Chapter II

Essentials of Fermion Masses and Mixings

To make the thesis self contained as well as to set the notations and conventions for the forthcoming chapters, in this chapter we have summarized the essentials of the formalism and the formulae used. To begin with, in Section (2.1), we have discussed in detail the quark mass matrices, their connection with the quark mixing matrix $V_{\text{CKM}}$ within the framework of the SM has been discussed in Section (2.2). After defining the mixing matrix, in Section (2.3) we have discussed various CP violation measuring parameters in quark sector, for example, CP violating phase $\delta$, Jarlskog’s rephasing invariant parameter $J$ and precisely known angle of unitarity triangle $\beta$. Having discussed the quark sector, in Section (2.4) we have presented the lagrangian terms containing Dirac and Majorana lepton mass matrices. The relationship of the lepton mass matrices and lepton mixing matrix $V_{\text{PMNS}}$ has been explored in Section (2.5). The mixing phenomenon leads to neutrino oscillations which are described in Section (2.6). The discussion of leptonic CP violation and its measuring parameter, Jarlskog’s rephasing invariant $J_{l}$ has been given in Section (2.7). The effective Majorana neutrino mass has been discussed in Section (2.8). Most of the material presented in this chapter has been adopted from references [15, 50, 66, 67, 73, 76, 100–104].
2.1 Quark mass matrices

The Lagrangian density of the Electroweak Model \[102\] can be symbolically written as

\[
\mathcal{L} = \mathcal{L}(f, G) + \mathcal{L}(f, H) + \mathcal{L}(G, H) + \mathcal{L}(G) - V(H), \tag{2.1}
\]

where \(f, G\) and \(H\) represent fermions, gauge bosons and Higgs doublet respectively. This Lagrangian is invariant under local (space-time dependent) symmetry group \(SU(2)_L \times U(1)_Y\).

The term \(\mathcal{L}(f, H)\) in equation (2.1) describes the interactions of fermions with the Higgs doublet and incorporates mass matrices, while the term \(\mathcal{L}(f, G)\) describes interactions of the fermions with the gauge bosons and incorporates the mixing matrix. The hadronic part of \(\mathcal{L}(f, H)\) is given below and the leptonic part may be written in the similar way if neutrinos are Dirac particles.

\[
\mathcal{L}(f, H) = \sum_{j,k=1}^n \left\{ Y_{jk}(U, D)_{3L} \begin{pmatrix} \phi(0) & \phi(0^*) \\ -\phi(-) & \phi(-) \end{pmatrix} U_{kjR} + Y'_{jk}(U, D)_{3L} \begin{pmatrix} \phi(+) & \phi(0) \\ \phi(0^*) & \phi(-) \end{pmatrix} D_{kjR} + h.c. \right\}, \tag{2.2}
\]

where \(U \equiv (u, c, t)\) and \(D \equiv (d, s, b)\) are the quark fields for the up and down sector respectively, the left and right-handed fields being

\[
U_L = \frac{1 - \gamma_5}{2} U, \quad U_R = \frac{1 + \gamma_5}{2} U, \tag{2.3}
\]

\[
D_L = \frac{1 - \gamma_5}{2} D, \quad D_R = \frac{1 + \gamma_5}{2} D. \tag{2.4}
\]

The doublets

\[
H = \begin{pmatrix} \phi(+) \\ \phi(0) \end{pmatrix} \quad \text{and} \quad H^C = \begin{pmatrix} \phi(0^*) \\ -\phi(-) \end{pmatrix},
\]

are the Higgs doublet and its conjugate respectively. \(Y_{jk}\) and \(Y'_{jk}\) are the quark Higgs coupling constants, referred to as the Yukawa couplings, related to quark mass matrices and are "completely arbitrary" within the SM.

If \(SU(2)_L \times U(1)_Y\) were an exact symmetry, all quarks would be massless and there would be no physical distinction between the interaction eigenbasis and the mass eigenbasis. In the physical world, however, this symmetry is spontaneously broken to \(U(1)_{EM}\) and the Higgs doublet assumes a non-zero vacuum expectation
value (VEV), and the quarks acquire mass. For example, after the spontaneous symmetry breaking, the up and down type quarks respectively acquire masses through the interactions

$$-\mathcal{L}_q = \overline{U}_L M_U U_R + \overline{D}_L M_D D_R,$$

(2.5)

where $M_U$ and $M_D$ respectively denote up quark and down quark mass matrices and are $n \times n$ dimensional for $n$ number of families. The scale of mass matrices is characterized by the electroweak symmetry breaking scale $\approx 174$ GeV. The CP non-conservation requires mass matrices to be complex.

