Chapter VI

Going Beyond Texture 5 Zero
Quark Mass Matrices

Having discussed texture 6 zero and texture 5 zero mass matrices in previous chapters, we have investigated in this chapter next lower order texture 4 zero mass matrices. It was concluded that in Chapter III that Fritzsch like texture 5 zero mass matrices are the only candidates amongst all hermitian texture 5 zero mass matrices, which can accommodate quark mixing data. Even these show limited viability in the sense that the viability is very much dependent on the values of light quark masses used. In this regard, it has been emphasized in the literature that Fritzsch like texture 4 zero quark mass matrices are quite successful in reconciling the strong hierarchy of quark masses and the smallness of flavor mixing angles [50,67,73]. Further, apart from being related to the Nearest Neighbor Interaction (NNI) form of the mass matrices through weak basis rotations [60], these texture 4 zero mass matrices are known to be compatible with specific models of GUTs, e.g., SO(10) [73,76,112,113] and these could be obtained using considerations of Abelian family symmetries [114]. Also, it may be added that the structure of these matrices is very much in agreement with the hypothesis of natural mass matrices advocated by [55].

It may be mentioned that in the case of quark mass matrices, usually the elements are assumed to follow ‘strong hierarchy’, whereas there is no such compulsion for the leptonic mass matrices. Therefore, in case we have to invoke quark-lepton unification [95], it becomes interesting to examine whether ‘weakly hierarchical’ quark mass
matrices are able to reproduce the mixing data which involves strongly hierarchical parameters. This is all the more important as the texture 4 zero mass matrices perhaps provide the simplest parallel structure for quark and lepton mass matrices which are compatible with the low energy data.

Realizing the importance of Fritzsch like Hermitian texture 4 zero mass matrices in the context of quarks, as emphasized above, a few years back Xing and Zhang [75] have attempted to find the parameter space of the elements of these mass matrices. Their analysis has provided good deal of information regarding the space available to various parameters as well as have provided valuable insight into the ‘structural features’ of texture 4 zero mass matrices. In this context, it may be noted that the hierarchy of the elements of the mass matrices is largely governed by the (2,2) element of the matrix. In their analysis, attempt has been made to go somewhat beyond the minimal values of this element, corresponding to the ‘strong hierarchy’ case, however, in case we have to consider the ‘weak hierarchy’ case as well then there seems a further need to consider a still larger range for this element. Further, their analysis has also given valuable clues about the phase structure of the mass matrices, in particular for the strong hierarchy case they conclude that only one of the two phase parameters plays a dominant role. Since, the phases of the mass matrices play a crucial role in giving information about the CP violating phase of the CKM matrix, therefore it would be interesting to find out the ranges of both the phases in the context of strong as well as weak hierarchy of the elements of the mass matrices.

It may be noted that in the last few years there have been considerable improvements in the measurement of CKM parameters and light quark masses. However, at present, even after recent updating by various groups [15,18–20], the situation regarding the element $|V_{ub}|$ is not as clear as for the other CKM matrix elements. For example, as per PDG 2010 [15] its exclusive and inclusive values respectively are $0.0338 \pm 0.0036$ and $0.0427 \pm 0.0038$. It may be noted that although the difference between the exclusive and inclusive values of $|V_{ub}|$ is not statistically significant, however recent unitarity based analyses [115,116] as well as results from a global fit [19] emphasize exclusive value of $|V_{ub}|$. Therefore, it becomes important to examine separately the implications of exclusive and inclusive values of $|V_{ub}|$ for the phenomenology of quark mass matrices.

