Chapter 3

Elastic waves at a corrugated interface between two dissimilar fibre-reinforced elastic half-spaces

3.1 Introduction

Singh and Singh (2004) explored the propagation of plane waves in an infinite fibre-reinforced elastic medium and showed that there can propagate longitudinal and transverse waves. These waves propagate with distinct velocity, but their phase speeds depend upon their angle of propagation. Due to the dependence of their phase velocity on their direction of propagation, these waves are quasi-waves. They have also investigated the phenomena of reflection of plane elastic waves from a stress free boundary surface of a fibre-reinforced elastic half-space and obtained the expressions of reflection coefficients, when a plane $qP/ qSV$– wave is made incident at the boundary surface. In this chapter, a model consisting of two dissimilar fibre reinforced elastic half-spaces exhibiting different elastic properties and separated by a corrugated interface, has been considered. These two fibre reinforced half-spaces possess anisotropic property due to the fibre reinforcement. In reference to the geophysical Earth model, the sedimentary rocks in their natural states may be anisotropic of that kind. Thus, the present model may be a good approximation to such geophysical earth model. As the seismic waves originated from Earthquake focus or from similar disturbances, travel through different

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layers inside the earth, they suffer reflection and refraction at the surfaces of separa­
tions. Therefore, it is important to study the mechanics of propagation of these elastic
waves in anisotropic media. This is also helpful in understanding the composition
of crustal layers and in exploration of valuable materials like minerals, ore metals and
crystals, etc. beneath the Earth surface. We have investigated the reflection and trans­
mission phenomena of a plane harmonic $q_{P}/q_{SV}$– wave incident at the corrugated
interface between two dissimilar fibre-reinforced elastic half-spaces. Using Rayleigh’s
method of approximation, the reflection and transmission coefficients are derived for a
general corrugated interface and for a periodic type of interface. The results of Singh
and Singh (2004), and Ben-Menaham and Singh (1981) have been recovered from the
present problem as particular cases.

3.2 Problem and solution

Let the equation of corrugated interface between two different homogeneous fibre­
reinforced elastic half-spaces $H_1[y \leq y < \infty]$ and $H_2[-\infty < y \leq \zeta(x)]$ be given by
$y = \zeta(x)$. Fourier series representation of the function $\zeta(x)$ is given by

$$\zeta(x) = \sum_{n=1}^{\infty} (\zeta_n e^{inpx} + \zeta_{-n} e^{-inpx}), \quad (3.1)$$

where $\zeta(x)$ is a periodic function of $x$ and independent of $z$, whose mean value is zero,
the coefficients $\zeta_n$, $\zeta_{-n}$ and the quantity $p$ are defined earlier in the previous chapter.
Introducing the constants $d$, $c_n$ and $s_n$ as in the previous chapter and putting into
equation (3.1), we obtain

$$\zeta = d \cos px + \sum_{n=2}^{\infty} (c_n \cos npx + s_n \sin npx). \quad (3.2)$$

The two dimensional equations of motion and constitutive relations in fibre-reinforced
elastic medium are given in (1.64) - (1.67) and are re-written here for the half-spaces
$H_l$, $(l = 1, 2)$ as

$$P_{\nu} \frac{\partial^2 u_l}{\partial x^2} + \mu_{kl} \frac{\partial^2 u_l}{\partial y^2} + Q_{kl} \frac{\partial^2 v_l}{\partial x \partial y} = \rho_{l} \frac{\partial^2 u_l}{\partial t^2}, \quad (3.3)$$
\[ P_2 \frac{\partial^2 v_l}{\partial y^2} + \mu_l \frac{\partial^2 v_l}{\partial x^2} + Q_{ul} \frac{\partial^2 u_l}{\partial x \partial y} = \rho \frac{\partial^2 v_l}{\partial t^2} \]  

(3.4)

and

\[ (\tau_{11})_l = P_{ul} \frac{\partial u_l}{\partial x} + T_{ul} \frac{\partial v_l}{\partial y}, \quad (\tau_{21})_l = T_{ul} \frac{\partial u_l}{\partial x} + P_{gl} \frac{\partial v_l}{\partial y}, \quad (\tau_{12})_l = \mu_{l_u} \left( \frac{\partial u_l}{\partial y} + \frac{\partial v_l}{\partial x} \right), \]  

(3.5)

where

\[ Q_{ul} = \alpha_l + \lambda_l + \mu_{l_u}, \quad P_{ul} = \lambda_l + 2\alpha_l + 4\mu_{l_u} - 2\mu_{T_l} - \beta_l, \quad P_{gl} = \lambda_l + 2\mu_{T_l}, \quad T_{ul} = \lambda_l + \alpha_l. \]

Consider a plane \( qP \) (or \( qSV \)) wave propagating through the medium \( H_1 \) and striking at the corrugated interface making an angle \( \theta_0 \) with the \( y \)-axis. Due to corrugated nature of the interface, the reflection and transmission of waves will be affected. Thus, an incident plane wave will give rise to non-specular waves (irregularly reflected and transmitted waves), in addition to specular waves (regularly reflected and transmitted waves) (see Asano, 1961). The geometry of the problem is shown in Figure 3.1.

\[ \text{Figure 3.1: Geometry of the Problem} \]
The \( x \)- and \( y \)- components of total displacement field in the half-space \( H_1 \) caused by the incident and reflected waves, can be written as

\[
u_1 = (m_1 + n_1)A_0e^{i\theta_0} + A_1e^{i\theta_1} + A_2e^{i\theta_2} + \sum_{n=1}^{\infty} \sum_{j=1}^{2} [A_{jn}e^{i\Omega_{jn}} + A_{jn}e^{i\Omega_{jn}}],\tag{3.6}
\]

\[
u_1 = (m_1 + n_1)B_0e^{i\theta_0} + B_1e^{i\theta_1} + B_2e^{i\theta_2} + \sum_{n=1}^{\infty} \sum_{j=1}^{2} [B_{jn}e^{i\Omega_{jn}} + B_{jn}e^{i\Omega_{jn}}],\tag{3.7}
\]

where \( \Omega_0 = \frac{\omega}{c_{1,2}}[c_{1,2}t - (x \sin \theta_0 - y \cos \theta_0)] \) is the phase factor of the incident \( qP \) (or \( qSV \)) wave at an angle \( \theta_0 \) and with \( A_0 \) and \( B_0 \) as displacement constants. In the case, when a plane \( qP \) wave is incident, we shall consider the quantity \( c_1 \) in \( \Omega_0 \), while the quantity \( c_2 \) will be considered in \( \Omega_0 \) only when a plane \( qSV \) wave is incident.

\( \Omega_1 = \frac{\pi}{c_1}[c_1t - (x \sin \theta + y \cos \theta)] \) is the phase factor for regularly reflected \( qP \) wave at an angle \( \theta \) and with \( A_1 \) and \( B_1 \) as displacement constants,

\( \Omega_{1n}^{\pm} = \frac{\pi}{c_1}[c_1t - (x \sin \theta^\pm_n + y \cos \theta^\pm_n)] \) are phase factors for irregularly reflected \( qP \) waves at angles \( \theta^\pm_n \) and with \( A_{1n}^{\pm} \) and \( B_{1n}^{\pm} \) as displacement constants,

\( \Omega_2 = \frac{\pi}{c_2}[c_2t - (x \sin \phi + y \cos \phi)] \) is the phase factor for regularly reflected \( qSV \) wave at an angle \( \phi \) and with \( A_2 \) and \( B_2 \) as the displacement constants,

\( \Omega_{2n}^{\pm} = \frac{\pi}{c_2}[c_2t - (x \sin \phi^\pm_n + y \cos \phi^\pm_n)] \) are phase factors of irregularly reflected \( qSV \) waves at angles \( \phi^\pm_n \) and with \( A_{2n}^{\pm} \) and \( B_{2n}^{\pm} \) as displacement constants.