The quark mass matrix $M_q (q = U, D)$ can be diagonalized by a bi-unitary transformation given as

$$M^\text{diag}_q = V_{qL}^\dagger M_q V_{qR} = \text{Diag} \{m_1, m_2, m_3\},$$

(2.6)

where $V_{qL}$ and $V_{qR}$ are complex matrices and $M^\text{diag}_q$ is a diagonal real matrix. The quantities $m_1, m_2, m_3$ etc. denote the eigenvalues of the quark mass matrices. For example, for the up and down sector of quarks, the eigenvalues $m_1, m_2, m_3$ are replaced by $m_u, m_c, m_t$ and $m_d, m_s, m_b$ respectively. One can evaluate $V_{qL}$ in the following manner. Multiplying (2.6) by its hermitian conjugate, one gets

$$V_{qL}^\dagger M_q V_{qR} V_{qR}^\dagger V_{qL} = (M_q^\text{diag})^2 \equiv \text{Diag} \{m_1^2, m_2^2, m_3^2\},$$

(2.7)

Thus $V_{qL}$ diagonalizes the hermitian matrix $M_q M_q^\dagger$. Similarly $V_{qR}$ diagonalizes the hermitian matrix $M_q^\dagger M_q$.

Using equation (2.6), one can rewrite the quark mass terms in equation (2.5) as

$$-\mathcal{L}_q = \overline{U}_L M^\text{diag}_U U_R + \overline{D}_L M^\text{diag}_D D_R,$$

(2.9)

which can be expressed in terms of the physical quark fields as

$$-\mathcal{L}_q = \overline{U}_L^{\text{phys}} M^\text{diag}_U U_R^{\text{phys}} + \overline{D}_L^{\text{phys}} M^\text{diag}_D D_R^{\text{phys}},$$

(2.10)
where the physical fields are

\[ U_L^{\text{phys}} = V_{UL} U_L = V_{UL}^t \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L, \]  

and

\[ D_L^{\text{phys}} = V_{DL}^t D_L = V_{DL}^t \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L. \]  

### 2.2 Quark mixing matrix

In SM, the term \( \mathcal{L}(f, G) \) in equation (2.1) describes the interactions of the fermions with the gauge bosons. This term includes the neutral current terms involving either the up kind quarks or the down kind quarks but not the both, as well as the charged current terms which do mix the up and the down kind quarks. The neutral currents conserve CP and there are no flavor changing neutral currents (FCNC's). The absence of FCNC's is referred to as the GIM (Glashow-Iliopoulos-Maiani) mechanism. The charged currents are mediated by the weak bosons \( W^\pm \) that carry one unit of charge and hence mix the up and the down kind quarks. The charged current term in the Lagrangian \( \mathcal{L}(f, G) \) in equation (2.1) is given as

\[
-\mathcal{L}_q^{\text{CC}} = \frac{g'}{\sqrt{2}} U_L \gamma^\mu (1 - \gamma_5) D_L W + h.c.
\]

\[
= \frac{g'}{\sqrt{2}} U_L^{\text{phys}} \gamma^\mu (1 - \gamma_5) V_{UL} V_{DL} D_L^{\text{phys}} W + h.c.
\]

\[
= \frac{g'}{\sqrt{2}} U_L^{\text{phys}} \gamma^\mu (1 - \gamma_5) V_{\text{CKM}} D_L^{\text{phys}} W + h.c., \tag{2.13}
\]

where \( V_{\text{CKM}} \) referred to as Cabibbo-Kobayashi-Maskawa matrix [13], is the quark mixing matrix connecting mass eigenstates to the weak interaction eigenstates and is given as

\[
V_{\text{CKM}} = V_{UL}^t V_{DL}, \tag{2.14}
\]
where $V_{ud}$ gives the amplitude of the process $u \rightarrow d + W^-$ and similarly, for the other elements. For hermitian mass matrices, the $V_{CKM}$ is given as

$$V_{CKM} = V_d^l V_D^r.$$

(2.16)

Thus, the elements of $V_{CKM}$ constitute the low energy observable quantities through which one can get clues about the structure of the mass matrices.

The direct relationship of $V_{CKM}$ with the mass matrices is given in equation (2.6). Thus, an experimental knowledge of $V_{CKM}$ would have implications for quark mass matrices. The best determined elements of the $V_{CKM}$ as given in PDG [15] are

$$|V_{ud}| = 0.97425 \pm 0.00022, \quad |V_{ud}| = 0.2255 \pm 0.0024, \quad |V_{ud}| = 0.230 \pm 0.011,$$

(2.17)

which have been measured in the nuclear $\beta$ decay, semi-leptonic kaon, tau and hyperon decays and neutrino and anti-neutrino interactions respectively. Semi-leptonic $D$ or leptonic $D_s$ decays give

$$|V_{cs}| = 1.023 \pm 0.036.$$

(2.18)

