CHAPTER III
JETS IN POROUS MEDIA

The study of free and wall jets has been an interesting subject for scientists and engineers over several decades because of their technical applications. The mathematical analysis of the flow in a laminar free jet of an incompressible fluid was first investigated on the basis of boundary layer theory by Schlichting (1933) and Bickley (1937). The corresponding turbulent jet was first studied by Tollmien (1926) who used Prandtl's mixing length hypothesis and later simple solution based on Prandtl second hypothesis, was obtained by Reichardt (1942) and Gortler (1942). In recent years Smith and Cambel (1965), Pozzi and Bianchini (1972) and Bansal (1978) have extended the analysis to include magneto-hydrodynamic effects on the flow of the jet. Then Peskin (1963) extended the Schlichting solution to the case of electrically conducting fluid by perturbing the stream function of the Schlichting model.

The analysis of the two dimensional wall jet was first given by Andrade (1937) and it was later extended by Glauert
(1956) to study the radial and plane wall jets involving flows in two and three dimensions.

The brief resume on the studies of the jets given above and the survey on jets in Chapter I points out that much attention has not been given to study the structure of jets in porous media.

Therefore, the aim of this chapter is to investigate the structure of free and turbulent jets issuing from a narrow long slit and mixing with the fluid in a saturated porous medium.

Section 3.1 deals with the study of free jets in a saturated porous medium and their general characteristics. The solutions are obtained by three different approaches. The analysis is then extended to cover turbulent jets using Prandtl second hypothesis.

In Section 3.2, the structure of jets in rotating systems using the generalised Ergun equation is presented. The problem involves three dissipation mechanisms which control the momentum transfer and the spreading of jets. The effect of each of these dissipation mechanisms on the jet is discussed in detail.
3.1 STRUCTURE OF LAMINAR AND TURBULENT FREE JETS IN A POROUS MEDIA.

The solution of jet equations are attempted by three approaches. First we discuss the approach of Schlichting and its limitations. In the second and third approaches we have used Von-Mises transformations of the type \( \Psi = H(x, \psi) \) and \( \psi = \eta \). We have discussed in detail, the characteristic of the general solution, the transition from a purely viscous jet to a purely Darcy jet, the momentum of transport and the demise of the jet.

3.1(a) Mathematical Formulation:

The physical model consists of a two dimensional jet emerging from a long narrow slit and mixing with the fluid in a saturated porous medium. The governing equations of motion for a two dimensional steady, incompressible, laminar jet in a porous media from equations (2.1.4) and (2.1.5) are,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1.1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \gamma \frac{\partial^2 u}{\partial y^2} - \frac{\gamma}{k} u \tag{3.1.2}
\]
where,

\[ K \] - permeability of the medium

\[ \nu \] - coefficient of kinematic viscosity

\[ u \] - velocity component in the X-direction

\[ v \] - velocity component in the Y-direction.

In equation (3.1.2) the first term on the R.H.S. represents the viscous resistance and the second term represents the Darcy resistance.

The boundary conditions are:

\[ \text{At } y = 0, \quad \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad v = 0 \quad \text{.. (3.1.3)} \]

\[ \text{At } y = \pm \infty, \quad u = 0 \quad \text{.. (3.1.4)} \]

The equations are non-dimensionalised using the width of the slit \( Y_0 \) as the scale in the Y-direction and \( X_0 \) as the scale in the X-direction. As there is no imposed downstream scale \( X_0 \) will be determined as the scale of dominant dissipation and it is discussed below. The velocity field is scaled using the velocity at the slit, \( U \). Using the following dimensionless variables,

\[ u^* = \frac{u}{U}; \quad v^* = \frac{vX_0}{U\nu Y_\infty}; \quad x^* = \frac{x}{X_\infty}; \quad y^* = \frac{y}{Y_\infty}; \]
the governing equations have the form,

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{3.1.5}
\]

\[
u^* \frac{\partial u^*}{\partial x^*} + \nu^* \frac{\partial u^*}{\partial y^*} + R_1 \frac{\partial^2 u^*}{\partial y^2} - R_2 u^* \tag{3.1.6}
\]

where,

\[
R_1 = \frac{1}{R_s \delta}; \quad \text{viscous parameter} \tag{3.1.6a}
\]

\[
R_2 = \frac{\sigma^2}{R_s \delta} = \sigma^2 R_1; \quad \text{Darcy parameter} \tag{3.1.6b}
\]

\[
R_s = \frac{U Y_0}{U}; \quad \text{Reynolds number based on the width of the slit}
\]

\[
\delta = \frac{\sigma^2}{K}; \quad \text{porous parameter} \tag{3.1.6c}
\]

The transformed boundary conditions are:

At \( y^* = 0 \), \( \frac{\partial u^*}{\partial y^*} = 0 \) and \( v^* = 0 \)

At \( y^* = \pm \infty \), \( u^* = 0 \) \tag{3.1.7}
For convenience we remove asterisks (*) in the above equations.

The suitable stream function \( \psi \) which satisfies the continuity equation (3.1.5) is,

\[
\begin{align*}
u &= - \frac{\delta \psi}{\delta y} \quad \ldots (3.1.8) \\
v &= \frac{\delta \psi}{\delta x} \quad \ldots (3.1.9)
\end{align*}
\]

The study of the equation (3.1.6) shows that there are two mechanisms of dissipation represented by two terms of the R.H.S. Each of these mechanisms gives a characteristic downstream scale, the scale corresponding to the more efficient process being shorter of the two. The downstream scales are determined as follows:

Case (i): Viscous resistance is very small compared to Darcy resistance \((R_1 \ll R_2)\).

In this case, to make the inertial terms of the same order as the Darcy resistance term \(R_2\) should be unity and hence from the equation (3.1.6b),

\[
\delta = \frac{\sigma^2}{R_s}; \quad X_0 = \frac{Y \sigma R}{\sigma^2} \quad \ldots (3.1.10)
\]

The downstream scale \(X_0\) emerges out as the ratio of product of
width of the slit and the Reynolds number to the square of the porous parameter.

Case (ii): If the viscous resistance is dominant \((R_1 \gg R_2)\), then to make the inertial terms of the same order as the viscous resistance term \(R_1\) should be unity. This leads to, from equations (3.1.6a) and (3.1.6c),

\[
\delta = \frac{1}{R_s} ; \quad X_0 = Y_0 R_s
\]

.. (3.1.11)

In this case the downstream scale is the product of slit width and Reynolds number \(R_s\).

Case (iii): If both the Darcy and the viscous resistances are dominant, then

\[
\frac{R_1}{R_2} = 1 = \frac{1}{\sigma^2}
\]

and hence,

\[
Y_o^2 = k \quad \text{or} \quad \bar{Y}^2 = k
\]

.. (3.1.12)

\(\bar{Y}\) in equation (3.1.12) gives the internal scale in the cross stream direction. The ratio of the local width of the jet and the scale \(\bar{Y}\) determines whether the viscous resistance or the Darcy resistance is dominant. If the width of the slit
is smaller than \( \tilde{V} \), then the dynamics near the slit is that of viscous or Schlichting type jet. As the distance from the slit increases, the local width of the jet increases, the Darcy resistance becomes more important and finally dominates viscous dissipation at large distances from the slit. From the above, we find that the Darcy resistance is dominant always away from the slit and viscous resistance is important only near the slit when the diameter of the slit is smaller than \( \tilde{V} \).

3.1(b): General Characteristics of Jets in Porous Media:

Before obtaining solutions of the equations (3.1.6) and (3.1.8) for various values of ratio \( R_2/R_1 \), it is important to consider some general properties of jet like solutions of (3.1.6).

At the axis of the jet \( (y = 0) \), the cross-stream velocity \( v \) vanishes. If \( R_1 \neq 0 \), then \( u(x,y) \) has a proper maximum at \( y = 0 \).

\[
\begin{align*}
u_y(y = 0) &= 0 ; \quad u_{yy}(y = 0) < 0 \\
& \quad \text{(3.1.13)}
\end{align*}
\]

Hence, for all \( R_2 \) and \( R_1 \), from equation (3.1.6) at \( y = 0 \), we have,

\[
\begin{align*}
u_x(0) &< 0 \\
& \quad \text{(3.1.14)}
\end{align*}
\]
Since $y = 0$ is a streamline, we can take $\Psi(y = 0) = 0$.