The expressions of \( c_{1,2} \) in the medium \( H_1 \) are given by

\[
2\rho_1 c_1^2 = \Psi_{11}(\theta) + \Psi_{21}(\theta) + \sqrt{[\Psi_{11}(\theta) - \Psi_{21}(\theta)]} + 4Q_{01}^2 \sin^2 \theta \cos^2 \theta,\tag{3.8}
\]

\[
2\rho_1 c_2^2 = \Psi_{11}(\phi) + \Psi_{21}(\phi) - \sqrt{[\Psi_{11}(\phi) - \Psi_{21}(\phi)]} + 4Q_{01}^2 \sin^2 \phi \cos^2 \phi,\tag{3.9}
\]

where

\[
\Psi_{11}(\theta) = P_{11} \sin^2 \theta + \mu_{L_1} \cos^2 \theta, \quad \Psi_{21}(\theta) = P_{21} \cos^2 \theta + \mu_{L_1} \sin^2 \theta,
\]

\[
\Psi_{11}(\phi) = P_{11} \sin^2 \phi + \mu_{L_1} \cos^2 \phi, \quad \Psi_{21}(\phi) = P_{21} \cos^2 \phi + \mu_{L_1} \sin^2 \phi.
\]
Similarly, the relevant components of total displacement field in the half-space $H_2$ are the sum of the displacements produced by the regularly and irregularly transmitted waves. Hence, we can write

$$u_2 = A_3e^{i\Omega_3} + A_4e^{i\Omega_4} + \sum_{n=1}^{\infty} [A_{3n}e^{iQ_{3n}} + A_{3n}e^{iQ_{3n}^*} + A_{4n}e^{iQ_{4n}} + A_{4n}e^{iQ_{4n}^*}], \quad (3.10)$$

$$v_2 = B_3e^{iQ_3} + B_4e^{iQ_4} + \sum_{n=1}^{\infty} [B_{3n}e^{iQ_{3n}} + B_{3n}e^{iQ_{3n}^*} + B_{4n}e^{iQ_{4n}} + B_{4n}e^{iQ_{4n}^*}], \quad (3.11)$$

where

$$\Omega_3 = \frac{\pi}{c_3}[c_3t - (x \sin \delta - y \cos \delta)]$$

is the phase factor of the regularly transmitted $qP-$ wave at an angle $\delta$ and with $A_3$ and $B_3$ as the displacement constants,

$$\Omega_{3n}^\pm = \frac{\pi}{c_3}[c_3t - (x \sin \delta_n^\pm - y \cos \delta_n^\pm)]$$

are phase factors for irregularly transmitted $qP-$ waves at angles $\delta_n^\pm$ and with $A_{3n}$ and $B_{3n}$ as displacement constants,

$$\Omega_4 = \frac{\pi}{c_4}[c_4t - (x \sin \gamma - y \cos \gamma)]$$

is the phase factor for regularly transmitted $qSV-$ wave at an angle $\gamma$ and with $A_4$ and $B_4$ as displacement constants,

$$\Omega_{4n}^\pm = \frac{\pi}{c_4}[c_4t - (x \sin \gamma_n^\pm - y \cos \gamma_n^\pm)]$$

are phase factors for irregularly transmitted $qSV-$ waves at angles $\gamma_n^\pm$ and with $A_{4n}$ and $B_{4n}$ as displacement constants.

The expressions of $\Omega_{3,4}$ in the medium $H_2$ are given by

$$2\rho c_3^2 = \Psi_{12}(\delta) + \Psi_{22}(\delta) + \sqrt{\{\Psi_{12}(\delta) - \Psi_{22}(\delta)\}^2 + 4Q_{12}^2 \sin^2 \delta \cos^2 \delta}, \quad (3.12)$$

$$2\rho c_4^2 = \Psi_{12}(\gamma) + \Psi_{22}(\gamma) - \sqrt{\{\Psi_{12}(\gamma) - \Psi_{22}(\gamma)\}^2 + 4Q_{12}^2 \sin^2 \gamma \cos^2 \gamma}, \quad (3.13)$$

where

$$\Psi_{12}(\delta) = P_{12} \sin^2 \delta + \mu_{L2} \cos^2 \delta, \quad \Psi_{22}(\delta) = P_{22} \cos^2 \delta + \mu_{L2} \sin^2 \delta,$$

$$\Psi_{12}(\gamma) = P_{12} \sin^2 \gamma + \mu_{L2} \cos^2 \gamma, \quad \Psi_{22}(\gamma) = P_{22} \cos^2 \gamma + \mu_{L2} \sin^2 \gamma.$$

We shall take $(m_1, n_1) = (1, 0)$ for the incident $qP-$ wave, and $(m_1, n_1) = (0, 1)$ for the incident $qSV-$ wave. The quantities $c_1$ and $c_3$ are the phase speeds of $qP-$ waves,
while \(c_2\) and \(c_4\) are the phase speeds of \(qSV\) waves in medium \(H_1\) and \(H_2\) respectively. Since the displacements of each of the incident waves, regularly as well as irregularly reflected and transmitted \(qP\) and \(qSV\) waves satisfy the equations (3.3) and (3.4), therefore, one can obtain (see Singh and Singh, 2004)

\[
A_0 = \eta_0 B_0, \quad A_i = \pm \eta_i B_i, \quad A_{jn}^\pm = \pm \eta_{jn}^\pm B_{jn}^\pm, \quad (3.14)
\]

where negative sign refers for \(i = 1, 2\) and positive sign refers for \(i = 3, 4\),

\[
\eta_{10} = \frac{Q_{01} \sin \theta_0 \cos \theta_0}{\Psi_{11}(\theta_0) - \rho_1 c_1^2}, \quad \eta_1 = \frac{Q_{01} \sin \theta \cos \theta}{\Psi_{11}(\theta) - \rho_1 c_1^2}, \quad \eta_{1n}^+ = \frac{Q_{01} \sin \theta_0^+ \cos \theta_0^+}{\Psi_{11}(\theta_0^+) - \rho_1 c_1^2},
\]

\[
\eta_{1n}^- = \frac{Q_{01} \sin \theta_0^- \cos \theta_0^-}{\Psi_{11}(\theta_0^-) - \rho_1 c_1^2}, \quad \eta_2 = \frac{Q_{01} \sin \phi \cos \phi}{\Psi_{11}(\phi) - \rho_1 c_2^2}, \quad \eta_{2n}^+ = \frac{Q_{01} \sin \phi_0^+ \cos \phi_0^+}{\Psi_{11}(\phi_0^+) - \rho_1 c_2^2},
\]

\[
\eta_{2n}^- = \frac{Q_{01} \sin \phi_0^- \cos \phi_0^-}{\Psi_{11}(\phi_0^-) - \rho_1 c_2^2}, \quad \eta_3 = \frac{Q_{02} \sin \delta \cos \delta}{\Psi_{12}(\delta) - \rho_2 c_3^2}, \quad \eta_{3n}^+ = \frac{Q_{02} \sin \delta_0^+ \cos \delta_0^+}{\Psi_{12}(\delta_0^+) - \rho_2 c_3^2},
\]

\[
\eta_{3n}^- = \frac{Q_{02} \sin \delta_0^- \cos \delta_0^-}{\Psi_{12}(\delta_0^-) - \rho_2 c_3^2}, \quad \eta_4 = \frac{Q_{02} \sin \gamma \cos \gamma}{\Psi_{12}(\gamma) - \rho_2 c_4^2}, \quad \eta_{4n}^+ = \frac{Q_{02} \sin \gamma_0^+ \cos \gamma_0^+}{\Psi_{12}(\gamma_0^+) - \rho_2 c_4^2},
\]

\[
\eta_{4n}^- = \frac{Q_{02} \sin \gamma_0^- \cos \gamma_0^-}{\Psi_{12}(\gamma_0^-) - \rho_2 c_4^2}.
\]