Semi-leptonic inclusive and exclusive $B$ decays give $|V_{cb}|$ and $|V_{ub}|$ as

$$|V_{cb}| = 0.0427 \pm 0.0038, \quad |V_{ub}| = 0.0338 \pm 0.0036.$$

(2.19)

The elements $|V_{ud}|$ and $|V_{us}|$ can be measured from the tree-level top quark decays, however, there are several quark loop mediated rare K and B decays such as $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixing give

$$|V_{td}| = 0.0084 \pm 0.0006, \quad |V_{ts}| = 0.0387 \pm 0.0021.$$

(2.20)

The determination of $|V_{tb}|$ from the top decays without assuming unitarity gives

$$|V_{tb}| = 0.88 \pm 0.07.$$

(2.21)
2.3 CP violating parameters in the quark sector

2.3.1 CP phase $\delta$

The $V_{\text{CKM}}$ matrix not only describes the three flavor mixings but also describes the CP violation in the SM. It can be shown that the complex matrix $V_{\text{CKM}}$ leads to CP non-conservation. In this context, it is important to examine the question whether the phases associated with a unitary matrix are real or not and how the $V_{\text{CKM}}$ can be parametrized. In general, a unitary $n \times n$ matrix has $n^2$ parameters, $n(n - 1)/2$ of these may be taken as Euler angles and the remaining parameters are phases. However, not all the phases are physical and the unphysical phases may be rotated away by the unitary transformations. For unitary $n \times n$ matrix, $(2n - 1)$ phases are not measurable, therefore one is left with only $(n - 1)(n - 2)/2$ measurable physical phases.

The $V_{\text{CKM}}$ by definition, is a unitary matrix and hence can be expressed as in terms of three mixing angles and six phases. Out of the six phases, five can be reabsorbed into the quark fields. Therefore, one is left with only one non-trivial phase which is responsible for CP violation in the SM. Many parameterizations of the $V_{\text{CKM}}$ have been proposed in the literature [15, 105–107]. One can show that in terms of three angles and one phase, the $V_{\text{CKM}}$ matrix can have as many as 36 different parameterizations, all of which are physically equivalent. For different purposes different parameterizations may prove to be convenient, however the most commonly used parametrization is the standard parametrization given by PDG, which is strongly recommended for numerical evaluations, given as

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i \delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i \delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i \delta} & -s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i \delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i \delta} & c_{23}c_{13} \end{pmatrix}, \quad (2.22)$$

with $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ for $i, j = 1, 2, 3$. The parameter $\delta$ is the CP violating phase. The real angles $\theta_{12}, \theta_{23}$ and $\theta_{13}$ can be chosen to lie in the first quadrant, whereas the quadrant of $\delta$ has physical significance, therefore cannot be fixed. In the PDG representation $\sin \delta \neq 0$ implies the existence of CP violation.
The present direct and indirect measurements constrain $\delta$ to be

$$\delta = (73^{+22}_{-25})^\circ. \quad (2.23)$$

### 2.3.2 Jarlskog’s rephasing invariant parameter $J$

As mentioned earlier, large number of representation of $V_{\text{CKM}}$ have been considered and for different purposes different parameterizations prove to be convenient. In terms of three angles and one phase, the $V_{\text{CKM}}$ can have as many as 36 different parameterizations, all of which are physically equivalent and are related to each other due to the facility of rephasing of quark fields. This brings forth the necessity to carry out $V_{\text{CKM}}$ phenomenology in terms of rephasing invariant quantities. Apart from the magnitude of the $V_{\text{CKM}}$ elements which are rephasing invariant, Jarlskog has found an important parameter, known as Jarlskog’s rephasing invariant parameter and denoted by $J$. The significance of $J$ lies in the fact that all the CP violating effects in the SM are proportional to it as well as it is independent of the representation of $V_{\text{CKM}}$. In terms of $V_{\text{CKM}}$ elements, $J$ can be written in a form that is explicitly parametrization independent, for example,

$$J \sum_{m,n=1}^3 \epsilon_{ilm} \epsilon_{jkn} = |\text{Im}(V_{ij}^* V_{ik} V_{jk}^* V_{lj})|, \quad (2.24)$$

for any choice of $i \neq l, j \neq k$. Thus knowing the $V_{\text{CKM}}$ elements, $J$ can easily be evaluated.

