) = \theta(f). \]

Thus \( \theta(\bar{g}) = \theta(f) = \theta(\bar{g})\theta(\bar{e}) \) and as \( \theta \) is an isomorphism, \( \bar{g} = \bar{f} \).

(d) \( \Rightarrow \) (a). We may assume that \( g \) is an idempotent of \( S \). Thus, \( Pg \) is a direct summand of \( P \) with
\[ Me = M\theta(\bar{g}) = p(P)\theta(\bar{g}) = p(Pg). \]

Last assertion is obvious. □

The following result improves upon [AF, Proposition 27.4].

**Corollary 1.8.** Let \( I \) be an ideal of a ring \( R \) such that \( I \ll R \). Let \( (R/I)\bar{g} \), where \( \bar{g} \) is an idempotent in \( R/I \), be a direct summand of left \( R \)-module \( R/I \). Then the following are equivalent:

(a) \( \bar{g} \) lifts modulo \( I \) in \( R \);

(b) \( (R/I)\bar{g} \) has a projective cover. □

**Corollary 1.9 ([AF, Exercise 27.2]).** For a ring \( R \) with \( J = J(R) \) the following are equivalent:

(a). For each natural number \( n \) if \( K \) is a direct summand of the left \( R \)-module \( (R/J)^{(n)} \) and \( \nu : R^{(n)} \to (R/J)^{(n)} \) is the canonical homomorphism then there exists a direct summand \( P \) of \( R^{(n)} \) such that \( \nu(P) = K \);
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(b) For each natural number \(n\) every direct summand of left \(R\)-module \(R^{(n)}/J^{(n)}\) has a projective cover;

(c) Idempotents lift modulo \(M_n(J)\) in \(M_n(R)\) for every natural number \(n\);

(d) Idempotents lift modulo the Jacobson radical of every ring that is Morita equivalent to \(R\).

**Proof.** (a) \(\Rightarrow\) (b). Let \(K \subseteq \oplus (R/J)^{(n)} \cong R^{(n)}/J^{(n)}\). By (a) there exists \(P \subseteq \oplus R^{(n)}\) such that \(\nu(P) = K\). Now, in view of Lemma 1.1, \((\nu|P) : P \to K\) is a projective cover.

(b) \(\Rightarrow\) (c). As \(\nu : R^{(n)} \to R^{(n)}/J^{(n)}\) is a projective cover so (c) follows from Theorem 1.7.

(c) \(\Rightarrow\) (d). A ring which is Morita equivalent to \(R\) is of the form \(eM_n(R)e\), for some idempotent \(e\) in \(M_n(R)\) and some natural number \(n\), (see [AF, Corollary 22.7]). Now (d) follows by Lemma 1.5.

(d) \(\Rightarrow\) (a). This follows from Theorem 1.7 and Lemma 1.6. □

The following is an open problem (see [AF, Exercise 27.2]):

**Question 1.10.** Let \(R\) and \(S\) be Morita equivalent rings. If idempotents lift modulo \(J(R)\) in \(R\) then do idempotents lift modulo \(J(S)\) in \(S\)?

Before proving our next result we recall the following two well known results.

**Lemma 1.11.** Let \(S\) be a subring of a ring \(R\) and \(I\) be a left quasi-regular left ideal of \(R\). If \(I \subseteq S\) then \(I\) is left quasi-regular left ideal of \(S\) as well. □
**Lemma 1.12** ([AF, Proposition 15.2]). For a left ideal $I$ of $R$ the following statements are equivalent:

(a) $I$ is left quasi-regular;
(b) $I$ is quasi-regular;
(c) $I \subseteq \mathcal{J}(R)$;
(d) $I \subseteq \mathcal{J}(R)$.

Now we are ready to prove the last result of this section.