The Snell’s law is the same as given in (2.10) and re-written here as

\[
\sin \theta_0 = \sin \theta \cdot \frac{c_1(\theta)}{c_1(\theta)} = \sin \phi \cdot \frac{c_2(\phi)}{c_2(\phi)} = \sin \delta \cdot \frac{c_3(\delta)}{c_3(\delta)} = \sin \gamma \cdot \frac{c_4(\gamma)}{c_4(\gamma)} = \frac{1}{c_n}, \quad (3.15)
\]

The Spectrum theorem is also similar to (2.11) and re-written here as

\[
\sin \theta_n^\pm - \sin \theta = \pm \frac{n p c_1}{\omega}, \quad \sin \phi_n^\pm - \sin \phi = \pm \frac{n p c_2}{\omega},
\]

\[
\sin \delta_n^\pm - \sin \delta = \pm \frac{n p c_3}{\omega}, \quad \sin \gamma_n^\pm - \sin \gamma = \pm \frac{n p c_4}{\omega}. \quad (3.16)
\]

### 3.3 Boundary conditions

The appropriate boundary conditions are the continuity of displacements and stresses at the corrugated interface \(y = \zeta(x)\). Mathematically, these boundary conditions can be expressed as
(i) corresponding to the continuity of displacement components

\[ v_1 = v_2, \quad u_1 = u_2, \]  

(3.17)

(ii) corresponding to the continuity of shear stress

\[ \zeta' \left[ P_{31} \frac{\partial v_1}{\partial x} + P_{41} \frac{\partial u_1}{\partial y} \right] + (1 - \zeta'^2) \mu L_1 \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) + \zeta \left[ P_{32} \frac{\partial u_2}{\partial x} + P_{42} \frac{\partial v_2}{\partial y} \right] + (1 - \zeta'^2) \mu L_2 \left( \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right), \]  

(3.18)

(iii) corresponding to the continuity of normal stress

\[ (\zeta'^2 P_{11} + T_{11}) \frac{\partial u_1}{\partial x} - 2\zeta' \mu L_1 \left( \frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) + (\zeta'^2 T_{12} + P_{21}) \frac{\partial v_1}{\partial y} \]

\[ \quad = (\zeta'^2 P_{12} + T_{12}) \frac{\partial u_2}{\partial x} - 2\zeta' \mu L_2 \left( \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) + (\zeta'^2 T_{12} + P_{22}) \frac{\partial v_2}{\partial y}, \]  

(3.19)

where \( P_{3l} = T_{ll} - P_{2l}, \quad P_{4l} = P_{2l} - T_{ll}. \)

Inserting the expressions of displacements from equations (3.6), (3.7), (3.10) and (3.11) and using (3.15) and (3.16), into the boundary conditions given by (3.17) - (3.19), we obtain

\[ (m_1 + n_1) B_0 e^{\kappa R_0} + B_1 e^{-\kappa R} + B_2 e^{-\kappa Q} + \sum_{n=1}^{\infty} \left\{ \left( B_{1n}^+ e^{-\kappa R_n^+} + B_{2n}^+ e^{-\kappa Q_n^+} \right) e^{-mpx} \right\} 

\right. 

\left. + \left( B_{1n}^- e^{-\kappa R_n^-} + B_{2n}^- e^{-\kappa Q_n^-} \right) e^{mpx} \right] = B_3 e^{\kappa S} + B_4 e^{\kappa L} + \sum_{n=1}^{\infty} \left\{ \left( B_{3n}^+ e^{\kappa S_n^+} 

\right. 

\left. + B_{4n}^+ e^{\kappa L_n^+} \right) e^{-mpx} + \left( B_{3n}^- e^{\kappa S_n^-} + B_{4n}^- e^{\kappa L_n^-} \right) e^{mpx} \right\}, \]  

(3.20)

\[ (m_1 + n_1) A_0 e^{\kappa R_0} + A_1 e^{-\kappa R} + A_2 e^{-\kappa Q} + \sum_{n=1}^{\infty} \left\{ \left( A_{1n}^+ e^{-\kappa R_n^+} + A_{2n}^+ e^{-\kappa Q_n^+} \right) e^{-mpx} 

\right. 

\left. + \left( A_{1n}^- e^{-\kappa R_n^-} + A_{2n}^- e^{-\kappa Q_n^-} \right) e^{mpx} \right] = A_3 e^{\kappa S} + A_4 e^{\kappa L} + \sum_{n=1}^{\infty} \left\{ \left( A_{3n}^+ e^{\kappa S_n^+} 

\right. 

\left. + A_{4n}^+ e^{\kappa L_n^+} \right) e^{-mpx} + \left( A_{3n}^- e^{\kappa S_n^-} + A_{4n}^- e^{\kappa L_n^-} \right) e^{mpx} \right\} \]
\[ e^{-\kappa R} \sum_{n=1}^{\infty} \frac{1}{(P_0 + np)} \left\{ (P_{01} + np) \right\} e^{-\kappa R} + \cdots \]
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\[ +P_{21} B_{2n} Q_{n} \text{e}^{-i\kappa Q_{n} x_{\text{mp}} x_{\text{mp}}} = \left\{ \left( P_{12} \zeta + T_{12} \right) A_{3} - 2 \mu \zeta B_{3} \right\}_{0} + \left\{ -2 \mu \zeta A_{3} \right\}_{0} + \left( \zeta^{2} T_{12} + P_{22} B_{3} \right) S_{0} \text{e}^{i\kappa S_{0} x_{\text{mp}}} + \left\{ \left( P_{12} \zeta^{2} + T_{12} \right) A_{4} - 2 \mu \zeta B_{4} \right\}_{0} + \left\{ -2 \mu \zeta A_{4} \right\}_{0} + \left( \zeta^{2} T_{12} + P_{22} B_{4} \right) S_{0} \right\} L_{e}^{\kappa L} - \sum_{n=1}^{\infty} \left\{ \left( P_{0} + np \right) \left\{ \left( P_{12} \zeta + T_{12} \right) A_{3n}^{+} - 2 \mu \zeta B_{3n}^{+} \right\}_{0} + \left\{ 2 \mu \zeta A_{3n}^{+} \right\}_{0} - \left( \zeta^{2} T_{12} + P_{22} B_{3n}^{+} \right) S_{n}^{+} \text{e}^{i\kappa S_{n}^{+} x_{\text{mp}} x_{\text{mp}}} + \left( P_{0} - np \right) \left\{ \left( P_{12} \zeta^{2} + T_{12} \right) A_{4n}^{+} - 2 \mu \zeta B_{4n}^{+} \right\}_{0} + \left\{ 2 \mu \zeta A_{4n}^{+} \right\}_{0} - \left( \zeta^{2} T_{12} + P_{22} B_{4n}^{+} \right) S_{n}^{+} \right\} L_{n}^{+} \text{e}^{i\kappa L_{n}^{+} x_{\text{mp}} x_{\text{mp}}} + \left( P_{0} - np \right) \left\{ \left( P_{12} \zeta^{2} + T_{12} \right) A_{4n}^{-} - 2 \mu \zeta B_{4n}^{-} \right\}_{0} + \left\{ 2 \mu \zeta A_{4n}^{-} \right\}_{0} - \left( \zeta^{2} T_{12} + P_{22} B_{4n}^{-} \right) S_{n}^{-} \right\} L_{n}^{-} \text{e}^{i\kappa L_{n}^{-} x_{\text{mp}} x_{\text{mp}}} \right\} \right\} \] (3.23)