Also $J$ is “invariant function” of the mass matrices, i.e. it is independent of any choice of basis for the mass matrices. All the measurable quantities must be invariant functions of mass matrices. The parameter $J$ is related to the mass matrices as

$$\text{Det} [M_U M_D^\dagger, M_D M_U^\dagger] = -2i J (m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_u^2 - m_t^2)$$

$$\times (m_b^2 - m_c^2)(m_c^2 - m_d^2)(m_d^2 - m_b^2). \quad (2.25)$$

A remarkable feature of the determinant in equation (2.25) is that it vanishes if and only if there is no CP non-conservation. Furthermore, the quantity $J$ in the
standard parametrization is given as

\[ J = \sin \theta_{12} \sin \theta_{23} \sin \theta_{13} \cos \theta_{12} \cos \theta_{23} \cos^2 \theta_{13} \sin \delta , \] (2.26)

where \( \theta_{12}, \theta_{23} \) and \( \theta_{13} \) are the mixing angles and \( \delta \) is the CP violating phase. It is clear from equation (2.25) that an evaluation of \( J \) is going to have important implications on texture specific mass matrices in the sense that it could provide valuable clues for searching the right texture for fermion mass matrices [51,61,63,73]. The latest value of \( J \) given by PDG [15] is

\[ J = (2.91^{+0.01}_{-0.11}) \times 10^{-5}. \] (2.27)

### 2.3.3 Angles of the unitarity triangle

The CP violating aspects of the \( V_{\text{CKM}} \) have been expressed through the angles of the unitarity triangle, related to the unitarity of the \( V_{\text{CKM}} \) which implies nine conditions, three of which connect the magnitudes of the \( V_{\text{CKM}} \) elements, while there are other six non-diagonal relations. In case, the CP violating phase is non-zero, each of the six non-diagonal relations expresses this fact through the \( V_{\text{CKM}} \) elements representing a triangle in the complex plane.

**Figure 2.1:** Representation of the unitarity triangle in the complex plane. The relevant B decay modes are indicated for the angles involved in the corresponding CP violating asymmetries.
In Figure (2.1), we have schematically shown the unitarity triangle whose sides are related to several decay rates while the angles are related to the CP asymmetries in various B decays and can be expressed through the relation

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0.$$  

(2.28)

The three angles of the triangle are given as

$$\alpha \equiv \arg \left( -\frac{V_{ud} V_{ub}^*}{V_{td} V_{tb}^*} \right), \quad \beta \equiv \arg \left( -\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*} \right), \quad \gamma \equiv \arg \left( -\frac{V_{td} V_{tb}^*}{V_{cd} V_{cb}^*} \right).$$  

(2.29)

The precisely measured angle $\beta$ by BABAR [21] and Belle [22] in the B-factory experiments in $b \rightarrow c\bar{c}s$ decays is given as

$$\sin 2\beta = (0.673 \pm 0.023).$$  

(2.30)

### 2.4 Lepton mass matrices

The mass eigenstates of charged leptons can always be chosen to coincide with their flavor eigenstates through an appropriate unitary transformation of the right-handed fields. The coincidence between mass and flavor eigenstates of neutrinos can also be achieved, if neutrinos are assumed to be exactly massless particles. Hence there is no lepton flavor mixing nor CP violation within the framework of the SM. If neutrinos are massive and their masses are non-degenerate, then it will be impossible to find a basis of the flavor space in which the coincidence between flavor and mass eigenstates holds both for charged leptons and for neutrinos. In other words, the flavor mixing phenomenon is naturally expected to appear between three charged leptons and three massive neutrinos, just like the flavor mixing between three up-type quarks and three down-type quarks. The possibility of flavor oscillations was originally examined by B. Pontecorvo and further generalized by Maki, Nakagawa and Sakata [14]. The emerging picture that neutrinos are massive and therefore mix [16] has been proved beyond any doubt with the outcome of a set of precise experiments and provides an unambiguous signal of NP.

In the Standard Model (SM) of electromagnetic and weak interactions, only the left-handed leptons take part in the charged current weak interactions and the
corresponding Lagrangian is given as

$$- \mathcal{L}^{\nu}_{\nu} = \frac{g}{\sqrt{2}} (\nu_{\alpha})_{L} \gamma_{\mu} (l)_{L} W_{\mu}^{\pm} + h.c. \quad (\alpha = e, \mu, \tau).$$ \hspace{1cm} (2.31)

After spontaneous symmetry breaking, the charged leptons acquire masses through the Yukawa interactions as

$$- \mathcal{L}_{l} = \bar{l}_{L} M_{l} l_{R} + h.c..$$ \hspace{1cm} (2.32)

In the case of neutrinos, the generation of masses is not straightforward as they may have either the Dirac masses or the more general Dirac-Majorana masses. A Dirac mass term can be generated by the Higgs mechanism with the standard Higgs doublet. In this case, the neutrino mass term can be written as

$$- \mathcal{L}_{D} = \bar{\nu}_{aL} M_{\nu D} \nu_{aR} + h.c.,$$ \hspace{1cm} (2.33)

where $M_{\nu D}$ is a complex $3 \times 3$ Dirac mass matrix and $\nu_{a}$ are the flavor eigenstates ($\equiv (\nu_{e}, \nu_{\mu}, \nu_{\tau})$). The mass term mentioned above would also be characterized by the same symmetry breaking scale such as that of charged leptons or quarks, therefore in this case, very small masses of neutrinos would be very unnatural from the theory point of view.