**Proposition 1.13.** Let $p : P \rightarrow M$ be a projective cover of a left $R$-module $M$ with $K = \text{Ker } p$ and $E, S, T$ and $\theta$ be as in Lemma 0.2. Let $\bar{g}$ be an idempotent in $S/T$. Then the direct summand $M\theta(\bar{g})$ of the left $R$-module $M$ has a projective cover if and only if the direct summand $(S/T)\bar{g}$ of the left $S$-module $S/T$ has a projective cover. In particular, every direct summand of the left $R$-module $M$ has a projective cover if and only if every direct summand of the left $S$-module $S/T$ has a projective cover.

**Proof.** Let $\alpha \in T$. As $\text{Im } \alpha \subseteq K \subseteq P$, $\alpha$ is in $\mathcal{J}(E)$ by Lemma 1.2. Thus $T$ is contained in $\mathcal{J}(E)$ and so by Lemma 1.11 and Lemma 1.12, $T \subseteq \mathcal{J}(S)$. Thus the result follows from Theorem 1.7 and Corollary 1.8. □

It is well known that if idempotents lift modulo an ideal contained in the Jacobson radical of a ring then a countable orthogonal set of idempotents also lifts to an orthogonal set of idempotents (see [La, Proposition 21.25]). Thus, in the notations of Lemma 0.2, if every direct summand of $M$ has a projective cover then countable orthogonal set of idempotents lifts to an orthogonal set.
2. LIFTING IDEMPOTENTS

Let \( L \) be a left ideal of a ring \( R \). Then \( \mathcal{I}(L) = \{ r \in R : Lr \subseteq L \} \), called the \textit{idealizer} of \( L \), is the largest subring of \( R \) in which \( L \) is a two sided ideal.

The following special case of Theorem 1.7 can be of independent interest as is demonstrated in this section.

Corollary 2.1. Let \( L \) be a superfluous left ideal of a ring \( R \), \( \mathcal{I}(L) \) be the idealizer of \( L \) and \( \theta : \mathcal{I}(L)/L \to \text{End}_R(R/L) \) be the canonical isomorphism. Let \( K \) be a direct summand of left \( R \)-module \( R/L \) and \( \bar{g} \) is an idempotent in \( \mathcal{I}(L)/L \) such that \( K = (R/L)\theta(\bar{g}) \). Then the following are equivalent:

(a) \( \bar{g} \) lifts modulo \( L \) in \( \mathcal{I}(L) \);

(b) \((R/L)\theta(\bar{g})\) has a projective cover.

In particular, every direct summand of left \( R \)-module \( R/L \) has a projective cover if and only if idempotents lift modulo \( L \) in \( \mathcal{I}(L) \). \( \square \)

Nicholson [N] defined a ring \( R \) to be \textit{suitable} if each of its elements satisfies the equivalent properties of the following proposition.

Proposition 2.2 (Nicholson [N, Proposition 1.1]) If \( R \) is a ring, the following conditions are equivalent for an element \( x \) of \( R \):

(a) There exists an idempotent \( e \) in \( R \) with \( e - x \in R(x - x^2) \);

(b) There exists an idempotent \( e \) in \( Rx \) and an element \( c \in R \) such that \((1 - e) - c(1 - x) \in J(R)\).
(c) There exists an idempotent $e$ in $R_x$ such that $R = Re + R(1 - x);

(d) There exist an idempotent $e$ in $R_x$ such that $1 - e \in R(1 - x)$. □

Nicholson [N, Corollary 1.3] proved that a ring $R$ is suitable if and only if idempotents lift modulo every one-sided ideal of $R$. We are interested in rings in which idempotents lift modulo every one-sided ideal contained in the Jacobson radical. In Proposition 2.4 below, we prove that such rings coincide with the rings in which idempotents lift modulo the Jacobson radical. First we prove the following

**Lemma 2.3.** For a ring $R$ with the Jacobson radical $J$ the following are equivalent:

(a) Idempotents lift modulo every left ideal contained in $J$;

(b) Every element $x \in R$ such that $x^2 - x \in J$ satisfies the following equivalent conditions:

(i) There exists an idempotent $e$ in $R$ with $e - x \in R(x - x^2)$;

(ii) There exists an idempotent $e$ in $R_x$ and an element $c \in R$ such that $(1 - e) - c(1 - x) \in J$;

(iii) There exists an idempotent $e$ in $R_x$ such that $R = Re + R(1 - x)$;

(iv) There exist an idempotent $e$ in $R_x$ such that $1 - e \in R(1 - x)$.