where

\[ \zeta^{c} = \sum_{n=1}^{\infty} \left( \zeta_{n} \text{e}^{i\kappa x_{\text{mp}}} - \zeta_{-n} \text{e}^{-i\kappa x_{\text{mp}}} \right) m_{n}, \quad P_{0} = \frac{\omega \sin \theta_{0}}{c_{1,2}} = \frac{\omega \sin \theta}{c_{1}} = \frac{\omega \sin \delta}{c_{2}} = \frac{\omega \sin \gamma}{c_{3}}, \]
\[ R_{0} = \frac{\omega \cos \theta_{0}}{c_{0}}, \quad R_{n}^{+} = \frac{\omega \cos \theta_{n}^{+}}{c_{1}}, \quad R_{n}^{-} = \frac{\omega \cos \theta_{n}^{-}}{c_{1}}, \quad Q = \frac{\omega \cos \phi}{c_{2}}, \]
\[ Q_{n}^{+} = \frac{\omega \cos \phi_{n}^{+}}{c_{2}}, \quad Q_{n}^{-} = \frac{\omega \cos \phi_{n}^{-}}{c_{2}} \]
\[ S = \frac{\omega \cos \delta}{c_{3}}, \quad S_{n}^{+} = \frac{\omega \cos \delta_{n}^{+}}{c_{3}}, \quad S_{n}^{-} = \frac{\omega \cos \delta_{n}^{-}}{c_{3}}, \]
\[ L = \frac{\omega \cos \gamma}{c_{4}}, \quad L_{n}^{+} = \frac{\omega \cos \gamma_{n}^{+}}{c_{4}}, \quad L_{n}^{-} = \frac{\omega \cos \gamma_{n}^{-}}{c_{4}}. \] (3.24)

Equations (3.20) - (3.23) will provide the ratios of the displacement constants and hence we shall be able to find the reflection and transmission coefficients for any order of approximation of the corrugation.

### 3.4 Solution of the first order approximation

As in the previous chapter, the amplitude and slope of the corrugated interface are assumed to be small so that

\[ \text{exp}(\pm i\zeta R_{0}) \approx 1 \pm i\zeta R_{0}. \] (3.25)
Plugging the equations (3.1), (3.14) and (3.25) into equations (3.20) - (3.23) and equating the terms independent of $x$ and $\zeta$ to both sides of the resulting equations, we obtain

$$B_1 + B_2 - B_3 - B_4 = -(m_1 + n_1)B_0,$$  \hspace{1cm} (3.26)

$$\eta_1B_1 + \eta_2B_2 + \eta_3B_3 + \eta_4B_4 = (n_1 + m_1)\eta_{10}B_0,$$  \hspace{1cm} (3.27)

$$(P_0 - \eta_1R)B_1 + (P_0 - \eta_2Q)B_2 + \frac{\mu_{L_2}}{\mu_{L_1}}(\eta_2S - P_0)B_3 + \frac{\mu_{L_2}}{\mu_{L_1}}(\eta_4L - P_0)B_4$$

$$= (\eta_{10}R_0 - P_0)(n_1 + m_1)B_0,$$  \hspace{1cm} (3.28)

$$\{T_1n\eta_1P_0 - P_{21}R\}B_1 + \{T_1n\eta_2P_0 + P_{21}Q\}B_2 + \{T_1n\eta_3P_0 - P_{22}S\}B_3 + \{T_1n\eta_4P_0 - P_{22}L\}B_4$$

$$= \{\eta_{10}T_1nP_0 - P_{21}R_0\}(n_1 + m_1)B_0.$$  \hspace{1cm} (3.29)

Next, equating the coefficients of $e^{-\mu x}$ to both sides, we get

$$B_{1n}^+ + B_{2n}^+ - B_{3n}^+ - B_{4n}^+ = \kappa_{-n}[(m_1 + n_1)\rho_0B_0 + RB_1 + QB_2 + SB_3 + LB_4],$$  \hspace{1cm} (3.30)

$$\eta_{1n}B_{1n}^+ + \eta_{2n}B_{2n}^+ + \eta_{3n}B_{3n}^+ + \eta_{4n}B_{4n}^+$$

$$= \kappa_{-n}[(m_1 + n_1)\rho_0B_0 + \rho RB_1 + \rho QB_2 - \eta_3SB_3 - \eta_4LB_4],$$  \hspace{1cm} (3.31)

$$\mu_{L_1}\{\eta_{1n}R_n^+ - P_0 - np\}B_{1n}^+ + \eta_{2n}Q_n^+ - P_0 - np)B_{2n}^+ \} - \mu_{L_2}\{\eta_{3n}S_n^+ - P_0 - np)B_{3n}^+$$

$$+(\eta_{4n}L_n^+ - P_0 - np)B_{4n}^+ \} = \kappa_{-n}[(m_1 + n_1)\{-npP_{31}P_0 + \mu_{L_1}R_n^2\}\eta_{1n} + \mu_{L_1}\rho_0\rho_0$$

$$+npP_{41}\rho_0\}B_0 + \{(npP_{31}R + \mu_{L_1}R_n^2)\eta_1 - \mu_{L_1}\rho_0P_0 - npP_{21}R\}B_1 + \{(npP_{31}P_0 + \mu_{L_1}Q_n^2)\eta_2$$

$$-\mu_{L_1}\rho_0Q - npP_{41}Q\}B_2 + \{(npP_{32}P_0 + \mu_{L_2}S_n^2)\eta_3 - \mu_{L_2}\rho_0S - npP_{42}S\}B_3$$

$$+\{(npP_{32}P_0 + \mu_{L_2}L_n^2)\eta_4 - \mu_{L_2}\rho_0L - npP_{42}L\}B_4],$$  \hspace{1cm} (3.32)

$$\{T_1n\eta_{1n}(P_0 + np) - P_{21}R_n^+\}B_{1n}^+ + \{\eta_{2n}T_1n(P_0 + np) - P_{21}Q_n^+\}B_{2n}^+ + \{\eta_{3n}T_1n(P_0 + np)$$

$$-P_{22}S_n^+\}B_{3n}^+ + \{\eta_{4n}T_1n(P_0 + np) - P_{22}L_n^+\}B_{4n}^+ = \kappa_{-n}[(m_1 + n_1)\{(T_1nP_0 - 2np\mu_{L_1})\rho_0\eta_{10}$$

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Similarly, equating the coefficients of $e^{in \pi r}$ to both sides, we have

$$B_{1n} + B_{2n} - B_{3n} - B_{4n} = \zeta_n \left[ \left( m_1 + m_1 \right) R_0 B_0 + \nu_{1n} R_1 B_1 + \zeta_2 Q_2 B_2 - \eta_2 S B_3 - \eta_4 L B_4 \right],$$

(3.34)

$$\eta_{1n} B_{1n} + \eta_{2n} B_{2n} + \eta_{3n} B_{3n} + \eta_{4n} B_{4n} = \zeta_n \left[ \left( m_1 + m_1 \right) \left( np P_0 - \nu_{1n} S - P_0 + np \right) B_{2n} \right],$$

(3.35)

$$\mu_{1n} \left[ \left( \eta_{2n} + \eta_{2n} \right) \left( np P_0 - \nu_{1n} S - P_0 + np \right) B_{2n} \right] + \mu_{2n} \left[ \left( \eta_{2n} + \eta_{2n} \right) \left( np P_0 - \nu_{1n} S - P_0 + np \right) B_{2n} \right] = \zeta_n \left[ \left( m_1 + m_1 \right) \left( np P_0 - \nu_{1n} S - P_0 + np \right) B_{2n} \right],$$

(3.36)

Equations (3.26) - (3.29) will give the ratios of the displacement constants of the regularly reflected and transmitted waves corresponding to a plane interface, while equations (3.30) - (3.37) will give the ratios of the displacement constants of the irregularly reflected and transmitted waves corresponding to the first order approximation of the
corrugated interface. These ratios will be used to find the formulae of the reflection and transmission coefficients. Now, we shall find the reflection and transmission coefficients corresponding to the plane interface, due to an incident plane \( qP^- \) (or \( qSV^- \)) wave. These coefficients corresponding to a periodic type interface will be obtained in the next section.