The total transport in the $X$-direction is,

$$T = \int u \, dy = -\int \Psi_y \, dy = \Psi(x, y = -\infty) - \Psi(x, y = +\infty)$$

...(3.1.15)

Here $\Psi(y = \pm \infty)$ are the values of the stream function at the two edges of the jet. If the total transport $T$ is in the positive $X$-direction, we must have

$$\Psi(y = -\infty) > 0; \quad \Psi(y = +\infty) < 0$$

...(3.1.16)

Note that the above equation does not rule out counter currents. To determine the nature of the solution near the two edges of the jet one can linearise about the stream function at the edges,

$$\Psi = \Psi(x) + \Psi_1(x, y); \quad \Psi_1 \ll \Psi$$

...(3.1.17)

Substituting in (3.1.6), (3.1.8) and (3.1.9) yields, upon neglecting the non-linear terms,

$$V \Psi_{yyy} = R_1 \Psi_{yyy} - R_2 \Psi_y$$

...(3.1.18)

Hence, $u_1 \propto e^{ky}$. Where $k$ at the two edges is given as

$$k = \frac{V(\pm \infty) + [V^2(\pm \infty) + 4R_2R_1]^{1/2}}{2R_1};$$
When viscous friction dominates Darcy friction, there is only one root for $k$,

$$R_2 \ll R_1 \sim 1 ; \quad k = V(\pm \infty) / R_1.$$  

Similarly, when Darcy friction is dominant,

$$R_1 \ll R_2 \sim 1 ; \quad k = - \frac{R_2}{V(\pm \infty)}.$$  

Thus the velocity $u$ will decay exponentially at the two edges $y = \pm \infty$ only, if

$$R_2 \ll R_1 \sim 1 ; \quad V(\pm \infty) = \mp$$  

$$R_1 \ll R_2 \sim 1 ; \quad V(\pm \infty) = \mp \quad (3.1.19)$$

This implies that the Darcy jet ejects fluid at both sides of the jet while the viscous jet entrains fluid at the edges. The total transport $T$ decreases in the former and increases in the latter case, as can be seen from

$$\frac{\delta T}{\delta x} = \frac{\delta}{\delta x} \int_{-\infty}^{\infty} - \Psi_y \, dy = V(-\infty) - V(+\infty) \quad (3.1.20)$$

If both viscous friction and Darcy friction are equally
important, exponentially decaying solutions exist for both ejecting and entraining jets.

The momentum across the jet at any given \( x \) is,

\[
J = \int_{-\infty}^{\infty} u^2 \, dy
\]  .. (3.1.21)

If a solution of (3.1.6) and (3.1.8) satisfies the boundary conditions for large \( y \) (3.1.4), it can be shown by integration of equation (3.1.6) that \( J \) satisfies

\[
\frac{\partial J}{\partial x} = -R_2 \int_{-\infty}^{\infty} u \, dy = -R_2 T
\]  .. (3.1.22)

Thus, if the Darcy resistance is low so that \( R_2 \ll R_1 \), the downstream momentum flux across the jet remains the same for all \( x \), whereas for the cases \( R_2 \approx R_1 \), \( R_2 \gg R_1 \) the momentum is dissipated and decreases with downstream distance.

3.1(c) Method of Solution:

(i) First Approach (Schlichting's type solutions):

In the present problem the downstream scale are not imposed externally which suggests the existence of the similarity solutions namely solutions in which the scale of
velocity at the axis and the width of the jet depend only on the distance from the slit. Similarity solutions reduces the two dimensional problem to one dimension and yields a solution which is independent of the details of the velocity profile at the inlet.

Assuming the stream function of the form

\[ \Psi = f(x) F(\eta) \]
\[ \eta = \gamma g(x) \] 

The equations (3.1.6), (3.1.8) and (3.1.9) for the case \( R_2 = 0 \), reduce to the form

\[ (f'g + fg')F' = -R_1gF'' \quad \ldots (3.1.24) \]

The boundary conditions (3.1.7) become

\[ F(0) = F''(0) ; \quad F'(\pm \infty) = 0 \quad \ldots (3.1.25) \]

The functions \( f(x) \) and \( g(x) \) are chosen such that equations (3.1.24) has coefficient which depend only on \( \eta \). Hence \( f(x) \) and \( g(x) \) are assumed to be of the form,

\[ f(x) = (a + bx)^l ; \quad g(x) = (a + bx)^m \quad \ldots (3.1.26) \]

The relations using equation (3.1.24) give,

\[ (l - m) = 1 \quad \ldots (3.1.27) \]
Equation (3.1.24) thus provides only one relation between the exponents \( L \) and \( m \).

A solution of (3.1.24) which satisfies the boundary condition (3.1.25) for large \( \eta \) must satisfy the condition on momentum across the jet given by

\[
\frac{\partial}{\partial x} \int_{-\infty}^{x} u^2 \, dy = 0 \quad \cdots (3.1.28)
\]

Introducing (3.1.23) and (3.1.26) in (3.1.28) gives another relation between \( L \) and \( m \) in the form,

\[ 2L + m = 0 \quad \cdots (3.1.29) \]

Solving (3.1.27) and (3.1.29) we find

\[ L = 1/3 \quad \text{and} \quad m = -2/3 \quad \cdots (3.1.30) \]

The sign of the constant \( b \) is determined from the condition

\[ u_x(y = 0) = \frac{b(L + m)}{a + bx} u(y = 0) < 0 \] discussed earlier. Thus,

\[ b > 0 \quad \cdots (3.1.31) \]

Substitution of equations (3.1.23), (3.1.26) and (3.1.30) in (3.1.24) gives

\[ F'^2 + FF'' = \frac{3R_1}{b} F'''' \quad \cdots (3.1.32) \]
Integrating the equation (3.1.32) and choosing the constants such that the width and the velocity component along the axis of the slit are unity, we get

$$\Psi = -(1 + 6R_1x)^{1/3} \tanh [y(1 + 6R_1x)^{-2/3}] \quad \text{(3.1.33)}$$

$$u = (1 + 6R_1x)^{-1/3} \text{sech}^2[y(1 + 6R_1x)^{-2/3}] \quad \text{(3.1.34)}$$

$$V(\pm \infty) = \mp 2R_1(1 + 6R_1x)^{-2/3} \quad \text{(3.1.35)}$$

The total transport in the X-direction is given by

$$T = \int_{-\infty}^{\infty} u \, dy = 2(1 + 6R_1x)^{1/3} \quad \text{(3.1.36)}$$

The velocity at the axis of the jet decreases with the downstream distance $x$ and the transport increases with $x$ due to entrainment of fluid at the edges of the jet.

Darcy jet:

We investigate similarity solutions of equations (3.1.6), (3.1.8) and (3.1.9) for the case $R_1 = 0$, i.e., when Darcy jet is dominant. Substitution of the similar form equation (3.1.23) in the equation yields,

$$(f'g + fg')F''^2 - gf'FF''' = R_2F' \quad \text{(3.1.37)}$$

with

$$F(0) = 0; \quad F'(\pm \infty) = 0 \quad \text{(3.1.38)}$$
As in the viscous case, if \( f(x) \) and \( g(x) \) are chosen so that the equation for \( F \) has coefficients that depend only on \( \eta \) they have to be of the form (3.1.26) with one relation between the exponents \( l \) and \( m \) given as,

\[
l + m = 1 \tag{3.1.39}
\]

The equation (3.1.22) cannot, however be used in this case to determine \( l \) and \( m \), because it is satisfied for all \( l \) and \( m \) satisfying equation (3.1.39). In order to restrict the allowable values of \( l \), i.e., the values of \( l \) which will give acceptable solutions, we have to consider the other general properties of Darcy jet discussed in general properties of jets in porous media (Section 3.1(b)).

In this case the stream function,

\[
\psi(x,y) = (a + bx)^l \frac{d}{y(a + bx)^{1-l}} \tag{3.1.40}
\]

Hence,

\[
u = -(a + bx) F'(\eta) \tag{3.1.41}
\]

\[
v(\eta = \pm \infty) = b \frac{1}{l} (a + bx)^{l-1} F(\eta = \pm \infty) \tag{3.1.42}
\]

\[
u_x(\eta = 0) = \frac{b}{(a + bx) u(\eta = 0)} \tag{3.1.43}
\]

at the axis,

Using the conditions (3.1.14) \( \wedge \) (3.1.16) and (3.1.19) at
large \eta we get

\[ b < 0 \; ; \; J > 0 \]

The equation for \( F \) is

\[ F'^2 - JF F'' = \left( R_2/b \right) F' \] \hspace{1cm} \ldots (3.1.44)

Since \( F'(O) = R_2/b \)

\[ F''(0) = 0 \] \hspace{1cm} \ldots (3.1.45)

we have,

\[ u (\eta = 0) = \left[ - \frac{aR_2}{b} - R_2 \right] \] \hspace{1cm} \ldots (3.1.46)

We may set \( b = -R_2 \) without loss of generality. The equation (3.1.44) can be integrated once to yield, in terms of arbitrary constant \( k \),

\[ F' = -1 + kF^{1/J} \] \hspace{1cm} \ldots (3.1.47)

i.e., \( F'' = \frac{k}{J} F^{1/(J-1)} F' \) \hspace{1cm} \ldots (3.1.48)

In general,

\[ \delta^{k+1} F = \frac{k}{J} \left( \frac{1}{J} - 1 \right) \ldots \left( \frac{1}{J} - k \right) F^{1/(J-k-1)} F_{,k+1} \] \hspace{1cm} \ldots (3.1.49)

If we expect that \( F'' \) and all higher derivatives of \( F \) remain
finite at the axis where $F$ vanishes, $L$ has to be reciprocal of an integer. Also, it is evident from the equation (3.1.47) that $F''$ and higher derivatives vanish with $F'$ which must occur at the two edges of the jet. The values of $F$ at these points at which $F', F''$ etc. vanish are proportional to the asymptotic stream function at the two edges and are given as,

$$F^{1/k} = \frac{1}{k}$$ \hspace{1cm} (3.1.50)

For the downstream velocity to vanish at both edges of the jet, the equation (3.1.50) must have at least two roots, the difference between the roots being the total transport in the $x$-direction. This implies that $1/k$ has to be an even number.