The purpose of the present chapter is to update and broaden the scope of the
analysis carried out by Xing and Zhang [75] as well as to examine the implications of recent precision measurements on the structural features of texture 4 zero mass matrices for exclusive and inclusive values of $|V_{ub}|$ separately [99]. Taking clue from the quark-lepton symmetry [95] it would also be desirable to explore the parameter space of the elements of the mass matrices, considering not only the usual ‘strong hierarchy’ amongst them but also for the ‘weak hierarchy’ case. In view of the more precise information regarding CP violating parameters, it would be interesting to find out the ranges of both the phases having their origin in the mass matrices which are compatible with the quark mixing data. Further, for the sake of completeness it would also be desirable to construct the $V_{CKM}$ mixing matrix as well as to evaluate the Jarlskog’s rephasing invariant parameter $J$ and the CP violating phase $\delta$.

The detailed plan of the chapter is as follows. In Section (6.1), we detail the essentials of the formalism regarding the texture specific mass matrices. Inputs used in the present analysis and the methodology of the calculations have been given in Section (6.2). The discussion of the results related to exclusive and inclusive values of $|V_{ub}|$ have been presented in Section (6.3). Finally, Section (6.4) summarizes our conclusions.

6.1 Formalism

To begin with, we define the modified Fritzsch like mass matrices, e.g.,

$$M_k = \begin{pmatrix} 0 & A_k & 0 \\ A_k^* & D_k & B_k \\ 0 & B_k^* & C_k \end{pmatrix}, \quad k = U, D, \quad (6.1)$$

$M_U$ and $M_D$, respectively corresponding to the mass matrix in the up sector and the down sector. It may be noted that each of the above matrix is texture 2 zero type with $A_k = |A_k|e^{i\alpha_k}$ and $B_k = |B_k|e^{i\delta_k}$. The various relations between the elements of the mass matrices $A_k, B_k, C_k, D_k$ essentially correspond to the structural features of the mass matrices including their hierarchies.

In the absence of any standard definition in the literature for ‘weak’ and ‘strong’ hierarchy of the elements of the mass matrices, for the purpose of present work we consider these as follows. As is usual, the element $|A_k|$ takes a value much
smaller than the other three elements of the mass matrix which can assume different 
relations amongst each other, defining different hierarchies. For example, in case 
\( D_k < |B_k| < C_k \) it would lead to a strongly hierarchical mass matrix whereas a 
weaker hierarchy of the mass matrix implies \( D_k \lesssim |B_k| \lesssim C_k \). It may also be 
added that for the purpose of numerical work, one can conveniently take the ratio 
\( D_k/C_k \sim 0.01 \) characterizing strong hierarchy whereas \( D_k/C_k \gtrsim 0.2 \) implying weak 
hierarchy. This can be understood by expressing these parameters in terms of the 
quark masses, in particular \( D_U/C_U \sim 0.01 \) implies \( C_U \sim m_t \) and \( D_D/C_D \sim 0.01 \) 
leads to \( C_D \sim m_b \).

To facilitate diagonalization, the complex mass matrix \( M_k \) \((k = U, D)\) can be 
expressed as
\[
M_k = Q_k M_k^* P_k
\]
or
\[
M_k^* = Q_k^* M_k P_k^*
\]
where \( M_k^* \) is a real symmetric matrix with real eigenvalues and \( Q_k \) and \( P_k \) are 
diagonal phase matrices. The matrix \( M_k^* \) can be diagonalized by the orthogonal 
transformation, e.g.,
\[
M_k^{\text{diag}} = O_k^T M_k^* O_k,
\]
where
\[
M_k^{\text{diag}} = \text{diag}(m_1, -m_2, m_3),
\]
the subscripts 1, 2 and 3 referring respectively to \( u, c \) and \( t \) for the \( U \) sector as well 
as \( d, s \) and \( b \) for the \( D \) sector. Using the invariants, \( \text{tr} M_k, \text{tr} M_k^* \) and \( \text{det} M_k \), the 
values of the elements of the mass matrices \( A_k, B_k \) and \( C_k \), in terms of the free 
parameter \( D_k \) and the quark masses are given as
\[
C_k = (m_1 - m_2 + m_3 - D_k),
\]
\[
|A_k| = (m_1 m_2 m_3/C_k)^{1/2},
\]
\[
|B_k| = [(m_3 - m_2 - D_k)(m_3 + m_1 - D_k)(m_2 - m_1 + D_k)/C_k]^{1/2}.
\]
The exact diagonalizing transformation $O_k$ is expressed as