(i) When \( qP^- \) wave is incident:

In this case, we have \( m_1 = 1, \ n_1 = 0, \ \theta_0 = \theta, \ \eta_{10} = \eta_1 \). Solving the equations (3.26) - (3.29), we get

\[
\frac{B_1}{B_0} = \frac{\Delta_{B1}}{\Delta}, \quad \frac{B_2}{B_0} = \frac{\Delta_{B2}}{\Delta}, \quad \frac{B_3}{B_0} = \frac{\Delta_{B3}}{\Delta}, \quad \frac{B_4}{B_0} = \frac{\Delta_{B4}}{\Delta},
\]

where

\[
\Delta = \begin{vmatrix}
1 & 1 & -1 & -1 \\
\eta_2 & \eta_1 & \eta_2 & \eta_1 \\
-a & a & a & \eta_1 \\
b & b & b & b
\end{vmatrix},
\]

\[
a = \left[ \sin \phi - \eta_2 \cos \phi \right] c_1, \quad a_1 = \frac{\eta_2 \cos \phi - \sin \phi \mu_{L2} c_1}{\eta_1 \cos \phi - \sin \phi \mu_{L1} c_1},
\]

\[
a_2 = \frac{\eta_4 \cos \gamma - \sin \gamma \mu_{L2} c_1}{\eta_1 \cos \theta - \sin \theta \mu_{L1} c_1}, \quad b = \frac{\eta_2 T_{11} \sin \phi - P_{21} \cos \phi \mu_{L1} c_1}{\eta_1 T_{11} \sin \phi - P_{21} \cos \phi \mu_{L1} c_1},
\]

\[
b_1 = \frac{\eta_4 T_{11} \sin \delta - P_{22} \cos \delta \mu_{L1} c_1}{\eta_1 T_{11} \sin \theta - P_{21} \cos \theta \mu_{L1} c_1}, \quad b_2 = \frac{\eta_4 T_{12} \sin \gamma - P_{22} \cos \gamma \mu_{L1} c_1}{\eta_1 T_{11} \sin \phi - P_{21} \cos \phi \mu_{L1} c_1},
\]

and values of \( \Delta_{B1}, \ \Delta_{B2}, \ \Delta_{B3} \) and \( \Delta_{B4} \) can be written by replacing the corresponding entries of the first, second, third and fourth columns of the determinant in \( \Delta \) with the column matrix \(-1 \quad 1 \quad 1 \quad 1\) respectively. Using equations (3.14) and (3.38), we get

\[
\frac{A_1}{A_0} = -\frac{\Delta_{B1}}{\Delta}, \quad \frac{A_2}{A_0} = -\frac{\Delta_{B2} \eta_2}{\Delta \eta_1}, \quad \frac{A_3}{A_0} = \frac{\Delta_{B3} \eta_2}{\Delta \eta_1}, \quad \frac{A_4}{A_0} = \frac{\Delta_{B4} \eta_4}{\Delta \eta_1}.
\]

The amplitude of the incident \( qP^- \) wave is given by \( \sqrt{A_0^2 + B_0^2} = B_0 \sqrt{1 + \eta_2^2} \). Similarly the amplitudes of reflected \( qP^- \) and \( qSV^- \) waves are given by \( B_1 \sqrt{1 + \eta_2^2} \) and \( B_2 \sqrt{1 + \eta_2^2} \) respectively, etc. Therefore, the reflection coefficients; \( R_{pp} \) (for reflected \( qP^- \) wave), \( R_{pm} \) (for reflected \( qSV^- \) wave) and the transmission coefficients; \( T_{pp} \) (for
transmitted \( qP \)-wave), \( T_{ps} \) (for transmitted \( qSV \)-wave) are given by

\[
R_{pp} = \frac{\Delta B_1}{\Delta}, \quad R_{ps} = \frac{\Delta B_2}{\Delta} \sqrt{\frac{1 + \eta_2^2}{1 + \eta_1^2}},
\]

\[
T_{pp} = \frac{\Delta B_3}{\Delta} \sqrt{\frac{1 + \eta_3^2}{1 + \eta_1^2}}, \quad T_{ps} = \frac{\Delta B_4}{\Delta} \sqrt{\frac{1 + \eta_4^2}{1 + \eta_1^2}}.
\]

It is clear from these expressions of coefficients in equation (3.40) that reflection and transmission coefficients corresponding to the plane interface are independent of corrugation and frequency of the incident wave.

(ii) When \( qSV \)-wave is incident:

When \( qSV \)-wave is incident, we have \( m_1 = 0, n_1 = 1, \theta_0 = \phi, \eta_{10} = \eta_2 \). The ratios of the amplitude constants corresponding to the vertical components of the reflected and transmitted waves at the plane interface are given by equation (3.38) and that of the horizontal components are given with some modifications as

\[
\frac{A_1}{A_0} = -\frac{\Delta B_1 \eta_1}{\Delta \eta_2}, \quad \frac{A_2}{A_0} = -\frac{\Delta B_2}{\Delta}, \quad \frac{A_3}{A_0} = \frac{\Delta B_3 \eta_3}{\Delta \eta_2}, \quad \frac{A_4}{A_0} = \frac{\Delta B_4 \eta_4}{\Delta \eta_2},
\]

where

\[
\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & a & b \\ \eta_2 \cos \phi - \sin \phi & \mu_2 c_1 & 1 & 0 \\ a & 1 & a_1 & a_2 \\ b & 1 & b_1 & b_2 \\ \end{vmatrix},
\]

\[
a = \frac{\sin \theta - \eta_1 \cos \theta}{\eta_2 \cos \phi - \sin \phi} c_2, \quad a_2 = \frac{\eta_2 \cos \gamma - \sin \gamma}{\eta_2 \cos \phi - \sin \phi} \mu_2 c_3,
\]

\[
b_1 = \frac{\eta_2 T_{12} \sin \delta - \eta_2 T_{11} \sin \phi - P_{22} \sin \phi}{\eta_2 T_{11} \cos \phi} c_2, \quad b_2 = \frac{\eta_2 T_{12} \sin \gamma - \eta_2 T_{11} \sin \phi - P_{22} \sin \phi}{\eta_2 T_{11} \cos \phi} c_3.
\]

and the values of \( \Delta B_1, \Delta B_2, \Delta B_3 \) and \( \Delta B_4 \) can be written by replacing the corresponding entries of the first, second, third and fourth columns of the determinant in \( \Delta \) with the column matrix \( [\begin{array}{c} -1 \ 1 \ 1 \ 1 \end{array}]^t \) respectively.

The amplitude of the incident \( qSV \)-wave is given by \( \sqrt{A_0^2 + B_0^2} = B_0 \sqrt{1 + \eta_2^2} \). Therefore, the reflection coefficients; \( R_{sp} \) (for reflected \( qP \)-wave), \( R_{ss} \) (for reflected \( qSV \)-wave) are given by

\[
R_{sp} = \frac{\Delta B_2}{\Delta} \sqrt{\frac{1 + \eta_2^2}{1 + \eta_1^2}}, \quad R_{ss} = \frac{\Delta B_3 \eta_3}{\Delta \eta_2} \sqrt{\frac{1 + \eta_2^2}{1 + \eta_1^2}}.
\]
wave), the transmission coefficients; \( T_{sp} \) (for transmitted \( qP \)-wave) and \( T_{ss} \) (for transmitted \( qSV \)-wave) are given by

\[
R_{sp} = \frac{\Delta B_1}{\Delta} \sqrt{\frac{1 + \eta_1^2}{1 + \eta_2^2}}, \quad R_{ss} = \frac{\Delta B_2}{\Delta},
\]

\[
T_{sp} = \frac{\Delta B_3}{\Delta} \sqrt{\frac{1 + \eta_3^2}{1 + \eta_2^2}}, \quad T_{ss} = \frac{\Delta B_4}{\Delta} \sqrt{1 + \eta_4^2}. \tag{3.42}
\]

We see that in case of an incident \( qSV \)-wave also, the reflection and transmission coefficients corresponding to the plane interface are independent of corrugation.