If the constant $k$ is chosen so that the total transport on each side of the jet is unity at the slit, the acceptable solutions of the Darcy jet are,

$$F' = -1 + F^{2n}$$ \hspace{1cm} (3.1.51)

The above approach, however, does not yield an acceptable solution for the general case when $R_1$ and $R_2$ are both present. Hence we use the Von-Mises transformation of the type

$$\Psi_v = H(x, \Psi)$$ in an attempt to obtain the general solution.
(ii) Second approach:

To obtain the solutions of equations (3.1.5) and (3.1.6) satisfying the boundary conditions (3.1.7), we use Von-Mises transformation in which the stream function $\Psi$ as one of the independent variables. The velocity components $u$ and $v$ are given in terms of stream function $\Psi$ in the form

$$u = -\frac{\partial \Psi}{\partial y} \quad \quad \quad \quad (3.1.52)$$

$$v = \frac{\partial \Psi}{\partial x}$$

Introducing the variable $\Psi_y = H(\Psi, x)$ in equations (3.1.5) and (3.1.6) gives,

$$H_x = R_2 - R_1(H\Psi \Psi H + H^2) \quad \quad \quad \quad (3.1.53)$$

The solution of the viscous jet obtained by first approach is of the form

$$H = -(1 + 6\alpha x)^{-1/3} + \Psi^2/(1 + 6\alpha x) \quad \quad \quad \quad (3.1.54)$$

The above solution suggests that we should look for the solution of (3.1.53) in the form

$$H = -\alpha(x) + \beta(x) \Psi^j \quad \quad \quad \quad (3.1.55)$$

From equations (3.1.53) and (3.1.55) we note that the nontrivial
solution for $\alpha(x)$ and $\beta(x)$ exists only for $J = 2$. For this value of $J$ equation (3.1.55) can be integrated to obtain $\psi$ and $u$ in the form,

$$\psi = \left[-\frac{\alpha(x)}{\beta(x)}\right]^{1/2} \tanh \left[y\{\alpha(x) \beta(x)^{1/2}\} \right] \quad \ldots (3.1.56)$$

$$u = \alpha(x) \text{sech}^2 \left[y\{\alpha(x) \beta(x)^{1/2}\} \right] \quad \ldots (3.1.57)$$

Combining the equations (3.1.55) and (3.1.53) we find that $\alpha(x)$ and $\beta(x)$ satisfy the relation:

$$-\alpha'(x) + \beta'(x)\psi^2 = R_2 - R_1[6\psi^2 \beta^2 - 2 \alpha\psi] \quad \ldots (3.1.58)$$

Equation (3.1.58) yields two first order equations for $\psi$ and $\beta$ in the form

$$\alpha' = R_2 - 2R_1\alpha\beta \quad \ldots (3.1.59)$$

$$\beta' = -6R_1\beta^2 \quad \ldots (3.1.60)$$

The above equations are solved for three different cases, namely $R_2 = 0$ and $R_1 \neq 0$, $R_1 \to 0$ and $R_2 \neq 0$ and $R_1 \neq R_2 \neq 0$, using the conditions $u(0,0)$ and the width of the jet at the slit are unity. The solutions for the above three cases are discussed below.
Case (i): Viscous Jet Solutions

If \( \sigma^2 = 0 \), i.e., \( R_2 = 0 \), the solutions of equations \((3.1.59)\) and \((3.1.60)\) together with equations \((3.1.56)\) and \((3.1.57)\) give viscous jet solutions. The expressions for \( \Psi \), \( u \), \( v(\pm \infty) \) and \( T \) are,

\[
\Psi = -(1 + 6R_1 x)^{1/3} \tanh [(1 + 6R_1 x)^{-2/3}] \quad \ldots (3.1.61)
\]

\[
u = (1 + 6R_1 x)^{-1/3} \text{sech}^2 [(1 + 6R_1 x)^{-2/3}] \quad \ldots (3.1.62)
\]

\[
v(\pm \infty) = \mp 2R_1/(1 + 6R_1 x)^{2/3} \quad \ldots (3.1.63)
\]

\[
T = \int_{-\infty}^{\infty} u \, dy = 2(1 + 6R_1 x)^{1/3} \quad \ldots (3.1.63a)
\]

Equation \((3.1.63)\) shows that in the limit \( R_2 = 0 \), the jet entrains the fluid at its edges over the entire length of the jet and the transport increases with downstream scale. The above expressions are same as obtained in the first approach.

Case (ii): Darcy Jet Solutions

If \( R_1 \to 0 \) and \( R_2 \) finite, the expressions for \( \Psi \), \( u \), \( v(\pm \infty) \) and \( T \) are,

\[
\Psi = -(1 - R_2 x)^{1/2} \tanh [(1 - R_2 x)^{1/2}] \quad \ldots (3.1.64)
\]

\[
u = (1 - R_2 x) \text{sech}^2 [(1 - R_2 x)^{1/2}] \quad \ldots (3.1.65)
\]
\[ v(\pm \infty) = \pm \frac{R_2}{2(1 - R_2x)^{1/2}} \quad \ldots (3.1.66) \]

\[ T = 2(1 - R_2x)^{1/2} \quad \ldots (3.1.66a) \]

For the Darcy jet we observe that the velocity at the axis of the jet decreases with \( x \) and becomes zero at \( x = 1/R_2 \).

The momentum flux in the \( x \)-direction is given by

\[ J = \frac{4}{3} (1 - R_2x)^{2/3} \quad \ldots (3.1.67) \]

From the above relation we find that the jet cannot penetrate beyond \( x = 1/R_2 \) and further the transport decreases with the downstream scale and becomes zero at \( x = 1/R_2 \). The same conclusion is also reached from the expression for \( u \). The form of solutions for the Darcy jet obtained by this method is entirely different from the first approach. However, the solution satisfies the condition that \( b < 0 \) and \( l > 0 \) as discussed in the first approach and provides more information regarding the structure of the Darcy jets.

Case (iii): When both viscous and Darcy resistances are dominant \((R_1 \neq R_2 \neq 0)\)

The solutions for \( \Psi \), \( u \) and \( v(\pm \infty) \) are,

\[ \Psi = - \left[ (1 + \frac{R_2}{8R_1})(1 + 6R_1x)^{2/3} - (R_2/8R_1)(1 + 6R_1x)^2 \right]^{1/2} \]

\[ \tanh \left[ y \left( \frac{1 + (R_2/8R_1)}{(1 + 6R_1x)^{1/3}} - R_2/8R_1 \right) \right]^{1/2} \]
We observe that the solution for the general case reduces to that of viscous jet in the limit of vanishing $R_2$. The general solution in the limit of small $R_1$ yields the similarity solution for the case $n = 1$ in Eq. (3.1.51) discussed in the first approach for the Darcy jet and in fact the profile which has a minimum momentum transport. We find from Eqn. (3.1.70) that $v(\pm \infty)$ vanishes at the $x$ given by

\[ (1 + 6R_1x)^{4/3} = \frac{1}{3}(1 + 8R_1/R_2) \]  \hspace{1cm} (3.1.71)

i.e., the jet entrains the fluid at its edges from the slit up to the point $x$ given by Eqn. (3.1.71). Beyond this point,
the jet ejects the fluid implying downstream decrease in transport. The transition between two regimes is smooth and all the features in the region of transition are intermediate between viscous side frictional and Darcy jets. Axial velocity of the jet decreases with \( x \) and vanishes at the point given by,

\[
(1 + 6R_1x)^{4/3} = (1 + 8R_1/R_2) \quad \text{.. (3.1.72)}
\]

At this point the cross stream velocity equals \( R_2y \) for finite \( y \) and becomes indeterminate as \( y \to \infty \). These features are characteristic of the Darcy jet.