**Proof.** The four properties listed under (b) are equivalent by Proposition 2.2.

(a) $\Rightarrow$ (b)(i). Let $x \in R$ be such that $x^2 - x \in J$. As $R(x^2 - x) \subseteq J$ is a left ideal of $R$ so by (a) there exists an idempotent $e \in R$ such that $e - x \in R(x^2 - x)$. 


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(b)(i) (a). Let $L \subseteq J$ be a left ideal of $R$ and $x^2 - x \in L$ for some $x \in R$. By (b)(i) there exists an idempotent $e \in R$ such that $e - x \in R(x^2 - x) \subseteq L$. □

Proposition 2.4. Let $R$ be a ring with $J = J(R)$. If idempotents lift modulo $J$ in $R$ then idempotents lift modulo every left ideal of $R$ contained in $J$.

Proof. Let $x \in R$ be such that $x^2 - x \in J$. In view of Lemma 2.3 we have to show that there exists an idempotent $e \in Rx$ and an element $c \in R$ such that

$$(1 - e) - c(1 - x) \in J.$$ 

Now as idempotents lift modulo $J$ in $R$ so there exists an idempotent $f \in R$ such that $f - x \in J$. So $u = 1 - f + x$ is an invertible element with $u = 1$ in $R/J$. Let $e = u^{-1}fu = u^{-1}f(1 - f + x) = u^{-1}fx$. Clearly $e$ is an idempotent in $Rx$ and $\bar{e} = \bar{f} = \bar{x}$. Thus $x - e = (1 - e) - 1(1 - x) \in J$. □

Remark 2.5. Suppose that idempotents lift modulo a left ideal $L \subseteq J(R)$ in $R$. On the same lines one can prove that idempotents lift in $R$ modulo every left ideal contained in $L$ also. Note that this does not hold for left ideals which are not contained in the Jacobson radical. For example, idempotents lift modulo $4\mathbb{Z}$ in $\mathbb{Z}$ but do not lift modulo $12\mathbb{Z}$ in $\mathbb{Z}$, where $\mathbb{Z}$ denotes the ring of integers.

The following result characterizes the rings over which direct summands of cyclic modules, with projective covers, have projective covers.

Theorem 2.6. The following conditions are equivalent for a ring $R$ with $J = J(R)$:

(a). Idempotents lift modulo $J$ in $R$;
(b) Idempotents lift modulo $L$ in $I(L)$ for every left ideal $L \subseteq J(R)$;

(c) For any left ideal $L \subseteq J(R)$ every direct summand of left $R$-module $R/L$ has a projective cover;

(d) Every direct summand of a cyclic left $R$-module, with projective cover, has a projective cover.

**Proof.** (a) $\Rightarrow$ (b). Let $L \subseteq J$ be a left ideal of $R$ and $x^2 - x \in L$ for some $x \in I(L)$. By Proposition 2.4 there exists an idempotent $e \in R$ such that $e - x \in L$. This implies that $e \in I(L)$.

(b) $\Rightarrow$ (c). This follows from Corollary 2.1.

(c) $\Rightarrow$ (d). Let $M$ be a cyclic left $R$-module with projective cover. By Lemma 1.4 $M \cong Re/L$ for some idempotent $e \in R$ and left ideal $L \subseteq Je$. Now $(Re/L) \oplus (R(1-e)/J(1-e)) \cong R/(L \oplus J(1-e))$. As $L_1 = L \oplus J(1-e) \subseteq J$ is a left ideal of $R$ so, by (c), every direct summand of $R/L_1$ has a projective cover. Now as direct summands of $M \cong Re/L$ are isomorphic to direct summands of $R/L_1$ so every direct summand of $Re/L$ has a projective cover.