The Dirac mass matrix $M_{\nu D}$ can be diagonalized by a bi-unitary transformation, e.g.,

$$M_{\nu D}^{\text{diag}} = V_{\nu L}^{\dagger} M_{\nu D} V_{\nu R} = \text{Diag}\{m_{1}, m_{2}, m_{3}\},$$ \hspace{1cm} (2.34)

where $V_{\nu L}$ and $V_{\nu R}$ are unitary matrices and $M_{\nu D}^{\text{diag}}$ is a diagonal matrix and $m_{1}$, $m_{2}$, $m_{3}$ are the eigenvalues of neutrino mass matrix. In a similar manner, one can also diagonalize the charged lepton mass matrices given as

$$M_{l}^{\text{diag}} = V_{l_{L}}^{\dagger} M_{l} V_{l_{R}} = \text{Diag}\{m_{e}, m_{\mu}, m_{\tau}\}.$$ \hspace{1cm} (2.35)

On the other hand, the neutrino might be a Majorana particle which is defined as its own antiparticle and is characterized by only two independent particle states of the same mass ($\nu_{L}$ and $\bar{\nu}_{R}$ or $\nu_{R}$ and $\bar{\nu}_{L}$). A Majorana mass term, which violates both the law of total lepton number conservation and that of individual lepton flavor...
conservation, can be written either as

\[-\mathcal{L}_{M_L} = \frac{1}{2} \bar{\nu}_{eL} M_L \nu_{eR}^c + h.c. \quad (2.36)\]

or as

\[-\mathcal{L}_{M_R} = \frac{1}{2} \bar{\nu}_{eL} M_R \nu_{eR} + h.c., \quad (2.37)\]

where \(M_L\) and \(M_R\) are complex matrices.

A simple extension of the SM is to include one right handed neutrino in each of the three lepton families, while the Lagrangian of the electroweak interactions is kept invariant under \(SU(2)_L \times U(1)_Y\) gauge transformations. This can be shown to lead to Dirac-Majorana mass terms which further lead to the famous seesaw mechanism [46] for the generation of small neutrino masses, e.g.,

\[M_\nu = -M_{L_D}^T (M_R)^{-1} M_{e_D}, \quad (2.38)\]

where \(M_{L_D}\) and \(M_R\) are respectively, the Dirac neutrino mass matrix and the right-handed Majorana neutrino mass matrix.

The seesaw mechanism is based on the assumption that, in addition to the standard Higgs mechanism of generation of the Dirac mass term, there exists a beyond the SM mechanism of generation of the right-handed Majorana mass term, which changes the lepton number by two and is characterized by a mass \(M \gg m\). The Dirac mass term mixes the left-handed field \(\nu_L\), the component of a doublet, with a single field \((\nu^c)_R\). As a result of this mixing the neutrino acquires Majorana mass, which is much smaller than the masses of leptons or quarks.

### 2.5 Lepton mixing matrix

The only term in the lagrangian where the lepton mixing matrix or PMNS matrix \(V_{PMNS}\) enters the charged-current (cc) interaction lagrangian is given by,

\[j^{cc}_{\rho} = 2 \sum_{l=e,\mu,\tau} \bar{l}_L \gamma_\rho \nu_{lL} = 2 \sum_{l=e,\mu,\tau} \sum_{k=1}^{3} \bar{l}_L \gamma_\rho V_{\nu_{lL}}. \quad (2.39)\]
As the phases of Dirac fields are arbitrary one can eliminate from the matrix $V_{PMNS}$ five phases by a redefinition of the phases of the charged lepton and neutrino fields, with one physical phase remaining in $V_{PMNS}$. The presence of this phase causes CP violation in the leptonic sector. This is possible when the neutrinos have only Dirac mass term.

The Lagrangian of charged weak interactions in the mass basis is expressed as

\[ -\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \begin{pmatrix} \nu_1, \nu_2, \nu_3 \end{pmatrix}_L V_{PMNS} \gamma^\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L W^+_\mu + \text{h.c.}, \quad (2.40) \]

wherein only the left-handed leptons take part. The mixing matrix, for Dirac neutrinos is related to the unitary transformations which diagonalize the mass matrices $M_l$ and $M_{\nu D}$ for charged leptons and neutrinos respectively and is given as

\[ V_{\text{Dirac}}^{PMNS} = V_{\nu_L}^\dagger V_{\nu_R} \quad . \quad (2.41) \]