(iii) Third approach:

The solutions for the governing equations (3.1.1) and (3.1.2) with boundary conditions (3.1.3) and (3.1.4) are obtained by using the variables:

\[
\xi = \psi x \quad \text{.. (3.1.73)}
\]

\[
\eta = \psi
\]

and defining the stream function \( \psi \) such that

\[
u = -\frac{\partial \psi}{\partial x} \quad \text{.. (3.1.73a)}
\]

\[
\nu = -\frac{\partial \psi}{\partial x}
\]
The momentum equation (3.1.2) and boundary condition (3.1.3) in terms of $\psi$ are transformed to,

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \eta} \left( u \frac{\partial u}{\partial \eta} \right) - \frac{1}{K} \ldots (3.1.74)$$

$$\eta = 0; \quad \frac{\partial u}{\partial \eta} = 0 \ldots (3.1.75)$$

To obtain the solution for the above equation we assume similarity solutions for $u$ (after Bansal (1977)) in the form,

$$u = \xi^{-1/3} f(\xi) + (A/K) \ldots (3.1.76)$$

where

$$\xi = \eta \xi^{-1/3} \ldots (3.1.77)$$

The function $f(\xi)$ and the constant $A$ satisfy the equations,

$$3 \xi \xi f' + \xi f' + f = 0 \ldots (3.1.78)$$

and

$$A = A \xi'' - 1 \ldots (3.1.79)$$

Solving for $f$ and $A$ and using the relation (3.1.76) we find that,

$$u = \frac{1}{6} \xi^{-1/3} \left( \alpha_1^2 - \xi^2 \right) - \frac{3}{4} \frac{\xi}{K} \ldots (3.1.80)$$
where $a_1$ is a constant still undetermined and it can be determined as follows. In the absence of a porous medium the above solution reduces to the form,

$$u_o = \frac{1}{8} \xi^{-1/3} (a_1^2 - J_o^2) \quad \ldots (3.1.81)$$

where

$$J_o = \eta_o \xi^{-1/3} \quad \ldots (3.1.82)$$

The constant $a_1$ is determined using the integral condition

$$J_o = \int_0^\infty u_o^2 \, dy \quad \ldots (3.1.83)$$

where $u_o$ is the velocity of the jet in the absence of the porous medium. After performing the integration we find that

$$\alpha = (9J_o/2\eta)^{1/3} \quad \ldots (3.1.84)$$

To get the stream function $\Psi$ and also the velocity components in the physical plane, we can make use of the equations (3.1.52), (3.1.73), (3.1.77) and (3.1.80) which yield,

$$\delta J = u \, dy \, \xi^{-1/3} \quad \ldots (3.1.85)$$

Incorporating the solution for $u$ in the above equation we have,

$$\delta J = \xi^{-1/3}[ (\xi^{-1/3}/6)(a_1^2 - J^2) - (3/4)(\xi/K)] \, dy \quad \ldots (3.1.86)$$
Now we can obtain the exact solution of the equation (3.1.86) by writing in the following form:

\[
\delta(J/\alpha_1) = [1 - (\Phi/\alpha_1)^2 - (9/2)(\frac{4}{3}/K\alpha_1^2)]d(\frac{1}{3}\alpha_1^2/6) \tag{3.1.87}
\]

\[
\delta(J/\alpha_1) = [1 - (s/2) - (J/\alpha_1)^2]d\lambda \tag{3.1.88}
\]

where \(\lambda = (\frac{1}{3}\alpha_1^2/6) \tag{3.1.88a}\)

and \(S = 9\frac{4}{3}/K\alpha_1^2 \tag{3.1.88b}\)

Now treating \(\frac{1}{3}\) as constant and integrating the equation (3.1.88) with boundary condition,

\[J = 0 \text{ at } \eta = 0\]

we get,

\[J = \alpha_1(1 - s/2)^{1/2} \tanh[\lambda(1 - s/2)^{1/2}] \tag{3.1.89}\]

Using the relation \(\psi = J\frac{1}{3} \) in equation (3.1.89), we have the expressions for the velocity \(u\) and \(v\) in the form,

\[u = \frac{1}{6}(1 - s/2) \sech^2 \left[\lambda(1 - s/2)^{1/2}\right] \tag{3.1.90}\]

Then

\[\left(\frac{u}{u_0}\right)_{\text{max}} = (1 - s/2) \sech^2 \left[\lambda(1 - s/2)^{1/2}\right] \tag{3.1.91}\]
\[ v = \frac{1}{3} \xi \frac{s^{-2/3} \alpha}{3} \sech^2(1 - s/2)^{1/2} \lambda - (1 - 3s/2) \]

\[ (1 - s/2)^{1/2} \tanh(1 - s/2)^{1/2} \lambda \]  \hspace{1cm} \text{.. (3.1.92)}

\[ \frac{v_0}{(\Omega_0)^{1/2}} = (1 - \frac{s}{3})(1 - \frac{s}{3})^{-1/2} \]  \hspace{1cm} \text{.. (3.1.92a)}

The solutions (3.1.91), (3.1.92) and (3.1.92a) obtained by the third approach formally resembles the Darcy jet solution of the second approach and hence it is not the complete solution of the problem.

Further, this method fails to give a solution in the absence of viscous resistance.

Solution in the absence of viscous resistance:

The equation (3.1.74) in the absence of viscous resistance term reduces to,

\[ \frac{du}{d\xi} = -\frac{1}{K} \]  \hspace{1cm} \text{.. (3.1.93)}

The similarity solution (3.1.76) leads to

\[ A = -1 \]  \hspace{1cm} \text{.. (3.1.94)}

and

\[ f' + (f/f) = 0 \]  \hspace{1cm} \text{.. (3.1.95)}

for \( f \) and \( A \).
The solution of (3.1.95) is of the form

\[ f = \frac{A}{j} \]  \hspace{1cm} (3.1.96)

This solution though exact does not satisfy the boundary condition \( f' = 0 \) at \( j = 0 \). Hence this approach does not yield solution when viscous resistance is absent.

From the above results we observe that the third approach yields the solution only for a particular case of \( R_1 \rightarrow 0 \) and \( R_2 \) remaining finite and hence the third approach provides less information about the structure of the jet and it is silent about the dominant downstream scales.

Results and Discussion

We have solved the problem of laminar free jet in a porous media by three methods. The methods provide the following information: (1) the downstream scales, (2) the point at which the transition from a purely viscous jet to Darcy jet occurs, (3) the dynamics of the jet and its demise.

The velocity profiles and streamlines are presented in Figs. 3.1.1 to 3.1.5, using the results of the second approach. From the Figs. 3.1.1 and 3.1.2 we observe that by increasing the value of \( R_2 \), increases the spreading of the jet. Further,
FIG. 3.1.1—VELOCITY DISTRIBUTION VS. $y$ WHEN BOTH D ARCY AND VISCOUS RESISTANCE TERMS ARE PRESENT.
FIG. 3.1.2 - VELOCITY DISTRIBUTION vs. y WHEN BOTH DARCY AND VISCOUS RESISTANCE TERMS ARE PRESENT
FIG. 3.1.3 - VELOCITY DISTRIBUTION vs. y FOR THE DARCY JET (i.e. $R_1=0$)
FIG. 3.1.4 - STREAMLINES FOR DARCY JET (i.e. $R_1 = 0$)
FIG 3.1.5 - STREAMLINES FOR SCHLICHTING JET (i.e., $R_2 = 0$)
when both Darcy and viscous resistances are present it is found that the magnitude of the viscous resistance \( R_1 \) should be very small compared to Darcy resistance \( R_2 \) to make the velocity positive. Figure 3.1.3 presents the velocity profiles for the Darcy jet whose behaviour is similar to Darcy-viscous jet. Figures 3.1.4 and 3.1.5 present streamlines for the Darcy and viscous jet respectively. It is found that the spreading of the Darcy jet is large compared to purely viscous jet.

3.1(d): Free Turbulent Jets:

Disregarding very small velocities of flow, it is found that the jet becomes completely turbulent at a short distance from the point of discharge. Owing to turbulence the emerging jet becomes partly mixed with the surrounding fluid at rest. Particles of fluid from the surroundings are carried away by the jet so that the mass flowing increases in downstream direction. Concurrently, the jet spreads out and its velocity decreases and finally the demise of the jet occurs.

Qualitatively such flows resemble similar flows but however, turbulent friction is much larger than laminar friction in the whole region of consideration. Consequently laminar
friction, may be wholly neglected in problems involving free turbulent flows. It is also noted that the problems in free turbulent flow are of boundary layer nature and it is convenient to study such problems with the aid of boundary layer equations.