(d) $\Rightarrow$ (a). As every direct summand of left $R$-module $R/J$ has a projective cover so by Corollary 1.8 idempotents lift modulo $J$ in $R$. \qed

**Remark 2.7.** From Remark 2.5 and the proof of Theorem 2.6 it is clear that if idempotents lift modulo an ideal $I \subseteq J(R)$ of $R$ and $L \subseteq I$ is a left ideal of $R$ then every direct summand of left $R$-module $R/L$ has a projective cover.

**Lemma 2.8.** Let $M$ be an $n$-generated left module. Then $M$ has a projective cover if and only if $M \cong R^{(n)}e/L$, where $e \in \text{End}_R(R^{(n)}) \cong M_n(R)$ is an
idempotent and \( L \subseteq J^{(n)}e \) is a submodule of left \( R \)-module \( R^{(n)}e \).

**Proof.** As \( M \) is \( n \)-generated, there exists an epimorphism \( p : R^{(n)} \to M \). By Lemma 1.4 there exists an idempotent \( e \in \text{End}_R(R^{(n)}) \cong M_n(R) \) such that \((p|R^{(n)}e) : R^{(n)}e \to M \) is a projective cover. Let \( L = \text{Ker}(p|R^{(n)}e) \). Then \( M \cong R^{(n)}e/L \) with \( L \ll R^{(n)}e \). Thus \( L \subseteq \text{Rad}(R^{(n)}e) = J^{(n)}e \). \( \square \)

In the following result we give a characterization of the rings over which direct summands of finitely generated modules, with projective covers, have projective covers.

**Theorem 2.9.** For a ring \( R \) with \( J = J(R) \) the following conditions are equivalent:

(a) Idempotents lift modulo \( M_n(J) \) in \( M_n(R) \) for every natural number \( n \);

(b) Every direct summand of any finitely generated left \( R \)-module, with projective cover, has a projective cover.

**Proof.** (b) \( \Rightarrow \) (a). This follows from Corollary 1.9.

(a) \( \Rightarrow \) (b). Let \( RM \) be a finitely generated module with projective cover. By Lemma 2.8 \( M \cong R^{(n)}e/L \), where \( e \) is an idempotent in \( E' = \text{End}_R(R^{(n)}) \cong M_n(R) \) and \( L \) is a superfluous submodule of left \( R \)-module \( R^{(n)}e \). Let \( E = \text{End}_R(R^{(n)}e) \cong eM_n(R)e \) and \( S = \{ \alpha \in E : L\alpha \subseteq L \} \) and \( T = \{ \alpha \in E : R^{(n)}\alpha \subseteq L \} \). In view of Theorem 1.7, it is enough to show that idempotents lift modulo \( T \) in \( S \).

Let \( S' = \{ \alpha \in E' : L\alpha \subseteq L \} \) and \( T' = \{ \alpha \in E' : R^{(n)}\alpha \subseteq L \} \). Then \( T' \) is a left ideal of \( E' \) and a two sided ideal of \( S' \). Also as \( Le = L, e \in S' \). As
L \ll R^{(n)}e, by Lemma 1.1, \ L \ll R^{(n)} and thus, by Lemma 1.2, \ T' \subseteq J(E').
So, by Lemma 1.11 and Lemma 1.12, \ T' \subseteq J(S').

Claim: \ S = eS'e and \ T = eT'e.
As \ e \in S', \ eS'e \subseteq S'. So if \ \beta \in S' \ then \ Le\beta e \subseteq L \ showing \ that \ e\beta e \in S. Conversely, if \ eae \in S, \ where \ \alpha \in M_n(R), \ then \ eae \in S' \ also. Thus \ eae \in eS'e \ showing \ that \ S = eS'e.