Similar to the quark mixing case, the $n \times n$ lepton mixing matrix $V_{PMNS}^{\text{Dirac}}$ can also be parametrized in terms of $n(n-1)/2$ rotation angles and $(n-1)(n-2)/2$ phases. The $3 \times 3$ unitary matrix $V_{PMNS}$ transforms the neutrino mass eigenstates $(\nu_1, \nu_2, \nu_3)$ to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ as

\[ \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (2.42) \]

where $V_{PMNS}$ is the flavor mixing Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [14] of charged leptons and active neutrinos. The analogue of $V_{PMNS}$ is the well known $V_{\text{CKM}}$ matrix [13] in the quark sector. For the case of three Dirac neutrinos, in the standard Particle Data Group (PDG) parametrization [15], involving three angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and the Dirac-like CP violating phase $\delta$, the matrix $V_{PMNS}$ has the form

\[ V_{\text{Dirac}}^{PMNS} = \begin{pmatrix} C_{12} C_{13} & S_{12} C_{13} & 0 \\ -S_{12} C_{13} - C_{12} S_{23} S_{13} e^{i\delta} & C_{12} C_{23} - S_{12} S_{23} S_{13} e^{i\delta} & S_{12} S_{23} C_{13} \\ S_{12} S_{23} - C_{12} C_{23} S_{13} e^{i\delta} & -C_{12} S_{23} S_{13} e^{i\delta} & C_{23} C_{13} \end{pmatrix}, \quad (2.43) \]
with $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$.

If neutrinos are Majorana particles, however, the situation is different. In this case the neutrino mass matrix $M_R$ has the property $(M_R)^T = M_R$, i.e., $M_R$ is, in general, a complex symmetric $n \times n$ matrix. The diagonalization of $M_R$ needs only a single unitary matrix $\hat{V}_\nu$, with $n=3$ it is given by

$$M_R^{\text{diag}} = \hat{V}_\nu^T M_R \hat{V}_\nu = \text{Diag}\{m_1, m_2, m_3\}. \quad (2.44)$$

Accordingly, the lepton flavor mixing matrix is given by

$$V_{\text{PMNS}}^{\text{Majorana}} = V_{\text{IL}}^T \hat{V}_\nu, \quad (2.45)$$

where $V_{\text{IL}}^T$ is the unitary matrix to diagonalize the charged lepton mass matrix $M_l$. The $n \times n$ flavor mixing matrix $V_{\text{PMNS}}^{\text{Majorana}}$ consists totally of $n^2$ real parameters, and $n(n-1)/2$ of them can always be chosen as rotation angles. Unlike the quark or Dirac neutrino mixing case, there is no freedom to redefine phases of the Majorana neutrino fields, as Majorana particles are their own antiparticles. The number of physical phase angles left in $V_{\text{PMNS}}^{\text{Majorana}}$ is $n(n+1)/2 - n = n(n-1)/2$. Thus $V_{\text{PMNS}}^{\text{Majorana}}$ can be parameterized in terms of $n(n-1)/2$ rotation angles and the same number of phase angles. There are extra phases which cannot be removed in the case of the Majorana neutrinos, therefore, the above matrix $V_{\text{PMNS}}^{\text{Majorana}}$ takes the following form

$$= \begin{pmatrix}
C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta_1} \\
-S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta_1} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_1} & S_{23}C_{13} \\
S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_1} & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta_1} & S_{23}C_{13}
\end{pmatrix} \begin{pmatrix}
\rho e^{i\alpha_1}/2 & 0 & 0 \\
0 & e^{i\alpha_2}/2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (2.46)$$

where $\alpha_1$ and $\alpha_2$ are the Majorana phases which do not play any role in neutrino oscillations. Thus, the observation of neutrino oscillations has triggered a lot of speculation regarding the existence or non existence of CP violation in the leptonic sector.
2.6 Neutrino oscillations

After briefly discussing the neutrino mixing phenomenon, it is interesting to explore its relationship with the neutrino oscillation phenomenon. The neutrino weak interaction flavor eigenstates are superposition of mass eigenstates which are not observable. Using the more usual notations, the three flavor eigenstates $\nu_a$ are related to the three mass eigenstates $\nu_i$ through the unitary transformation i.e.,

$$\nu_a = \sum_i V_{ai} \nu_i,$$

(2.47)

with $i = 1, 2, 3$ and $a = e, \mu, \tau$. $V$ being unitary, the following relation also holds,

$$\nu_i = \sum_a V_{ai} \nu_a.$$

(2.48)

From equation (2.47), it follows that the time evolution of a neutrino with momentum $p$ produced in the state $\nu_a$ at time $t = 0$ is given by

$$\nu(t) = e^{i \phi} \sum_i V_{ai} e^{-i E_i t} \nu_i,$$

(2.49)

where $E_i = \sqrt{m_i^2 + p^2}$ is the energy of the neutrinos. Since $m_i << E_i$,

$$E_i \approx p + \frac{m_i^2}{2p}.$$  

(2.50)

If the masses $m_i$ are not all equal, the three terms of the sum in equation (2.49) get out of phase and the state $\nu(t)$ acquires components $\nu_b$ with $b \neq a$. So neutrino oscillations depict a periodic behaviour which is a typical interference effect and which requires the different mass eigenstates to be coherent. Thus neutrino oscillations are observable only if the neutrino masses are non vanishing and non degenerate. For the sake of simplicity we first discuss the two flavor case.