In two dimensional case, the momentum boundary layer equation is,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau}{\partial y}
\]

where \( \tau \) denotes the turbulent shear stress. Using Prandtl's second hypothesis \( \tau \) can be replaced by

\[
\tau = \rho \theta \frac{\partial u}{\partial y}
\]

where \( \theta \) is the virtual kinematic viscosity, assumed constant over the whole width and is equal to \( x_1bu \).

\( x_1 \) is an empirical constant

\( b \) - width of the mixing zone

\( u \) - the central line velocity.

Denoting the central line velocity and the width of the jet, at a fixed characteristic distance \( S \) from the slit by \( u_s \) and \( b_s \) respectively. We may write,
after Schlichting (1966) and consequently \( \theta \) becomes equal to \( \theta_s(x/S)^{1/2} \).

Then the momentum equation for two dimensional incompressible turbulent free jet in the porous medium in terms of virtual kinematic viscosity is of the form,

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \theta \frac{\partial^2 u}{\partial y^2} - \frac{\theta}{K} \tag{3.1.99}
\]

The boundary conditions are,

at \( y = 0 \), \( \frac{\partial u}{\partial y} = 0 \) and \( v = 0 \) \( \tag{3.1.100} \)

at \( y = \pm \infty \), \( u = 0 \) \( \tag{3.1.101} \)

and the integral condition is,

\[
\int_{-\infty}^{\infty} u_o^2 \, dy = J_0 \tag{3.1.102}
\]

where \( u_o \) — velocity in the Schlichting model, in the absence of porous medium. The Prandtl's-Von-Mises variables \( \zeta_T \) and \( \eta_T \) are defined as,

\[
\zeta_T = \int_0^x \theta \, dx \tag{3.1.103}
\]
and it implies,

\[ \frac{1}{3} \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \eta} - \frac{1}{K} \quad \text{(3.1.104)} \]

where,

\[ K_o = (3\ell_s^2 / 25)^{1/2} \]

To have a qualitative idea of the structure of turbulent jets we use the third approach to obtain the solutions. We use,

\[ \eta_T = \Psi_T \quad \text{(3.1.105)} \]

the momentum equation (3.1.91) then reduces to,

\[ \frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \eta} \left( u \frac{\partial u}{\partial \eta} \right) - \frac{1}{K} \quad \text{(3.1.106)} \]

Following the similar analysis as in laminar free jet, we get,

\[ u = \frac{\xi^{-1/3}}{6} (\alpha^2 - \eta_T^2) - (3/4K) \xi \quad \text{(3.1.107)} \]

\[ J_T = \eta_T \xi^{1/3} \quad \text{(3.1.108)} \]

\[ \alpha = (9J/2\eta)^{1/3} \quad \text{(3.1.109)} \]

Using equations (3.1.105), (3.1.107) and (3.1.108), we can arrive at,

\[ \delta(T_T/\alpha) = [1 - (S/2) - (T_T/\alpha)^2] d\lambda \quad \text{(3.1.110)} \]
where,

\[ S = - \left( 9 \frac{t'^{4/3}}{\delta_T} / \kappa \alpha^2 \right) \]
\[ = \left( \frac{t'^{-2/3}}{\delta_T} / 6 \right) \]

By integrating the equation (3.1.110) by treating \( t' \) as constant, with boundary conditions

\[ J_T = 0 \quad \text{at} \quad \eta = 0 \quad \ldots \quad (3.1.111) \]

we get,

\[ u = \frac{t'^{-1/3} \alpha}{6} \left( 1 - S/2 \right) \text{sech}^2 \left[ (1 - S/2)^{1/2} \lambda_T \right] \quad \ldots \quad (3.1.112) \]

and

\[ (u/u_o)_{max} = \left( 1 - S/2 \right) \text{sech}^2 \left[ (1 - S/2)^{1/2} \lambda_T \right] \quad \ldots \quad (3.1.113) \]

The velocity profiles of the turbulent jet for different values of \( S \) are shown in Figure 3.1.d.1) and the qualitative effect of the porous medium in a turbulent free jet is similar to that of laminar free jet.
3.2 STRUCTURE OF JETS IN ROTATING POROUS SYSTEM

In this section, the structure of jets in rotating porous system is studied with the motive of gaining some insight on the effect of permeability on Ekman layer and the effect of bed friction on the spreading of the jets.

3.2.1 Formulation of the Problem:

The physical model consists of a container having a large horizontal dimensions filled with spherical particles whose diameters are large compared to narrow vertical slit which is in one of the vertical walls through which the jet enters or leaves the container. The top and bottom of the container are taken to be rigid and all the vertical walls except that containing the slit are taken to be porous so as to allow the jet to entrain or eject the fluid without producing secondary circulations. The container is placed on a rotating table with a vertical axis of rotation. The origin of the coordinate system is chosen to be the centre of the line of intersection of the slit and the horizontal bottom plane. The y-axis is chosen along this line and the x-axis is normal to it on the
bottom plane and along the direction of the incoming jet. The z-axis is taken to be vertical which is also the axis of rotation.

For steady flows, the governing equations of motion (2.1.4) is modified by including the inertial resistance term and Coriolis force term to study Ekman layers and jet solutions.

The equations are of the form:

\[
\begin{align*}
 uu_x + vv_y + wu_z - 2\omega v &= -(p_x/p_o) + \nu \nabla^2 u - (\gamma/K)u - c_b u^2/d_p \\
 uv_x + vv_y + wv_z + 2\omega u &= -(p_y/p_o) + \nu \nabla^2 v - (\gamma/K)v - c_b v^2/d_p \\
 p_z &= 0 \\
 u_x + v_y + w_z &= 0
\end{align*}
\]

where,

\begin{align*}
 u &= \text{velocity component in the } x\text{-direction} \\
 v &= \text{velocity component in the } y\text{-direction} \\
 w &= \text{velocity component in the } z\text{-direction} \\
 p_x &= \text{pressure gradient in } x\text{-direction} \\
 p_y &= \text{pressure gradient in } y\text{-direction} \\
 p_z &= \text{pressure gradient in } z\text{-direction} \\
 \omega &= \text{rate of rotation} \\
 \theta_o &= \text{porosity}
\end{align*}
\[ \rho_0 = \text{density} \]
\[ \nu = \text{kinematic viscosity} \]
\[ K = \text{permeability} \]
\[ c_b = \text{dimensionless constant} \left( 1.75 \times \frac{1 - \theta}{\theta^3} \right) \]
\[ d_p = \text{particle diameter} \]
\[ g = \text{acceleration due to gravity}. \]

Equation (3.2.1) and (3.2.2) represent the generalised Ergun equation including the Coriolis term for the motion of fluid in rotating systems involving porous media (Chandrasekhara and Vortmeyer, 1979; Manoj Choudhary et al., 1976). The last two terms on the R.H.S. in equations (3.2.1) and (3.2.2) represent the Darcy resistance and bed friction respectively. The bed friction term has the form given by the third term of the equation (2.1.3), when \( f_2 \) is replaced by \( c_b/d_p \).

The pressure in the system is that due to the relative motion of the fluid and is given by

\[ \frac{p}{\rho_0} = p_{\text{total}}/\rho_0 + gz + \frac{1}{2} \kappa^2 (x^2 + y^2) \quad \ldots (3.2.5) \]

The above equations (3.2.1) to (3.2.4) can be solved subjected to the boundary conditions,

\[ u = v = w = 0 \quad \ldots (3.2.6) \]
together with the conditions

\[
\begin{align*}
\nu(y = 0) &= 0, \\
u = u_\gamma = u_{\gamma\gamma} &= 0; \quad \text{at} \quad y \pm \infty
\end{align*}
\]  \hspace{1cm} \text{(3.2.7)}