$$
O_k = \begin{pmatrix}
\pm \sqrt{\frac{m_3 m_2 (C_k - m_1)}{(m_3 - m_1)(m_2 + m_1)}} & \pm \sqrt{\frac{m_1 m_3 (C_k + m_2)}{C_k (m_2 + m_1)(m_3 + m_2)}} & \pm \sqrt{\frac{m_1 m_3 (m_3 - C_k)}{C_k (m_3 + m_2)(m_3 - m_1)}} \\
\pm \sqrt{\frac{m_1 (C_k - m_1)}{(m_3 - m_1)(m_2 + m_1)}} & \mp \sqrt{\frac{m_2 (C_k + m_2)}{(m_3 + m_2)(m_2 + m_1)}} & \pm \sqrt{\frac{m_3 (m_3 - C_k)}{(m_3 + m_2)(m_3 - m_1)}} \\
\mp \sqrt{\frac{m_1 (m_2 - C_k)(C_k + m_2)}{C_k (m_2 + m_1)(m_3 + m_2)}} & \pm \sqrt{\frac{m_2 (C_k - m_1)(m_3 - C_k)}{C_k (m_2 + m_1)(m_3 + m_2)}} & \pm \sqrt{\frac{m_3 (C_k - m_1)(C_k + m_2)}{C_k (m_3 + m_2)(m_3 - m_1)}}
\end{pmatrix}.
$$

(6.7)

It may be noted that while finding the diagonalizing transformation $O_k$, one has the freedom to choose several equivalent possibilities of phases. Similarly, while normalizing the diagonalized matrix to quark masses, one again has the freedom to choose the phases for the quark masses. This is due to the fact that the diagonalizing transformations of $M_U$ and $M_D$ occur in a particular manner in the weak charge current interactions of quarks to give the $V_{\text{CKM}}$ mixing matrix. As is usual, we have chosen the phase of $m_2$ to be negative facilitating the diagonalization process as well as the construction of the $V_{\text{CKM}}$ matrix. This is one of the possibilities considered by Xing and Zhang [75], in particular it corresponds to their $(\eta_u, \eta_d) = (-1, -1)$. The other possibilities considered by them are related and are all equivalent as well, these only redefine the phases $\phi_1$ and $\phi_2$ which in any case are arbitrary. For the present work, we have chosen the possibility

$$
O_i = \begin{pmatrix}
O_i(11) & O_i(12) & O_i(13) \\
O_i(21) & -O_i(22) & O_i(23) \\
-O_i(31) & O_i(32) & O_i(33)
\end{pmatrix}.
$$

(6.8)

The mixing matrix $V_{\text{CKM}}$ which measures the non-trivial mismatch between diagonalizations of $M_U$ and $M_D$ can be obtained using $O_{U(D)}$ through the relation

$$
V_{\text{CKM}} = O_U^T P_D P_U^T O_{D}.
$$

(6.9)

Explicitly, the elements of the $V_{\text{CKM}}$ mixing matrix can be expressed as

$$
V_{\text{Im}} = O_U^T_1 O_{1m}^D e^{-i\phi_1} + O_U^T_2 O_{2m}^D + O_U^T_3 O_{3m}^D e^{i\phi_2},
$$

(6.10)
where the subscripts $l$ and $m$ run respectively over $u, c, t$ and $d, s, b$ with $\phi_1 = \alpha_U - \alpha_D, \phi_2 = \beta_U - \beta_D$.