In this case, the mixing matrix $V$ is described by one real parameter $\theta$ and three phases. The three phases, however, can be rotated away by absorbing them in the neutrino fields. Thus, the mixing matrix is given by
Equation (2.47) implies
\[ V = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \] (2.51)

Equation (2.47) implies
\[ \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \] (2.52)

This gives
\[ \nu_a = \cos \theta \nu_1 + \sin \theta \nu_2, \] (2.53)
\[ \nu_b = -\sin \theta \nu_1 + \cos \theta \nu_2. \] (2.54)

Similarly, equation (2.48) yields
\[ \begin{align*}
\nu_1 &= \cos \theta \nu_a - \sin \theta \nu_b, \\
\nu_2 &= \sin \theta \nu_a + \cos \theta \nu_b.
\end{align*} \] (2.55)

Using the above expressions, equation (2.49) becomes
\[ \nu(t) = e^{i\theta \tau} (\cos \theta e^{-iE_1 t} \nu_1 + \sin \theta e^{-iE_2 t} \nu_2), \] (2.57)

where \( \nu(0) = \nu_a \).

The transition probability to detect a neutrino flavor state \( \nu_b \) at time \( t \) can then be easily calculated to be
\[ P_{ab}(t) = |\langle \nu_b | \nu(t) \rangle|^2 = \sin^2 2\theta \sin^2 \left( \frac{(m_2^2 - m_1^2)t}{4p} \right) \] (2.58)
\[ = \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \frac{\Delta m^2 t}{2p} \right). \] (2.59)

For the survival probability we have
\[ P_{aa} = 1 - \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \frac{\Delta m^2 t}{2p} \right). \] (2.60)
In terms of more familiar units the transition probability can be rewritten as

\[ P_{ab} = \sin^2 \theta \sin^2 \left( \frac{1.267 \Delta m^2 L}{E} \right) = \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \frac{2.53 \Delta m^2 L}{E} \right). \]  

(2.61)

Likewise, the survival probability is given by,

\[ P_{aa} = 1 - \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \frac{2.53 \Delta m^2 L}{E} \right). \]  

(2.62)

In the above equations \( L = ct \) is the distance from the source in metres, \( E \approx p \) is the neutrino energy in MeV and \( \Delta m^2 \) is neutrino mass squared difference in eV^2. The same relation holds if \( L \) is measured in km and \( E \) in GeV.

The transition probability \( P_{ab} \), in terms of neutrino oscillation length \( L_0 \), defined as

\[ L_0 = 4\pi \frac{E}{\Delta m^2}, \]  

(2.63)

becomes

\[ P_{ab} = \frac{1}{2} \sin^2 2\theta \left( 1 - \cos \frac{L}{L_0} \right) \quad (a \neq b). \]  

(2.64)

The expression for the oscillation length can also be written in the form

\[ L_0 = 2.47 \frac{E(\text{MeV})}{\Delta m^2(\text{eV}^2)} \text{ m.} \]  

(2.65)

Neutrino oscillations cannot be observed if the oscillation length is much larger than the distance \( L \) between the neutrino source and the neutrino detector. In order to observe neutrino oscillations, oscillation length must be smaller or of the order of magnitude of \( L \).

In the case of three flavors, the mixing matrices for Dirac neutrinos as well as Majorana neutrinos have already been defined. Coming to the transition probability, we express it in terms of the flavor states \( \nu_\alpha \) and \( \nu_\beta \), as is usually done. The general solution for the probability to detect a neutrino state \( \nu_\beta \) in an initial beam of \( \nu_\alpha \) at a time \( t \) can be given as \[ P(\nu_\alpha \to \nu_\beta) = \left| \sum_{i=1}^{3} V_{\beta i} e^{-iE_i t} V^*_{\alpha i} \right|^2 \]
\[
= \sum_{i=1}^{3} |V_{\beta i}|^2 |V_{\alpha i}|^2 + \text{Re} \sum_{i \neq j} V_{\beta i} V_{\beta j}^* V_{\alpha i}^* V_{\alpha j} \left( e^{-i \theta^2 - \phi^2} \right), \tag{2.66}
\]

where \((\alpha, \beta)\) run over \((e, \mu), (\mu, \tau)\) or \((\tau, e)\). The first term is just the 'classical' probability, all of the phase information is in the last term. In terms of the Jarlskog's rephasing invariant parameter in the leptonic sector \(J_i\), the above conversion probability of a neutrino \(\nu_{\alpha}\) to another neutrino \(\nu_{\beta}\) in vacuum is given by \([101]\)