The equations and the boundary conditions are made dimensionless by choosing proper scales for distances and velocities. The scales in the \( y \) and \( z \) directions can be taken as the slit-width \( Y_0 \) and the total height \( H \) of the container. There is no externally imposed downstream scale and hence \( x \) is scaled by \( X_0 \) which should be determined later as the scale of the dominant dissipation. The velocity at the slit along the axis can be taken to scale the velocity field (i.e., \( U, UY_0/X_0, UH_0/X_0 \)) so that no parameters appear in the equation of continuity. The pressure is made dimensionless using the geostrophic scale \( p_0 \text{ } 2^{-\gamma} U Y_0 \). Finally, equations of motion have the form,

\[
\begin{align*}
\Theta(uu_x + vv_y + wu_z) - v &= -p_x + (\Theta/\delta)u_{zz} + (\sigma/\delta)(u_{yy} + \sigma^2 u_{xx}) \\
&\quad - \frac{\sigma_0^2 u}{\delta} - \frac{c_b \Theta}{\delta} \delta^' u^2 \quad \cdots \text{(3.2.8)}
\end{align*}
\]

\[
\begin{align*}
\delta^2 \Theta[uu_x + vv_y + wu_z] + u &= -p_y + E \delta v_{zz} + \sigma \delta(v_{yy} + \sigma^2 v_{xx}) \\
&\quad - \sigma_0^2 \delta v - c_b v^2 \delta^2 \Theta \delta^' \quad \cdots \text{(3.2.9)}
\end{align*}
\]
\[ p_z \quad \text{(3.2.10)} \]

\[ u_x + v_y + w_z = 0 \quad \text{(3.2.11)} \]

where:

\[ \Theta = \text{Rossby number, } \frac{U}{2A_Y} \]

\[ \delta = \frac{Y_o}{X_o} \]

\[ Y_o = \text{slit width} \]

\[ X_o = \text{downstream scale} \]

\[ \sigma = \text{Ekman number based on the slit width, } \frac{\nu}{2\lambda Y_o^2} \]

\[ \delta' = \frac{Y_o}{d_p} \text{ (dimensionless constant)} \]

\[ \sigma_1^2 = \text{porous parameter, } \frac{Y_o^2}{K} \]

\[ E = \frac{\nu}{2\lambda H^2} \text{, Ekman number} \]

\[ H = \text{function defined in (3.2.48)} \]

\[ H_o = \text{total height of the container.} \]

The boundary conditions are,

\[ u = v = w = 0; \text{ at } z = 0; \text{ l } \quad \text{(3.2.12)} \]

\[ v(y = 0) = 0 \quad \text{(3.2.13)} \]

\[ u = u_y = u_{yy} = 0; \text{ at } y \pm \infty \quad \text{(3.2.14)} \]

3.2.2 Geostrophic and Ekman Layer Solutions:

The flow under consideration will be quasi-geostrophic when,
The condition $E^{1/2}/\delta \ll 1$ together with the equation (3.2.10) implies that the vertical shear will be confined to Ekman layers near the top and the bottom. If the local Rossby number, $(U/2\Delta y) << 1$ the conventional linear Ekman theory holds. However, it has been shown that jet like solutions to the above set of equations can be obtained only if the scale $X_0$ is such that $\sigma/\delta << \Theta$ and $\delta' < 1$. Hence in the interior, away from the Ekman layers, the velocity and pressure fields are expanded in powers of Rossby number. For example,

$$u = u_0 + \Theta u_1 + \ldots \quad \ldots (3.2.15)$$

The zeroth order equations or geostrophic equations are

$$v_0 = p_{ox} \quad \ldots (3.2.16)$$

$$u_0 = -p_{oy} \quad \ldots (3.2.17)$$

$$u_{ox} + v_{oy} + w_{oz} = 0 \quad \ldots (3.2.18)$$

Eliminating pressure using the equation (3.2.18) and using the boundary conditions (3.2.12) we find that

$$w_\Theta = 0 \quad \ldots (3.2.19)$$

everywhere in the interior of the flow field. The equations
valid in the Ekman boundary layers are,

\[
\begin{align*}
\nu^E_0 &= \rho_0 x - (E/\delta)u^E_0 \sigma^2_1 + (\sigma^2_1/\delta)u^E_0 \quad \ldots (3.2.20) \\
u^E_0 &= -\rho_0 y + E \delta v^E_0 \sigma^2_1 - \sigma^2_1 \delta v^E_0 \quad \ldots (3.2.21)
\end{align*}
\]

The solutions of equations (3.2.20) and (3.2.21) satisfying no-slip conditions at the top and bottom are,

\[
\begin{align*}
u^E_0 &= \frac{u_0 + e^{-m\frac{1}{2}} [-u_0 \cos q \frac{1}{2} - v_0 \delta (1 + \sigma^2_1) + u_0 \sigma^2_1 \sin q \frac{1}{2}] \quad \quad \ldots (3.2.22) \\
u^E_0 &= \frac{u_0 + e^{-m\frac{1}{2}} [-u_0 \cos q \frac{1}{2} + \{u_0/\delta\} (1 + \sigma^2_1) - \sigma^2_1 v_0 \sin q \frac{1}{2} \}]}{(1 + \sigma^2_1) \quad \ldots (3.2.23)}
\end{align*}
\]

where,

\[
\begin{align*}
m &= (S + \sigma^2_1)^{1/2} \quad q = (S - \sigma^2_1)^{1/2} \quad \text{and} \\
s &= (\sigma^2_1 + 1)^{1/2}
\end{align*}
\]

The variable \( \frac{1}{2} \) has the value,

\[
\frac{1}{2} = z/(2E)^{1/2} \quad 1 - z/(2E)^{1/2} \text{ at } z = 0, 1, \text{ respectively.}
\]

The axial velocity profiles of Ekman layers are presented in Fig. 3.2.1.
FIG. 3.2.1 - AXIAL VELOCITY IN THE EKMAN LAYER VS. ($\xi$) DISTANCE FROM THE BOUNDARY

$$\xi = \frac{Z}{(2E)^{1/2}}$$

$$\delta = 0.1$$

$$\frac{E^{1/2}}{\delta} = 0.2$$
It is observed from the figure that as \( \sigma_1 \) increases the magnitude of the velocity is greatly reduced and in fact is the characteristics of the flow through a porous medium. The thickness of the boundary layer is of the order \( (E/\sigma_1)^{1/2} \) and is thinner compared to the boundary layer of thickness \( (2E)^{1/2} \) obtained in the absence of porous medium (Gadgil, 1977). It may be concluded that the effect of permeability or the porosity of the medium represented by the parameter \( \sigma_1 \) is to reduce the magnitude of the velocity and to reduce the thickness of the Ekman boundary layer. The similar behaviour is observed for the velocity component \( v \).

The effect of the Ekman layer on interior flow is to produce a vertical velocity at the edge of the flow by the divergence in the layer. This velocity can be obtained by integrating the equation of continuity across the layer, (Robinson, 1965) and has the form,

\[
w(z = 1, 0) = \pm(1/\delta)(E/2)^{1/2}(u_{ey} - \delta^2 v_{ox}) \quad .. (3.2.24)
\]

As \( u \) and \( v \) are independent of \( z \) in the interior, equation (3.2.18) implies,

\[
w_{zz} = 0 \quad .. (3.2.25)
\]
Therefore, the solution for $w$ satisfying equation (3.2.24) and (3.2.25) is,

$$w = -\frac{1}{\delta} (E/2)^{1/2} (u_{oy} - \delta^2 v_{ox})(1 - 2z) \quad (3.2.26)$$

The vertical velocity given by equation (3.2.26) is of the order $(E^{1/2}/\delta)$ which is $<< 1$ and hence appears in the first order equations. The first order equations obtained from equations (3.2.8) to (3.2.11) are,

$$(u_0 u_{ox} + v u_{oy}) - v_1 = p_{1x} + (\sigma/\delta)u_{oyy} - (c_b/\delta)u_o^2 \delta' + O(\delta^2) \quad (3.2.27)$$

$$u_1 = -p_{1y} + O(\delta^2) \quad (3.2.28)$$

$$O = p_{1z} \quad (3.2.29)$$

$$u_{1x} + v_{1y} + [(2E)^{1/2}/\delta]u_{oy} = O(\delta^2) \quad (3.2.30)$$

Eliminating the pressure, using equations (3.2.28) and (3.2.30), the above set of equations reduce to the vorticity equation,

$$(u_0 u_{ox} + v u_{oy})_y = (\sigma/\delta)u_{oyy} - [(2E)^{1/2}/\delta]u_{oy} + (c_b \delta'/\delta)(u_o^2)y \quad (3.2.31)$$

Equation (3.2.31) is integrated once to give,

$$(u_0 u_{ox} + v u_{oy}) = (\sigma/\delta)u_{oyy} - [(2E)^{1/2}/\delta]u_o + (c_b \delta'/\delta)u_o^2 \quad (3.2.32)$$
The integration constant which is an arbitrary function of \( x \) is seen to be zero on account of boundary conditions \((3.2.14)\).