Also as \ e \in S', \ eT'e \subseteq T'. Let \ \gamma \in T'. Then \ R^{(n)}e(e\gamma e) = R^{(n)}e\gamma e \subseteq L \ and \ so \ e\gamma e \in T. Conversely, if \ eae \in T, \ where \ \alpha \in M_n(R), \ then \ R^{(n)}eae = R^{(n)}e(eae) \subseteq L. Thus \ eae \in T'. This establishes the claim.

Let \ I(T') \ denote \ the \ idealizer \ of \ T' \ in \ E'. Obviously, \ S' \subseteq I(T'). Conversely, suppose that \ \alpha \in I(T'). Then by definition \ T'\alpha \subseteq T' \ and \ so \ R^{(n)}T'\alpha \subseteq L. Now as \ R^{(n)} \ is \ a \ generator \ in \ the \ category \ of \ left \ R-modules, \ R^{(n)}T' = L. So \ R^{(n)}T'\alpha = L\alpha \subseteq L \ implying \ that \ \alpha \in S'. Thus \ S' = I(T'). As \ idempotents \ lift \ modulo \ J(E') \ in \ E' \ (by \ (a)) \ and \ T' \subseteq J(E') \ is \ a \ left \ ideal \ of \ E', \ by \ Theorem 2.6, \ idempotents \ lift \ modulo \ T' \ in \ S' = I(T'). Now as \ e \in S' \ and \ T' \subseteq J(S'), \ by \ Lemma 1.5, \ idempotents \ lift \ modulo \ T = eT'e \ in \ S = eS'e. □

In view of Theorem 2.6 and Theorem 2.9, one \ asks \ that \ does \ there \ exist \ a \ characterization \ of \ rings \ over \ which \ if \ a \ module \ has \ a \ projective \ cover \ then \ so \ does \ its \ every \ direct \ summand. In next section we give a sufficient condition for this to happen (Corollary 3.3).
3. SOME SUFFICIENT CONDITIONS

In this section we give some sufficient conditions so that direct summands of modules, with projective covers, have projective covers.

**Proposition 3.1.** Let \( p : P \to M \) be a projective cover such that idempotents lift modulo the Jacobson radical of \( \text{End}_R(P) \) and \( P \) generates \( L = \text{Ker} \, p \). Then every direct summand of \( M \) has a projective cover.

**Proof.** Let \( E = \text{End}(P) \), \( S = \{ \alpha \in E : L\alpha \subseteq L \} \) and \( T = \{ \alpha \in E : P\alpha \subseteq L \} \).

In view of Theorem 1.7 we have to show that idempotents lift modulo \( T \) in \( S \).

As \( T \) is two sided ideal in \( S \) so \( S \subseteq \mathcal{I}(T) \), the idealizer of \( T \) in \( E \).

Conversely suppose that \( \alpha \in \mathcal{I}(T) \). Then \( T\alpha \subseteq T \) so \( PT\alpha \subseteq L \). But as \( P \) generates \( L \) so \( PT = L \) and thus \( L\alpha \subseteq L \). Hence \( \mathcal{I}(T) \subseteq S \) and thus \( \mathcal{I}(T) = S \). Now as idempotents lift modulo \( J(E) \) in \( E \) so, by Theorem 2.6, idempotents lift modulo \( T \) in \( S \). \( \square \)

**Proposition 3.2.** Let \( M \) be a left \( R \)-module with projective cover and \( F \) be a free left \( R \)-module such that \( F \) maps onto \( M \) and idempotents lift modulo the Jacobson radical of \( \text{End}_R(F) \). Then every direct summand of \( M \) has a projective cover.

**Proof.** By Lemma 1.4 \( M \cong Fe/L \), where \( e \in E' = \text{End}_R(F) \) is an idempotent and \( L \ll_{RFe} \). By hypothesis idempotents lift modulo \( J(E') \) in \( E' \).

Let \( S' = \{ \alpha \in E' : L\alpha \subseteq L \} \) and \( T' = \{ \alpha \in E' : Fe\alpha \subseteq L \} \). Now as in the proof of Theorem 2.9, it can be seen that idempotents lift modulo \( eT'e \) in \( eS'e \).