\[
P(\nu_{\alpha} \rightarrow \nu_{\beta}) = -4 \sum_{i<j} \text{Re}(V_{\alpha i} V_{\beta j} V_{\alpha j} V_{\beta j}^*) \cdot \sin^2 F_{ij} + 8J_i \prod_{i<j} \sin F_{ij}, \tag{2.67}
\]

where \((\alpha, \beta)\) run over \((e, \mu), (\mu, \tau)\) or \((\tau, e)\), and \(F_{ij} \equiv \frac{\Delta m_{ij}^2}{L/E}\) with \(L\) being the baseline length (in unit of km) and \(E\) being the neutrino beam energy (in unit of GeV).

### 2.7 CP violation in the leptonic sector

The observation of neutrino oscillations has triggered a lot of speculation regarding the existence or non existence of CP violation in the leptonic sector. CP is violated in neutrino oscillations if transition probabilities are different for neutrinos and antineutrinos of the same flavor.

As in \(V_{\text{CKM}}\) phenomenology, the \(V_{\text{PMNS}}\) matrix can have several representations because of the freedom of rephasing the charged lepton fields. These different representations are all physically equivalent, therefore, any of these can be used for carrying out the analysis of the neutrino mixing. To estimate the magnitude of CP violation in leptonic sector, one can define the Jarlskog’s rephasing invariant parameter \(J\) \([102]\) as

\[
J \sum_{m,n=1}^{3} \epsilon_{ilm} \epsilon_{jkn} = |Im(V_{ij} V_{ik} V_{kj}^*)|, \tag{2.68}
\]

for any choice of \(i \neq l, j \neq k\). It may be emphasized that \(J\) is an 'invariant function' of mass matrices and is related to the mass matrices as

\[
\text{Det} C = -2J(m_{\nu_e}^2 - m_{\nu_s}^2)(m_{\nu_s}^2 - m_{\nu_s}^2)(m_{\nu_s}^2 - m_{\nu_s}^2)(m_{\nu_s}^2 - m_{\nu_s}^2)(m_{\nu_s}^2 - m_{\nu_s}^2), \tag{2.69}
\]

\[35\]
where \( C = -i[M_2M_1^*, M_1M_2^*] \). It may be mentioned that in the case of neutrino oscillations, the effect of Majorana phases does not appear in the PMNS matrix \([109]\). Therefore, it can be constructed completely in case we know three mixing angles and the Dirac-like phase. Thus the matrix can in principle be constructed by considering the unitarity of the mixing matrix.

Further, similar to quark sector, all CP violating effects are directly proportional to \( J \) in the leptonic sector which, in the PDG representation, is related to Dirac like CP violating phase \( \delta_i \), through the relation

\[
J_i = s_{12} s_{23} s_{13} c_{12} c_{23} c_{13}^2 \sin \delta_i,
\]

with \( s_{ij} = \sin \theta_{ij}, \ c_{ij} = \cos \theta_{ij} \).

### 2.8 The effective Majorana neutrino mass

It is not possible to determine the nature of massive neutrinos by studying the phenomenon of neutrino oscillations. Since neutrino masses are tiny, all observables capable of addressing the Majorana versus Dirac question are very, very rare. The best probe of the Majorana nature of the neutrino and of the conservation of the lepton number is neutrinoless double-beta decay. \( \langle m_{ee} \rangle \) is the effective neutrino mass to which neutrinoless double-beta decay experiments are sensitive, and agrees with the \( ee \)-element of the Majorana neutrino mass matrix in the weak basis where the charged-current and the charged-lepton mass matrices are diagonal, thus \( \langle m_{ee} \rangle \) is a weighted average of the different neutrino masses which is given as

\[
\langle m_{ee} \rangle = m_{\nu_1} V_{e1}^2 + m_{\nu_2} V_{e2}^2 + m_{\nu_3} V_{e3}^2,
\]

where \( m_{\nu_1}, m_{\nu_2} \) and \( m_{\nu_3} \) are the neutrino masses and \( V_{e1}^2, V_{e2}^2 \) and \( V_{e3}^2 \) are the elements of the mixing matrix. The most stringent upper bound on the \( \langle m_{ee} \rangle < (0.35 - 1.05)\text{eV} \) was obtained by Heidelberg-Moscow Ge experiment \([35]\). The IGEX collaboration \([36]\) has obtained the value of \( \langle m_{ee} \rangle < (0.33 - 1.35)\text{eV} \). However, in future, a large number of experiments such as CUORE \([39]\), MOON \([40]\), etc. aim at a sensitivity to \( \langle m_{ee} \rangle \sim (0.01 - 0.05)\text{eV} \).