3.2.3 Discussion of the Governing Equations of the Jet:

The basic equations for jet flow are the geostrophic equations \((3.2.16), (3.2.17)\) and the vorticity equation \((3.2.32)\). Dropping the subscript zero for convenience the equations take the form,

\[
\begin{align*}
\frac{u}{u_x} + \nu u_y &= \alpha u_{yy} - Ru - R'u^2 \\
\nu &= -p_y \\
v &= p_x
\end{align*}
\]

where,
\[
R = [(2E)^{1/2}/\sigma] = 2Y_0[(\nabla)^{1/2}/\sigma UH]
\]

= bottom friction parameter \((3.2.36)\)
\[
\alpha = \sigma/\Theta \delta = \nu/\sigma UY_0
\]

= side friction parameter \((3.2.37)\)
\[
R' = c_b \delta'/\delta = \text{bed friction parameter} \quad (3.2.38)
\]
\[
R/\alpha = (2E)^{1/2}/\sigma = (2Y_0^2/\nu)(\mu \nabla H^2)^{1/2}
\]

It can be seen from equation \((3.2.33)\) that there are three mechanisms of dissipation of vorticity represented by three terms appearing on the R.H.S. of \((3.2.33)\). The first term
represents the lateral dissipation due to the viscosity of the fluid and will be referred to as side-friction. The second term is due to dissipation in the Ekman layer near the top and bottom walls and will be referred to as bottom friction. The third term represents bed-friction which arises due to the structure of the bed. Each of these mechanisms gives a characteristic downstream scale of which the scale corresponding to the more efficient process being the shorter of the three. The downstream scale \( X_o \) of the jet therefore has to be that corresponding to the dominant dissipation. In other words, the parameter \( \delta \) is so determined as to make the dominant frictional term of the same order as the inertial terms. We now discuss different physical situations and the corresponding downstream scales.

Case (a):

If the rate of rotation is slow and the porosity is large so that \( R \) and \( R' \ll \alpha \), then from Eqn. (3.2.37) we have,

\[
\delta = \frac{\nu}{UY_o} = \frac{1}{R_e}
\]

where \( R_e \) is the Reynolds number based on slit-width and it leads to

\[
X_o = \frac{Uy^2_o}{\nu}
\]
Thus, for this case the downstream scale is the product of the slit width and the Reynolds number of the slit $R_e$ and the set of equations (3.2.33) to (3.2.35) reduce to those equations which govern the structure of the jets in nonrotating systems.

Case (b):

If $R \gg \alpha$ and $R'$ then from equation (3.2.36) we have,

$$\delta = \frac{2\gamma_0 (\Delta V)^{1/2}}{UH}$$

and hence,

$$X_o = \frac{UH}{2(\Delta V)^{1/2}}$$ .. (3.2.42)

In this case it is interesting to note that the downstream scale $X_o$ is independent of the slit width.

Case (c):

When $R' \gg R$ and $\alpha$, we find from equation (3.2.38) that $\delta = c_b \delta'$ and hence,

$$X_o = \frac{d_p}{c_b}$$ .. (3.2.43)

The downstream scale $X_o$, in this case is independent of the slit width but depends on the diameter of the particles.

The above discussion points out that whenever any one of
the three dissipation mechanisms become dominant then it gives rise to a downstream scale \( X_0 \).

We now consider the nature of downstream scale when two of the three mechanisms become comparable in magnitude at a time. To study this we introduce an internal scale \( \tilde{Y} \) to represent the cross-stream scale instead of \( Y_0 \) because the width of the side friction is known to increase with the distance downstream from the slit.

Case (d):

The side-friction and bottom-friction are of the same order, and the bed friction is small, i.e., \( R/\alpha = 1 \) and \( R' \) is small, then we have,

\[
\tilde{Y}^2 = \frac{\nu H}{2(\ln \tilde{Y})^{1/2}} \quad \ldots (3.2.44)
\]

For this choice of the cross-stream scale, the downstream scales (3.2.40) and (3.2.41) coincide and hence we have a single downstream scale. Further, if the width of the slit is smaller than \( \tilde{Y} \), the dynamics near the slit is that of a side-frictional jet. As the distance from the slit increases, bottom friction becomes more important and finally dominates the lateral dissipation at large distance from the slit.
If the slit width is larger than $\bar{Y}$, the bottom-friction dominates the side friction near the slit. We observe, that the bottom-friction always dominates at sufficiently large distances from the slit. However, the side friction is important near the slit only when the slit is narrower than $\bar{Y}$.

Case (e):

If $R/R' = 1$ and $\alpha$ is small, we find,

$$\frac{c_b}{d_p} = 2(\pi - \gamma)^{1/2}/UH \quad \ldots (3.2.45)$$

for this choice of $c_b/d_p$, the downstream scales (3.2.41) and (3.2.42) coincide.

Case (f):

If $R'/\alpha = 1$ and $R$ is small, we get,

$$\bar{Y}^2 = \gamma d_p / c_b U \quad \ldots (3.2.46)$$

for the above choice of cross stream scale, the downstream scales (3.2.40) and (3.2.42) coincide. Further, we observe that if the width of the slit is more narrow, than $\bar{Y}$, the dynamics near the slit are that of a side-frictional jet. As the distance from the slit increases bed friction becomes more important and finally dominates the lateral dissipation.
and the jet behaves like a bottom-frictional jet discussed earlier.

Case (g):

When all the three dissipation mechanisms becomes important we have to choose,

\[ \tilde{\gamma}^2 = \gamma H / 2 (\lambda \nu)^{1/2} \quad \text{and} \quad \frac{d_p}{c_b} = U H / 2 (\lambda \nu)^{1/2} \]

so that the downstream scales \((3.2.41), (3.2.42)\) and \((3.2.43)\) become identical and are equal to,

\[ x_0 = U H / 2 (\lambda \nu)^{1/2} \quad \ldots (3.2.49) \]

3.2.4 Similarity Solutions:

To obtain the solution of equation \((3.2.33)\) we use Von-Mises transformation \((Goldstein, 1960)\) in which the pressure \(p\) is used as one of the independent variables. Introducing,

\[ p_Y = H(p, x) \quad \ldots (3.2.48) \]

in equations \((3.2.33)\) to \((3.2.35)\) gives

\[ H_x = R - \alpha \left( H_p^2 + H H_{pp} \right) - R'H \quad \ldots (3.2.49) \]

Equation \((3.2.49)\) is similar to the form of equation \((3.1.53)\)
except for the last term and hence we will look for the solution of $H$ in the form

$$H = -\phi(x) + \psi(x)p^2 \quad \ldots (3.2.50)$$

Following analysis of the previous section (3.1(c)(ii)), we find,

$$\phi' = -R - 2\alpha \phi \psi - R' \phi \quad \ldots (3.2.51)$$

$$\psi' = -6\alpha \psi^2 - R' \psi \quad \ldots (3.2.52)$$

The arbitrary constants involved in the integration of (3.2.51) and (3.2.52) are chosen so that $u(0,0)$ and the width of the jet at the slit are unity and the following set of solutions are obtained for the different physical situations.

**Solutions for Side-frictional jet (i.e., $R = 0$):**

The solutions for the side frictional jet is obtained by putting $R = 0$ in (3.2.51) and solving for $\phi$ and $\psi$.

The expressions for $p, u$ and the cross-stream velocity $v(\pm \infty)$ are obtained in the form,

$$v(\pm \infty) = \frac{(R' + 6\alpha)e^{R'x} - 6\alpha}{R'} \right)^{1/3} \tanh \left[ y \left( \frac{(R' + 6\alpha)e^{R'x} - 6\alpha}{R'} \right)^{-2/3} \right] \quad \ldots (3.2.53)$$
The total transport in the \( x \)-direction is,

\[
I = \int_{-\infty}^{\infty} u \, dy = 2 \left[ \frac{\left( R' + 6\alpha \right)e^{R'x} - 6\alpha}{R'} \right]^{1/3} \quad .. (3.2.56)
\]

It can be observed from equations (3.2.54) and (3.2.56) that the velocity at the axis of the jet (\( u \)) decreases with the downstream distance \( x \) and the total transport (\( I \)) increases with \( x \) due to entrainment of the fluid at the edges. The streamlines for this case are presented in Fig. 3.2.2. In the limit \( R' \to 0 \) equations (3.2.53) to (3.2.56) reduce to the following forms:

\[
p = -(1 + 6\alpha x)^{1/3} \tanh[y(1 + 6\alpha x)^{-2/3}] \quad .. (3.2.57)
\]

\[
u = (1 + 6\alpha x)^{-1/3} \text{sech}^2[y(1 + 6\alpha x)^{-2/3}] \quad .. (3.2.58)
\]

\[
\nu(\pm \infty) = \pm 2\alpha(1 + 6\alpha x)^{-2/3} \quad .. (3.2.59)
\]
The streamlines for this case are shown in Fig. 3.2.3.

Comparing the Figures 3.2.2 and 3.2.3, we observe that in rotating porous system the streamlines diverge near the slit, indicating that the spreading of the jet is faster compared to jets in non-porous rotating systems (Fig. 3.2.2). The experiments on jets in porous systems also point out this fact.