6.2 Inputs used and calculations

The inputs used for carrying out the calculations have already been defined in Chapter III. For ready reference, we reproduce the following ranges of quark masses [17] at the $m_Z$ energy scale, e.g.,

$$m_u = 1.27^{+0.52}_{-0.42} \text{ MeV}, \quad m_d = 2.90^{+4.24}_{-1.19} \text{ MeV}, \quad m_s = 55^{+16}_{-13} \text{ MeV},$$

$$m_c = 0.619 \pm 0.084 \text{ GeV}, \quad m_b = 2.89 \pm 0.09 \text{ GeV}, \quad m_t = 171.7 \pm 3.0 \text{ GeV}. \quad (6.11)$$

The light quark masses $m_u, m_d$ and $m_s$ have been further constrained using the following mass ratios given by [110]

$$m_u/m_d = 0.553 \pm 0.043, \quad m_s/m_d = 18.9 \pm 0.8. \quad (6.12)$$

Further, we have given full variation to the phases $\phi_1$ and $\phi_2$, the parameters $D_U$ and $D_D$ have been given wide variation in conformity with the hierarchy of the elements of the mass matrices e.g., $D_i < C_i$ for $i = U, D$. The extended range of these parameters allows one to carry out the calculations for the case of weak hierarchy of the elements of the mass matrices as well. Also, it needs to be mentioned that the present range of $D_U$ and $D_D$ is much wider than the one considered by Xing and Zhang [75]. In particular, we have considered $D_i/C_i \sim 0.05 - 0.8$, whereas Xing and Zhang have emphasized $D_i/C_i \sim 0.1$ which corresponds to the case of strong hierarchy amongst the elements of the mass matrices. Furthermore, we have imposed the following constraints due to the latest PDG 2010 values [15],

$$|V_{us}| = 0.2253 \pm 0.0007, \quad |V_{cb}| = (0.0410^{+0.0007}_{-0.0011}),$$

$$|V_{ub}|(\text{excl.}) = 0.0338 \pm 0.0036, \quad |V_{ub}|(\text{incl.}) = 0.0427 \pm 0.00380,$$

$$\sin 2\beta = (0.673 \pm 0.023). \quad (6.13)$$

It may be noted that the calculations have been carried out separately for both exclusive and inclusive values of $|V_{ub}|$. 

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6.3 Results and discussion

In view of the fact that one of the aims of the present analysis is to update as well as to extend the analysis of Xing and Zhang [75], for exclusive and inclusive value of $|V_{ub}|$, we have carried out a detailed analysis regarding the structural features of the mass matrices by incorporating the extended ranges of the elements $D_i$ ($i = U, D$) as well as by imposing the constraints given in equation (6.13). To this end, we first present the results pertaining to exclusive value of $|V_{ub}|$, the case of its inclusive value will be discussed later.

6.3.1 Exclusive value of $|V_{ub}|$

As a first step, one can easily determine the relative magnitudes of four non zero matrix elements of $M_{U,D}$ by using the equations (6.6). To begin with, we have plotted the allowed ranges of relative magnitudes of matrix elements $D_u/B_u$ against $D_D/B_D$, $D_u/C_u$ against $D_D/C_D$ and $A_u/D_u$ against $A_D/D_D$ in Figure (6.1). Figure (6.1(a)) clearly brings out that the ratio $D_u/B_u \sim 0.2 - 0.98$ whereas the ratio $D_D/B_D \sim 0.15 - 0.9$. The upper limit of these ranges correspond to the weak hierarchy of the elements of the mass matrices and vice versa. One finds that when the strong hierarchy assumption is considered then the ratio $D_u/B_u$ takes value around 0.25 whereas $D_D/B_D \sim 0.15$ as given by [75].