### 3.5 Special case: A periodic interface

To study the effect of the corrugation of the interface on the reflection and transmission coefficients, let us consider a simple periodic interface represented by only one cosine term, \( \zeta = d \cos px \), where \( d \) as the amplitude of corrugated interface. Here, the wavelength of the corrugation will be \( \frac{2\pi}{p} \). We shall obtain the closed form expressions of the reflection and transmission coefficients for the first order approximation of such a corrugated interface. Comparing this equation with the equation given in (3.2), we see that

\[
\zeta_n = \zeta_n = \begin{cases} 
0 & \text{if } n \neq 1, \\
\frac{d}{2} & \text{if } n = 1.
\end{cases}
\]

Putting these values in the equations given in (3.30) - (3.37) and the resulting equations give the values of the ratios of amplitude constants \( \frac{B_{11}^+}{B_0}, \frac{B_{11}^-}{B_0}, \frac{B_{21}^+}{B_0}, \frac{B_{21}^-}{B_0}, \frac{B_{31}^+}{B_0}, \frac{B_{31}^-}{B_0}, \frac{B_{41}^+}{B_0}, \frac{B_{41}^-}{B_0} \) for the incident \( qP \)-wave by putting \((m_1, n_1) = (1, 0), (\theta_0, \eta_{10}) = (\theta, \eta_1)\), and for the incident \( qSV \)-wave by putting \((m_1, n_1) = (0, 1), (\theta_0, \eta_{10}) = (\phi, \eta_2)\).

(i) For the incident \( qP \)-wave, we obtain

\[
\begin{align*}
\frac{B_{11}^+}{B_0} &= \frac{\Delta B_{11}^+}{\Delta_1^+}, & \frac{B_{21}^+}{B_0} &= \frac{\Delta B_{21}^+}{\Delta_1^+}, & \frac{B_{31}^+}{B_0} &= \frac{\Delta B_{31}^+}{\Delta_1^+}, & \frac{B_{41}^+}{B_0} &= \frac{\Delta B_{41}^+}{\Delta_1^+}, \\
\frac{B_{11}^-}{B_0} &= \frac{\Delta B_{11}^-}{\Delta_1^-}, & \frac{B_{21}^-}{B_0} &= \frac{\Delta B_{21}^-}{\Delta_1^-}, & \frac{B_{31}^-}{B_0} &= \frac{\Delta B_{31}^-}{\Delta_1^-}, & \frac{B_{41}^-}{B_0} &= \frac{\Delta B_{41}^-}{\Delta_1^-}.
\end{align*}
\tag{3.43}
\]
The ratios of the amplitude constants corresponding to the horizontal components of the displacement are given by

\[
\frac{A_{11}}{A_0} = -\frac{\Delta_{111}^+ \eta_{11}}{\Delta_1^+ \eta_1}, \quad \frac{A_{21}^+}{A_0} = -\frac{\Delta_{212}^+ \eta_{21}^+}{\Delta_1^+ \eta_1}, \quad \frac{A_{31}}{A_0} = \frac{\Delta_{313}^+ \eta_{31}}{\Delta_1^+ \eta_1}, \quad \frac{A_{41}^+}{A_0} = \frac{\Delta_{414}^+ \eta_{41}}{\Delta_1^+ \eta_1},
\]

where

\[
\Delta_1^+ = \begin{vmatrix}
1 & 1 & -1 & -1 \\
\eta_{11}^+ & \eta_{21}^+ & \eta_{31}^+ & \eta_{41}^+ \\
e & e_1 & e_2 & e_3 \\
f & f_1 & f_2 & f_3
\end{vmatrix}, \quad \Delta_1^- = \begin{vmatrix}
1 & 1 & -1 & -1 \\
\eta_{11}^- & \eta_{21}^- & \eta_{31}^- & \eta_{41}^- \\
e' & e_1' & e_2' & e_3' \\
f' & f_1' & f_2' & f_3'
\end{vmatrix},
\]

and values of \(\Delta_{111}^+\), \(\Delta_{212}^+\), \(\Delta_{313}^+\) and \(\Delta_{414}^+\) are respectively obtained by replacing first, second, third and fourth column of the determinant in \(\Delta_1^+\) by the column matrix

\[
\begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14}
\end{bmatrix}^T,
\]

\[
f_{11} = \frac{id}{2} \left[ -R + R \frac{B_1}{B_0} + Q \frac{B_2}{B_0} + S \frac{B_3}{B_0} + L \frac{B_4}{B_0} \right], \quad f_{12} = \frac{id}{2} \left[ \eta_1 R + \eta_1 R \frac{B_1}{B_0} + \eta_2 Q \frac{B_2}{B_0} - \eta_3 S \frac{B_3}{B_0} \right],
\]

\[
-\eta_4 L \frac{B_4}{B_0}, \quad f_{13} = \frac{id}{2} \left[ -\left( pP_{31} P_0 + \mu_L R^2 \right) \eta_1 + \mu_L P_0 R + P_{41} p R + \{ (pP_{31} P_0 + \mu_L R^2) \eta_1 - \mu_L P_0 Q - P_{41} p Q \} \right] \frac{B_1}{B_0} + \{ (pP_{32} P_0 - P_{41} p R) \} \frac{B_2}{B_0} + \{ (pP_{32} P_0
\]

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\[ f_{14} = \frac{id}{2} (T_{11} P_0 - 2 \mu L_1) R \eta_1 + 2 \mu L_1 P_0 - P_{21} R^2 + \{(T_{11} P_0 - 2 \mu L_1) R \eta_1 + 2 \mu L_1 P_0 - P_{21} R^2\} B_1 B_0 + \{(T_{11} P_0 - 2 \mu L_1) Q \eta_2 + 2 \mu L_1 P_0 - P_{21} Q^2\} B_2 B_0 - \{(T_{12} P_0 - 2 \mu L_2) S \eta_3 + 2 \mu L_2 P_0 - P_{22} S^2\} B_3 B_0 + \{(T_{12} P_0 - 2 \mu L_2) L \eta_4 + 2 \mu L_2 P_0 - P_{22} L^2\} B_4 B_0 \]

and values of the determinant \( \Delta_{B11}, \Delta_{B21}, \Delta_{B31} \) and \( \Delta_{B41} \) can be written by replacing first, second, third and fourth column of the determinant in \( \Delta_1 \) with the column matrix \( \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \end{bmatrix} \) respectively,

\[ f'_{13} = \frac{id}{2} (p P_{31} P_0 - \mu L_1 R^2) \eta_1 + \mu L_1 P_0 - P_{41} P R + \{(p P_{31} P_0 + \mu L_1 R^2) \eta_1 + \mu L_1 P_0 R + P_{41} P R\} B_1 B_0 + \{(p P_{32} P_0 + \mu L_1 R^2) \eta_2 - \mu L_1 R P_0 + P_{41} P R\} B_2 B_0 + \{(p P_{32} P_0 + \mu L_2 S^2) \eta_3 - \mu L_2 S P_0 + P_{42} P S\} B_3 B_0 + \{(p P_{32} P_0 + \mu L_2 S^2) \eta_4 - \mu L_2 S P_0 + P_{42} P L\} B_4 B_0, \]