Solution for the Bottom-frictional jet (i.e., when α = 0):

The expression for $P$, $u$, $v(\pm \infty)$ and $T$ for the bottom-frictional jet are obtained using equations (3.1.16), (3.1.17), (3.2.51) and (3.2.52) and written in the form,

$$p = -\left[\frac{(R' + R) - R e^{R'x}}{R'}\right]^{1/2}$$

$$\tanh \left[y \left\{ \frac{(R' + R)e^{-2R'x} - R e^{-R'x}}{R'} \right\}^{1/2} \right]$$

$$u = (R' + R) e^{-R'x} - R$$

$$\text{sech}^{2} \left[y \left\{ \frac{(R' + R)e^{-2R'x} - R e^{-R'x}}{R'} \right\}^{1/2} \right]$$

$$'(\pm \infty) = \pm (R/2) e^{R'x} \left\{ \frac{(R' + R) - Re^{R'x}}{R'} \right\}^{-1/2}$$
FIG. 3.2.2-STREAMLINES FOR THE SIDE-FRICTIONAL JET \((R' = 3.0, R = 0, \alpha = 1.0)\)
FIG. 3.2.3 - STREAMLINES FOR THE SIDE-FRICTIONAL JET \( (R = 0) \)
The total transport in the $x$ direction is,

$$
T = \int_{-\infty}^{\infty} u \, dy \left[ \frac{(R' + R) - R \, e^{-R'x}}{R'} \right]^{1/2} \quad \text{.. (3.2.63)}
$$

In this case the velocity at the axis as well as the transport decreases with the downstream distance $x$ and vanishes at the point,

$$
x = -\frac{1}{R'} \log \frac{R}{R' + R} \quad \text{.. (3.2.64)}
$$

The downstream distance $x$ is measured from the slit and is considered to be positive. Therefore, the values of $\log[R/(R+R')]$ should always be less than unity to make $x$ positive. This imposes a restriction on the values of $R$ and $R'$. Further, the jet cannot penetrate beyond this point since all the momentum is dissipated in Ekman layers. The cross-stream velocity at this point is $R_y/2$ for finite $y$ and becomes indeterminate for infinite $y$ which can be seen from equation (3.2.62). The equations for the bottom-frictional jet yield solutions for two limiting cases, $R \to 0$ and $R' \to 0$, and in what follows we discuss these limiting cases.
Case (i):

When $R' \to 0$ the expressions for $p$, $u$, $v(\pm \infty)$ and $T$ reduce to the forms:

$$p = -(1 - Rx)^{1/2} \tanh [y(1 - Rx)^{1/2}] \quad \cdots (3.2.65)$$

$$u = (1 - Rx) \text{sech}^2 [y(1 - Rx)^{1/2}] \quad \cdots (3.2.66)$$

$$v(\pm \infty) = \pm \frac{R}{2(1 - Rx)^{1/2}} \quad \cdots (3.2.67)$$

and the total transport in the $x$ direction becomes,

$$T = \int_{-\infty}^{\infty} u \, dy = 2(1 - Rx)^{1/2} \quad \cdots (3.2.68)$$

It can be seen from (3.2.66) and (3.2.68) that the velocity and the transport decrease with the downstream distance $x$ and vanish at $x = 1/R$.

Case (ii):

When $R \to 0$, the expression (3.2.60) to (3.2.63) reduce to the following forms:

$$p = - \tanh [y \, e^{-R'x}] \quad \cdots (3.2.69)$$

$$u = \text{sech}^2 [y \, e^{-R'x}] \quad \cdots (3.2.70)$$

$$v(\pm \infty) = 0 \quad \cdots (3.2.71)$$
It is observed from Eqn. (3.2.72) that the total transport is independent of the downstream distance and remains constant. It is interesting to note that the velocity never vanishes as $x$ increases but approaches asymptotically the value unity. The streamlines for the bottom-frictional jet are represented in Figures 3.2.4, 3.2.5 and 3.2.6. It is seen that as in the case of the side-frictional jet, bed-friction causes the jet to spread wider nearer the slit when $R$ and $R'$ are both present and the spreading is less when $R'$ is alone present.

Case (iii) - General case:

If all the three dissipation mechanisms become dominant (i.e., when $R \neq 0$, $\alpha \neq 0$ and $R' \neq 0$), the expression for the pressure $p$ has the form,

$$
\left\{ \frac{(R' + 6\alpha)e^{R'x} - 6\alpha}{R'} \right\}^{2/3} (1 - R/8\alpha)
- (R/8\alpha) \left\{ \frac{(R' + 6\alpha)e^{R'x} - 6\alpha}{R'} \right\}^2
$$

$$
\tanh \left[ v \left\{ \frac{(1 + R/8\alpha)}{[((R' + 6\alpha)e^{R'x} - 6\alpha)/ R']^{4/3} - R/8\alpha} \right\}^{1/2} \right]
$$

\[ (3.2.73) \]
FIG. 3.2.4 - STREAMLINES FOR THE BOTTOM FRICTIONAL JET ($\alpha = 0$, $R = 1.0$, $R' = 1.0$)
FIG. 3.2.5 - STREAMLINES FOR THE BOTTOM FRICTIONAL JET ($\alpha = 0$, $R' = 0$, $R = 1.0$)
FIG. 3.2.6 - STREAMLINES FOR THE BOTTOM FRICTIONAL JET ($\alpha = 0$, $R = 0$, $R' = 1.0$)
In the limit $R' \to 0$, the above expression takes the form,

$$p = -[(1 + 6ax)^{2/3} (1 + R/8a) - (R/8a)(1 + 6ax)^2]^{1/2}$$

$$\tanh \left[ y \left\{ \frac{1 + R/8a}{(1 + 6ax)^{4/3}} \frac{R}{8a} \right\}^{1/2} \right] \quad \ldots (3.2.74)$$

The cross-stream velocity at the edges of the jet is given by,

$$v(\pm \infty) \left[ ((R' + 6a)e^{R'x} - 6a)/R' \right]^{-1/3}$$

$$= \pm (R/4a)e^{R'x} \left[ ((R' + 6a)e^{R'x} - 6a)/R' \right]^{1/2} (R' + 6a)$$

$$\frac{[[((R' + 6a)e^{R'x} - 6a)/R']^{2/3}]}{((1 + R/8a) - (R/8a)\left[ ((R' + 6a)e^{R'x} - 6a)/R' \right]^{2})^{1/2}}$$

$$\ldots (3.2.75)$$

In the limit $R' \to 0$, the above equation reduces to the form,

$$v(\pm \infty) = \frac{([(R/2) + 4a](1 + 6ax))^{-1/3} \pm [(3/2)R](1 + 6ax)]}{2 \left\{ ((1 + R/8a)(1 + 6ax))^{2/3} - (R/8a)(1 + 6ax)^2 \right\}^{1/2}}$$

$$\ldots (3.2.76)$$
From equation (3.2.75) it is clear that jet entrains the fluid at its edges from the slit \((x = 0)\) up to a distance \(x\) given by

\[
\left[ (R' + 6\alpha)e^{R'x} - 6\alpha \right]^{4/3} = \frac{1}{3} \left( 1 + 8\alpha/R \right) \quad \text{.. (3.2.77)}
\]

and behaves as a side-frictional jet. Beyond this point the jet ejects the fluid causing a downstream decrease in transport and the jet behaves as a bottom-frictional jet. The transition between the two regimes is smooth. The axial velocity of the jet decreases with \(x\) and vanishes at a point given by

\[
\left[ \frac{(R' + 6\alpha)e^{R'x} - 6\alpha}{R'} \right]^{4/3} = (1 + 8\alpha/R) \quad \text{.. (3.2.78)}
\]

The point at which the downstream velocity of the jet becomes zero is represented in Figure 3.2.7, for different values of \(R'\) and it can be seen from the figure that the distance from the slit at which the velocity vanishes decreases with the increase in the value of \(R'\), the bed friction parameter. At this point the cross-stream velocity equals \(R_y/2\) and becomes indeterminate as \(y \to \infty\). These features are characteristic of the bottom-frictional jet discussed earlier and represent the behaviour of the jet in a strongly rotating...
FIG. 3.2.7 - DISTANCE (x) FROM THE SLIT VS. R'
3.2.5 Results and Conclusions:

The structure of jets in rotating porous system is investigated using generalised Ergun equations. The results of the investigation reveal that the permeability of the medium influences the Ekman boundary layer flow and reduces the magnitude of the velocity when the porous parameter $\sigma_1$ is increased. Further, the Ekman boundary layer thickness is reduced from $(2E)^{1/2}$ (in the absence of a porous medium) to $(E/\sigma_1^2)^{1/2}$ in the present case.

Von-Mises transformation is used to obtain similarity solutions for the jet flow. The solutions indicate that the jet behaves as a side-frictional jet up to a downstream distance, defined by equation (3.5.78) and acts as a bottom-frictional jet beyond that point. The role of bed-friction in all these cases is to quicken the process of spreading the jets. Other interesting result is that the three dissipation mechanisms provide different characteristic downstream scales depending upon the physical situations.