Similarly, Figure (6.1(b)) clearly brings out that the ratio $D_u/C_u \sim 0.08 - 0.96$ whereas the ratio $D_D/C_D \sim 0.01 - 0.8$, in contrast to the case when the strong hierarchy assumption is considered then the ratio $D_u/C_u$ and $D_D/B_D$ both takes the value around 0.20. In the Figure (6.1(c)) the most favorable values for the relative magnitudes of $A_u/D_u \sim 0.001$ and $A_D/D_D \sim 0.025$. From these figures one may conclude that the ranges of these ratios corresponding to the weak hierarchy cases are much wider in comparison to ratios obtained by [75]. The ranges of the parameters plotted in Figure (6.1) not only have implications for the structural features of the mass matrices, but also indicate that there are large number of possibilities for which one can achieve compatibility of texture 4 zero mass matrices with the $V_{CKM}$ mixing data.
Further, to obtain the numerical results of $C_U/m_t$ versus $C_D/m_b$ as well as to emphasize the role of the hierarchy defining parameters $D_U$ and $D_D$, in Figure (6.2) we have plotted $C_U/m_t$ versus $C_D/m_b$ and $D_U/m_t$ versus $D_D/m_b$, representing an extended range of the parameters $D_U$ and $D_D$.

A look at the Figure (6.2(a)) reveals that both $C_U/m_t$ as well as $C_D/m_b$ take values from $\sim 0.5 - 0.95$, which interestingly indicates the ratios being almost proportional. Also, the figure gives interesting clues regarding the role of strong and weak hierarchy. In particular, one finds that in case one restricts to the assumption of strong hierarchy, then these ratios take large values around 0.95. However, for the case of weak hierarchy, the ratios $C_U/m_t$ and $C_D/m_b$ take much larger number of values, in fact almost the entire range mentioned above, which are compatible with the data.
Figure 6.2: Plot showing the allowed range of (a) $C_U/m_t$ versus $C_D/m_b$ and (b) $D_U/m_t$ versus $D_D/m_b$ for exclusive $|V_{ub}|$.

As a next step, we would like to emphasize the role of the hierarchy defining parameters $D_U$ and $D_D$. To this end, in Figure (6.2(b)) we have plotted $D_U/m_t$ versus $D_D/m_b$, representing an extended range of the parameters $D_U$ and $D_D$. A closer look at the figure reveals both $D_U/m_t$ as well as $D_D/m_b$ take values ~ 0.05 – 0.5. The lower limit of the range i.e. when the ratios $D_U/m_t$ and $D_D/m_b$ are around 0.05 corresponds to strong hierarchy amongst the elements of the mass matrices, whereas when the elements have weak hierarchy then these ratios take a much larger range of values.

From this, one may conclude that in the case of strongly hierarchical elements of the texture 4 zero mass matrices, we have limited compatibility of these matrices with the quark mixing data, whereas the weakly hierarchical ones indicate the compatibility for much broader range of the elements. Further, a comparison of Figure (6.2(a)) with Figure (6.2(b)) shows that there is anti co-relation between the two. For example, weak hierarchy of the mass matrices implies $C_U/m_t$ corresponding to the lower limit of its allowed range, whereas the ratio $D_U/m_t$ corresponding to the higher limit of its allowed range and vice versa.

In Figure (6.3), we have plotted phases $\phi_1$ versus $\phi_2$ by giving full variation to the free parameters $D_U$ and $D_D$, corresponding to both strong as well as weak hierarchy cases. Interestingly, one can conclude from the figure that the present refined inputs limit the ranges of the two phases to $\phi_1 \sim 76^\circ - 92^\circ$ and $\phi_2 \sim 1^\circ - 11^\circ$, which come out to be very narrow. In particular, for the strong hierarchy case one gets $\phi_2 \sim 10^\circ$, whereas for the case of weak hierarchy $\phi_2$ takes almost its entire range mentioned above. Further, it may be mentioned that this figure should not be
directly compared with the corresponding $\phi_1$ versus $\phi_2$ plot given by [75] as they have considered different initial phases. Also, our analysis indicates that although $\phi_1 \gg \phi_2$, still both the phases are required for fitting the mixing data. It may also be noted that the phases $\phi_1$ and $\phi_2$ and the elements of the $V_{\text{CKM}}$ mixing matrix can be easily used to obtain angles of the unitarity triangle.