The reflection coefficients: \( R_{pp^+} \) and \( R_{pp^-} \) (for irregularly reflected \( qP^- \) waves at angles \( \theta_1^+ \) and \( \theta_1^- \) respectively), \( R_{ps^+} \) and \( R_{ps^-} \) (for irregularly reflected \( qSV^- \) waves at angles \( \phi_1^+ \) and \( \phi_1^- \) respectively) and the transmission coefficients: \( T_{pp^+} \) and \( T_{pp^-} \) (for irregularly transmitted \( qP^- \) waves at angles \( \delta_1^+ \) and \( \delta_1^- \) respectively), \( T_{ps^+} \) and \( T_{ps^-} \) (for irregularly transmitted \( qSV^- \) waves at angles \( \gamma_1^+ \) and \( \gamma_1^- \) respectively) are given by

\[ R_{pp^+} = \frac{\Delta_{B11}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \quad R_{pp^-} = \frac{\Delta_{B11}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \quad R_{ps^+} = \frac{\Delta_{B21}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \quad R_{ps^-} = \frac{\Delta_{B21}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \]

\[ R_{ps^+} = \frac{\Delta_{B31}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \quad T_{pp^+} = \frac{\Delta_{B11}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \quad T_{pp^-} = \frac{\Delta_{B11}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_{11}^{-2}}{1 + \eta_1^2}}, \]
(ii) Similarly, for the incident \(qSV^-\) wave, the reflection coefficients: \(R_{sp}^r\) and \(R_{sp}^r\) 
(for irregularly reflected \(qP^-\) waves at angles \(\theta_1^+\) and \(\theta_1^-\) respectively), \(R_{ss}^r\) and \(R_{ss}^r\) 
(for irregularly reflected \(qSV^-\) waves at angles \(\phi_1^+\) and \(\phi_1^-\) respectively) and the transmission coefficients: \(T_{sp}^1\) and \(T_{sp}^1\) 
(for irregularly transmitted \(qP^-\) waves at angles \(\delta_1^+\) and \(\delta_1^-\) respectively), \(T_{ss}^1\) and \(T_{ss}^1\) 
(for irregularly transmitted \(qSV^-\) waves at angles \(\gamma_1^+\) and \(\gamma_1^-\) respectively) are given by

\[
R_{sp}^r = \frac{\Delta_{B11}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_1^{+2}}{1 + \eta_2^2}}, \quad R_{sp}^r = \frac{\Delta_{B31}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_1^{+2}}{1 + \eta_2^2}}, \quad R_{ss}^r = \frac{\Delta_{B21}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_1^{+2}}{1 + \eta_2^2}},
\]

\[
R_{sp}^r = \frac{\Delta_{B21}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_1^{-2}}{1 + \eta_2^2}}, \quad R_{sp}^r = \frac{\Delta_{B31}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_1^{-2}}{1 + \eta_2^2}}, \quad R_{ss}^r = \frac{\Delta_{B21}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_1^{-2}}{1 + \eta_2^2}},
\]

\[
T_{sp}^1 = \frac{\Delta_{B41}^+}{\Delta_1^+} \sqrt{\frac{1 + \eta_4^{+2}}{1 + \eta_2^2}}, \quad T_{sp}^1 = \frac{\Delta_{B41}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_4^{-2}}{1 + \eta_2^2}}, \quad T_{ss}^1 = \frac{\Delta_{B41}^-}{\Delta_1^-} \sqrt{\frac{1 + \eta_4^{-2}}{1 + \eta_2^2}},
\]

where, the values of \(\Delta_1^+, \Delta_1^-, \Delta_{B11}^+, \Delta_{B21}^+, \) etc. are given earlier wherein the following modified values of \(f_{11}, \ f_{12}, \ f_{13}, \ f_{14}, \ f_{13}'\) and \(f_{14}'\) are used

\[
f_{11} = \frac{id}{2} [-Q + R^1 B_1^1 B_0 + Q B_2^1 B_0 + S B_3^1 B_0 + L B_4^1 B_0], \quad f_{12} = \frac{id}{2} \eta_3 Q + \eta_1 R^1 B_1^1 B_0 + \eta_2 Q B_2^1 B_0 - \eta_3 S B_3^1 B_0 - \eta_4 L^1 B_0^1 B_0, \quad f_{13} = \frac{id}{2} (-p P_3^1 P_0 + \mu L_4 Q^2) \eta_2 + \mu L_4 P_0 Q + P_4^1 P Q + \{(p P_3^1 P_0 + \mu L_4 R^2) \eta_1
\]

\[
- \mu L_4 P_0 R - P_3^1 P R^1 \frac{B_1^1}{B_0} + \{(p P_3^1 P_0 + \mu L_4 Q^2) \eta_2 - \mu L_4 P_0 Q - P_4^1 P Q) \frac{B_2^1}{B_0} + \{(p P_3^1 P_0 + \mu L_4 S^2) \eta_3 - \mu L_4 P_0 S - P_4^1 P S \frac{B_3^1}{B_0} + \{(p P_3^1 P_0 + \mu L_4 L^2) \eta_4 - \mu L_4 P_0 L - P_4^1 P L \frac{B_4^1}{B_0}],
\]

\[
f_{14} = \frac{id}{2} \{(T_1 P_0 - 2 p \mu L_4) Q \eta_2 + 2 \mu L_4 P_0 P_0 - P_2^1 Q^2 + \{(T_1 P_0 - 2 p \mu L_4) R \eta_1 + 2 \mu L_4 P_0 P_0
\]

\[
- P_2^1 R^1 \frac{B_1^1}{B_0} + \{(T_1 P_0 - 2 p \mu L_4) Q \eta_2 + 2 \mu L_4 P_0 P_0 - P_2^1 Q^2 \frac{B_2^1}{B_0} - \{(T_1 P_0 - 2 p \mu L_4) S \eta_3 + 2 \mu L_4 P_0 P_0
\]

\[
- P_2^1 S^2 \frac{B_3^1}{B_0} - \{(T_1 P_0 - 2 p \mu L_4) L \eta_4 + 2 \mu L_4 P_0 P_0 - P_2^1 L^2 \frac{B_4^1}{B_0}], \quad f_{13}' = \frac{id}{2} \{(p P_3^1 P_0 + \mu L_4 Q^2) \eta_2
\]

\[
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\]
It is clear from the formulae for reflection and transmission coefficients given in (3.45) and (3.46) that these coefficients depend on the amplitude of corrugation, elastic properties of the half-spaces and frequency of the incident wave.

We note that when the amplitude of corrugated interface is put equal to zero in the formulae given in (3.45) and (3.46), we see that

\[ R_{sp} = R_{ps} = \frac{B_1}{B_0} \]

This means that when the interface is perfectly plane, all the coefficients corresponding to the irregularly reflected and transmitted waves vanish, except the coefficients given in (3.40) and (3.42) which are the reflection and transmission coefficients due to the incident \( qP \) and \( qS \) waves, respectively corresponding to a plane interface between two dissimilar fibre-reinforced elastic half-spaces.

### 3.6 Particular cases

(a) If we neglect the upper medium \( H_2 \), then the problem would reduce to the problem of reflection of elastic waves from a stress free boundary surface of a fibre-reinforced elastic half-space. In this case all the elastic parameters concerning the medium \( H_2 \) will vanish. Thus, in the absence of upper half-space, the reflection coefficients due to an incident \( qP \) wave at a free plane surface of a fibre-reinforced half-space can be obtained by putting \( \beta_2 = \alpha_2 = \mu_L = \mu_T = 0 \) into equation (3.40), we obtain the reflection coefficients for incident \( qP \) wave as

\[
R_{pp} = \frac{\frac{\Delta B_1}{\Delta}}{\), \quad R_{ps} = \frac{\frac{\Delta B_2}{\Delta}}{\sqrt{1 + \frac{\eta_2^2}{\eta_1^2}}},
\]

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and from equation (3.42), we obtain the reflection coefficients for incident $qSV-$ wave as

$$R_{sp} = \frac{\Delta_{B2}}{\Delta} \sqrt{\frac{1 + \eta_1^2}{1 + \eta_2^2}}, \quad R_{st} = \frac{\Delta_{B4}}{\Delta},$$

where

$$\Delta = \frac{1}{c_1 c_2} \left[ (\eta_1 \cos \theta - \sin \theta) \{ P_{21} \cos \phi - \eta_2 T_{11} \sin \phi \} \right. \left. + (\eta_2 \cos \phi - \sin \phi) \{ -P_{21} \eta_1 \cos \theta + \eta_1 T_{11} \sin \theta \} \right],$$

$$\Delta_{B1} = \frac{1}{c_1 c_2} \left[ - (\eta_1 \cos \theta - \sin \theta) \{ P_{21} \cos \phi - \eta_2 T_{11} \sin \phi \} + (\eta_2 \cos \phi - \sin \phi) \{ -P_{21} \eta_1 \cos \theta + \eta_1 T_{11} \sin \theta \} \right],$$

$$\Delta_{B2} = - \frac{4}{c_1^2} \left[ (\eta_1 \cos \theta - \sin \theta) \{ -P_{21} \cos \theta + \eta_1 T_{11} \sin \theta \} \right],$$

$$\Delta_{B3} = - \frac{4}{c_2^2} \left[ (\eta_2 \cos \phi - \sin \phi) \{ P_{21} \cos \phi - \eta_2 T_{11} \sin \phi \} \right],$$

$$\Delta_{B4} = \frac{1}{c_1 c_2} \left[ (\eta_1 \cos \theta - \sin \theta) \{ P_{21} \cos \phi - \eta_2 T_{11} \sin \phi \} - (\eta_2 \cos \phi - \sin \phi) \{ -P_{21} \eta_1 \cos \theta + \eta_1 T_{11} \sin \theta \} \right].$$

These reflection coefficients are exactly same as obtained by Singh and Singh (2004) for the relevant problems.