Let \( E = \text{End}_R(Fe) \), \( S = \{ \alpha \in E : L\alpha \subseteq L \} \) and \( T = \{ \alpha \in E : Fe\alpha \subseteq L \} \).
It can easily be checked that $S = eS'e$ and $T = eT'e$. So idempotents lift modulo $T$ in $S$. Thus by Theorem 1.7 every direct summand of $M$ has a projective cover.

**Corollary 3.3.** Let $R$ be a ring such that idempotents lift modulo the Jacobson radical of endomorphism ring of every free left $R$-module. Then every direct summand of any left $R$-module, with projective cover, has a projective cover.

Recall that a ring is called *left perfect* if every left $R$-module has a projective cover. Since over semiprimitive rings only projective modules have projective covers, direct summands of left modules, with projective covers, over semiprimitive or left perfect rings, have projective covers. In both these cases idempotents lift modulo the Jacobson radical of endomorphism ring of every free left $R$-module (see [N]). Also if direct summands of left $R$-modules, with projective covers, have projective covers then idempotents lift modulo the Jacobson radical of endomorphism ring of every finitely generated free left $R$-module (see Theorem 2.9). This makes one ask the following question.

**Question 3.4.** Let $R$ be a ring such that every direct summand of any left $R$-module, with projective cover, has a projective cover. Then do idempotents lift modulo the Jacobson radical of endomorphism ring of every free left $R$-module?
4. PROJECTIVE COVERS AND SUPPLEMENT SUBMODULES

Let $V$ be a submodule of a left $R$-module $M$. A submodule $U$ of $M$ is said to be a supplement of $V$ in $M$ if $U$ is a minimal element in the set of submodules $L$ of $M$ with $V + L = M$. In this case we say that $U$ is a supplement submodule of $M$. A module whose every submodule has a supplement is said to be supplemented. Mares [M] defined a projective module $P$ to be semiperfect if every factor module of $P$ has a projective cover. It is well known that a projective module $P$ is semiperfect if and only if it is supplemented (see e.g. [MM, Corollary 4.43]).

Let $U$ be a supplement submodule of a projective module $P$. We study that when does $P/U$ have a projective cover.

Lemma 4.1 ([K, Lemma 5.2.4]). Let $U$ and $V$ be submodules of a module $M$. Then

(a) $V$ is a supplement of $U$ if and only if

$$U + V = M \text{ and } U \cap V < V.$$

(b) If $U$ is a supplement submodule and $V$ is a supplement of $U$, then $U$ is a supplement of $V$. \hfill $\Box$

Proposition 4.2. Let $U$ be a supplement submodule of a projective module $P$. Then the following are equivalent:

(a) $P/U$ is projective;

(b) $U$ is a direct summand of $P$.

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Proof. The implications (a) \( \Rightarrow \) (b) is obvious because the natural map \( P \rightarrow P/U \) splits.

(b) \( \Rightarrow \) (a). Let \( P/U \) have a projective cover and \( \nu : P \rightarrow P/U \) be the natural projection. By Lemma 1.4 there exists a decomposition \( P = P' \oplus P'' \) such that

\[
P'' \subseteq \ker \nu = U
\]

and \( (\nu|P') : P' \rightarrow P/U \) is a projective cover. So \( \nu P' = (P' + U)/U = P/U \).

Thus we have

\[
P' + U = P \quad \text{and} \quad \ker (\nu|P') = P' \cap U \ll P'.
\]

This, in view of Lemma 4.1(a), implies that \( P' \) is a supplement of \( U \). Since \( U \) is a supplement submodule, by Lemma 4.1(b), \( U \) is a supplement of \( P' \). Now as \( P' + P'' = P \) and \( P'' \subseteq U \) (see (1)), \( U = P'' \). Thus \( U \) is a direct summand of \( P \). \( \square \)