![Plot showing the allowed range of $\phi_1$ versus $\phi_2$ for exclusive $|V_{ub}|$.](image)

**Figure 6.3:** Plot showing the allowed range of $\phi_1$ versus $\phi_2$ for exclusive $|V_{ub}|$.

Further, a comparison of all these figures with the corresponding plots given by [75] immediately reveals that in the specified ranges of the parameters our results are compatible with theirs. Further, it may be noted that a direct comparison of the ranges of various parameters considered by us and those given by [75] is not possible, however, one may be able to compare the ranges of the elements of the mass matrices, which will be discussed later.

The above discussion can also be understood by the construction of the mass matrices. However, as the phases of the elements of the mass matrices can be separated out, as can be seen from equation (6.3), one needs to consider real mass matrices $M_i^r$ ($i = U, D$) instead of hermitian $M_i$. The ranges of the elements of these matrices $M_U^r$ and $M_D^r$ are as follows

$$M_U^r = m_e \begin{pmatrix} 0 & 0.00017 - 0.00025 & 0 \\ 0.00017 - 0.00025 & 0.0464 - 0.4870 & 0.2184 - 0.5017 \\ 0 & 0.2184 - 0.5017 & 0.5094 - 0.9500 \end{pmatrix}, \quad (6.14)$$
These matrices lead to interesting consequences regarding the structural features characterized by relative magnitudes of the elements of the mass matrices. It may be noted that the elements of the mass matrices $A_i, B_i, C_i$ and $D_i$ satisfy the relation $|B_i|^2 - C_iD_i \approx m_2m_3$ for both the strong and the weak hierarchy cases characterized respectively by $D_i < |B_i| < C_i$ and $D_i \lesssim |B_i| \lesssim C_i$. This relation can be easily derived by using expressions mentioned in equation (6.6) as well as can be numerically checked from the above mentioned mass matrices in equations (6.14) and (6.15). The above constraint on the elements of the mass matrices as well as the ranges of various ratios, particularly in the case of weak hierarchy, provide an interesting possibility for checking the viability of various mass matrices formulated at the GUTs scale or obtained using horizontal symmetries. From a different point of view, this can also provide vital clues for the formulation of mass matrices which are in agreement with the low energy data.

6.3.2 Inclusive values of $|V_{ub}|$

Coming to the results pertaining to inclusive values of $|V_{ub}|$, in Figure (6.4) we have plotted relative magnitudes of non zero matrix elements of $M_{U,D}$ using inclusive value of $|V_{ub}|$. A look at the Figure (6.4(a)) reveals that the ratio $D_U/B_U \sim 0.24 - 0.99$ whereas the ratio $D_D/B_D \sim 0.2 - 0.9$. Similarly, in Figure (6.4(b)) the ratio $D_U/C_U \sim 0.08 - 0.99$ whereas the ratio $D_D/C_D \sim 0.08 - 0.8$. In Figure (6.4(c)) the most favorable values for the relative magnitudes of the following mass matrix elements ratios are, $A_U/D_U \sim 0.0005$ and $A_D/D_D \sim 0.01$. We find that the figures so obtained by using inclusive values $|V_{ub}|$ do not show much change compared to the earlier figures plotted using exclusive values $|V_{ub}|$.
Figure 6.4: Plot showing the allowed ranges of $D_U/B_U$ versus $D_D/B_D$, $D_U/C_U$ versus $D_D/C_D$ and $A_U/D_U$ versus $A_D/D_D$ for inclusive $|V_{us}|$.

In Figure (6.5(a)), a plot of $C_U/m_t$ versus $C_D/m_b$ for $|V_{ub}|$ inclusive values has been shown. The figure indicates that both $C_U/m_t$ as well as $C_D/m_b$ take values from $\sim 0.5 - 0.92$, which interestingly indicates the ratios being almost proportional. Also, the figure gives interesting clues regarding the role of strong and weak hierarchy. In particular, one finds that in case one restricts to the assumption of strong hierarchy then these ratios take large values around 0.92. However, for the case of weak hierarchy, the ratios $C_U/m_t$ and $C_D/m_b$ take much larger number of values.

In Figure (6.5(b)) we have plotted $D_U/m_t$ versus $D_D/m_b$, representing an extended range of the parameters $D_U$ and $D_D$. A closer look at the figure reveals that both $D_U/m_t$ as well as $D_D/m_b$ take values $\sim 0.08 - 0.5$. The lower limit of the range i.e. when the ratios $D_U/m_t$ and $D_D/m_b$ are around 0.08 corresponds to strong hierarchy amongst the elements of the mass matrices, whereas when the elements have weak hierarchy then these ratios take a much larger range of values. From figures (6.5(a)) and (6.5(b)) one concludes that there is anti-correlation between the
two, as already concluded for $|V_{ub}|$ exclusive values also.

Figure 6.5: Plot showing the allowed range of (a) $C_U/m_t$ versus $C_D/m_b$ and (b) $D_U/m_t$ versus $D_D/m_b$ for inclusive $|V_{ub}|$.

In Figure (6.6), we have presented the plot of $\phi_1$ versus $\phi_2$ by giving full variation to the free parameters $D_U$ and $D_D$. The ranges obtained for the two phases are, $\phi_1 \sim 76^\circ - 92^\circ$ and $\phi_2 \sim 1^\circ - 11^\circ$. Interestingly, the result so obtained for the ranges of two phases are exactly similar as obtained with $|V_{ub}|$ exclusive values.

Figure 6.6: Plot showing the allowed range of $\phi_1$ versus $\phi_2$ for inclusive $|V_{ub}|$. 
For comparison with the exclusive $|V_{ub}|$ case, we present below the real mass matrices $M_U$ and $M_D$ constructed by using the inclusive value of $|V_{ub}|$:

$$M_U = m_t \begin{pmatrix} 0 & 0.000179 - 0.000266 & 0 \\ 0.000179 - 0.000266 & 0.0696 - 0.4928 & 0.2610 - 0.5018 \\ 0 & 0.2610 - 0.5018 & 0.5036 - 0.9268 \end{pmatrix},$$  

(6.16)$$

$$M_D = m_b \begin{pmatrix} 0 & 0.003703 - 0.006462 & 0 \\ 0.003703 - 0.006462 & 0.0552 - 0.4586 & 0.2686 - 0.5065 \\ 0 & 0.2686 - 0.5065 & 0.5280 - 0.9236 \end{pmatrix}. $$  

(6.17)$$

A comparison of these matrices with the ones mentioned in equations (6.14) and (6.15) reveals that the (2,2) element $D_i$ of these matrices appear to be quite different for the corresponding $M_U$ and $M_D$ matrices for the case of exclusive and inclusive values of $|V_{ub}|$. Similarly, the lower limits of the element $B_i$ of the mass matrices are quite different for both the $M_U$ and $M_D$ matrices corresponding to exclusive and inclusive values of $|V_{ub}|$. Therefore, it seems that refinements in the evaluation of exclusive and inclusive values of $|V_{ub}|$ would have implications for the hierarchy of the elements of the texture 4 zero mass matrices.

As mentioned earlier, it seems interesting to compare the ranges of the elements of the above mentioned mass matrices with those constructed by Xing and Zhang [75]. The comparison immediately reveals that in the present work we have been able to achieve agreement with the $V_{CKM}$ mixing data for much wider ranges of the elements of the mass matrices. A closer scrutiny of our results reveals that these wider ranges are essentially due to wider ranges for the hierarchy defining parameters $D_U$ and $D_D$. It may be added that in case we restrict ourselves to the strong hierarchy case, then we are able to reproduce the matrices given by [75].

After constructing the mass matrices, it is desirable to construct the corresponding $V_{CKM}$ mixing matrix using the average value of $|V_{ub}|$ and compare it with the one arrived through global analysis. To this end, we have considered the average value of $|V_{ub}|$ given by PDG 2010 [15], the other input parameters remain the same.
as given by equations (6.13). The $V_{\text{CKM}}$ mixing matrix so obtained is as follows

\[
V_{\text{CKM}} = \begin{pmatrix}
0.9738 - 0.9747 & 0.2236 - 0.2274 & 0.0036 - 0.0043 \\
0.2234 - 0.2274 & 0.9729 - 0.9739 & 0.0401 - 0.0423 \\
0.0057 - 0.0114 & 0.0388 - 0.0420 & 0.9991 - 0.9992
\end{pmatrix}.
\] (6.18)

A general look at the matrix reveals that the ranges of $V_{\text{CKM}}$ elements obtained here are quite compatible with those obtained by recent global analysis [15,18-20]. We have also evaluated the Jarlskog’s rephasing invariant parameter $J$ using the average value of $|V_{ud}|$ which comes out to be

\[ J = (1.807 - 3.977) \times 10^{-5}. \] (6.19)

Further, using this value of $J$ we obtain the following range of the CP violating phase $\delta$

\[ \delta = 28.8^\circ - 110.4^\circ. \] (6.20)

The above mentioned ranges of the parameter $J$ and the phase $\delta$ accommodate the values given by PDG (2010) [15].

### 6.4 Summary and conclusions

In the light of recent precision measurements, in this chapter we have made an attempt to update and broaden the scope of the analysis carried out by Xing and Zhang [75] as well as have carried out a detailed analysis regarding the structural features of the mass matrices. The implications of these measurements on the texture 4 zero mass matrices have been examined by considering not only the usual ‘strong hierarchy’ amongst the elements of these matrices, defined as $D_i < |B_i| < C_i$, but also for the ‘weak hierarchy’ case given by $D_i \lesssim |B_i| \lesssim C_i$. For both the exclusive and inclusive values of $|V_{ud}|$, the analysis has been carried out by giving wide variation to the hierarchy defining parameters $D_U$ and $D_D$. We found that in case we assume strong hierarchy of elements of the texture 4 zero mass matrices, we have limited compatibility of these matrices with the quark mixing data, whereas for the weakly hierarchical mass matrices, the compatibility is obtained with much broader range of the elements. Using $|V_{ud}|$ exclusive values, figures obtained do not
show much difference as plotted by using $|V_{ub}|$ inclusive values. The ranges obtained for the magnitudes of matrix elements not only have implications for the structural features of the mass matrices, but also indicate that there are large number of possibilities for which one can achieve compatibility of texture 4 zero mass matrices with $V_{CKM}$ mixing data.

Further, in view of the more precise information regarding CP violating parameters, the ranges of both the phases $\phi_1$ and $\phi_2$, having their origin in the mass matrices, have been found. We found that despite considering weak hierarchy, still both the phases are required to fit the data, in particular these come out to be $\phi_1 \sim 76^\circ - 92^\circ$ and $\phi_2 \sim 1^\circ - 11^\circ$. Also for both the exclusive and inclusive values of $|V_{ub}|$, the texture 4 zero mass matrices are compatible with recent results obtained from global fits [15,18–20] for weak as well as strong hierarchy of the elements of the mass matrices. A comparison of $M^U_i$ and $M^D_i$ matrices corresponding to exclusive and inclusive values of $|V_{ub}|$ reveals that the parameters $D_i$ and $B_i$ $(i = U, D)$ would have implications for these values of $|V_{ub}|$.

In conclusion, we would like to state that even weakly hierarchical mass matrices can explain the quark masses and mixing matrix, which are strongly hierarchical. This, in turn, would have important implications for model building of the fermion mass matrices.