(b) By neglecting the parameters corresponding to fibre reinforcement in both the half-spaces, we shall be left with two dissimilar isotropic homogeneous elastic half-spaces. Thus, putting $\beta_1 = \alpha_1 = \beta_2 = \alpha_2 = 0, \mu_{11} = \mu_{11} = \mu_{11}, \mu_{22} = \mu_{22} = \mu_2, \mu_3 = \mu_3 = 2\mu_1 + \lambda_1, \eta_1 = \eta_1 = -\tan \theta, \eta_{11} = -\tan \theta_{11}, \eta_2 = \cot \phi, \eta_{21} = \cot \phi_{21}, \eta_3 = -\tan \delta, \eta_{31} = -\tan \delta_{31}, \eta_4 = \cot \gamma, \eta_{41} = \cot \gamma_{41}, \mu_3 = \mu_3 = -2\mu_2, P_{41} = P_{42} = 2\mu_2, c_1 = \sqrt{(\lambda_1 + 2\mu_1)/\rho_1}, c_2 = \sqrt{\mu_1/\rho_1}, c_3 = \sqrt{(\lambda_2 + 2\mu_2)/\rho_2}, c_4 = \sqrt{\mu_2/\rho_2},$ the reflection and transmission coefficients corresponding to the plane interface, when a $qP-$ wave is incident are given by equation (3.40) with the following modified values

$$a = \frac{c_1 \cos 2\phi}{2c_2 \sin \phi \sin \theta}, \quad a_1 = \frac{\mu_2}{\mu_1}, \quad a_2 = -\frac{\mu_2 c_1 \cos 2\gamma}{2\mu_1 c_4 \sin \theta \sin \gamma},$$

$$b = \frac{2\mu_1 c_1 \cos \theta \cos \phi}{(\lambda_1 + 2\mu_1 \cos^2 \theta)c_2}, \quad b_1 = \frac{(\lambda_2 + 2\mu_2 \cos^2 \delta) c_1 \cos \theta}{(\lambda_1 + 2\mu_1 \cos^2 \theta)c_3 \cos \delta}, \quad b_2 = \frac{2c_1 \mu_2 \cos \gamma \cos \theta}{(\lambda_1 + 2\mu_1 \cos^2 \theta)c_4}.$$

In the same way, by putting the above values in the equation (3.45), one can obtain the reflection and transmission coefficients of irregular waves for first order approximation of the corrugated interface.

Similarly, the reflection and transmission coefficients corresponding to the plane inter-
face due to an incident $qSV-$ wave, are given by the equation (3.42) with the following modified expressions

$$a = \frac{2c_2 \sin \theta \sin \phi}{c_1 \cos 2\phi}, \quad a_1 = \frac{2c_2 \mu_2 \sin \delta \sin \phi}{c_3 \mu_1 \cos 2\phi}, \quad a_2 = \frac{c_2^2 \mu_2 \cos 2\gamma}{c_4 \mu_1 \cos 2\phi},$$

$$b = \frac{(\lambda_1 + 2\mu_1 \cos^2 \theta)c_2}{2\mu_1 c_1 \cos \phi \cos \theta}, \quad b_1 = \frac{(\lambda_2 + 2\mu_2 \cos^2 \theta)c_2}{2c_3 \mu_1 \cos \phi \cos \delta}, \quad b_2 = \frac{c_2 \mu_2 \cos \gamma}{c_4 \mu_1 \cos \phi}.$$

These results are equivalent to the corresponding results given by Ben-Menahem and Singh (1981). By making appropriate substitutions, the reflection and transmission coefficients of the irregular waves for the first order approximation of the corrugated interface can be obtained from equation (3.46) for the case of incident $qSV-$ wave.

### 3.7 Numerical results and discussion

Here, our aim is to compute the reflection and transmission coefficients numerically for a specific model when a plane quasi-wave is made incident obliquely at a corrugated interface given by $\zeta = d \cos px$. For this purpose, one shall need the angles of propagation of reflected and transmitted quasi-waves for a given angle of incidence. Now, given the angle of incidence, $\theta_0$, we shall calculate the angles of reflection $\theta$ and $\phi$ of reflected $qP-$ and reflected $qSV-$ waves respectively, the angles of transmission $\delta$ and $\gamma$ of transmitted $qP-$ and transmitted $qSV-$ waves respectively. These angles will be calculated with the help of Snell’s law given by equation (3.15). In Snell’s law, $c_a$ is apparent phase velocity, which can be made dimensionless by defining a non-dimensional velocity $\bar{\theta}$ given by $\bar{\theta} = \frac{c_a}{\beta}$. Substituting the value of $c_a$ from (3.15), we see that $\bar{\theta} = \frac{c(\theta_0)}{\beta \sin \theta_0} = \frac{c(\theta_0)}{p_2 \beta}$. With this, the equation (1.70) for the half- space $H_1$ becomes

$$\bar{\theta}^4 - (\bar{\psi}_{11} + \bar{\psi}_{21}) \bar{\theta}^2 + \bar{\psi}_{11} \bar{\psi}_{21} - \bar{\psi}_{00} \bar{p}_0^2 = 0,$$

where

$$\bar{\psi}_{11} = \frac{P_{11}}{\mu L_1} + \bar{p}_0^2, \quad \bar{\psi}_{21} = \frac{P_{21}}{\mu L_1} \bar{p}_0^2 + 1, \quad \bar{\psi}_{01} = \frac{Q_{01}}{\mu L_1}, \quad \bar{p}_0 = \frac{p_3}{p_2}, \quad \beta = \sqrt{\frac{\mu L_1}{\rho L}}.$$
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\((p_2, p_3) = (\sin \theta_0, \cos \theta_0)\).

Substituting the values of \(\Psi_{11}, \Psi_{21}\) and \(Q_{01}\) into equation (3.47), we obtain

\[ g_0 p_0^4 + g_2 p_0^2 + g_4 = 0, \]  

where

\[
g_0 = \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\mu_{T_1}}{\mu_{L_3}} \quad \text{and} \quad g_2 = 1 + \left( \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\alpha_1}{\mu_{L_3}} + 4 - 2 \frac{\mu_{T_1}}{\mu_{L_1}} + \frac{\beta_1}{\mu_{L_1}} \right) \left( \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\mu_{T_1}}{\mu_{L_1}} \right) - (1 + \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\mu_{T_1}}{\mu_{L_1}}) c^2 \left( \frac{\lambda_1}{\mu_{L_1}} + \frac{\alpha_1}{\mu_{L_3}} + 1 \right)^2, \quad g_4 = c^4 - (5 + \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\alpha_1}{\mu_{L_3}} - 2 \frac{\mu_{T_1}}{\mu_{L_1}} + \frac{\beta_1}{\mu_{L_1}}) c^2 + \frac{\lambda_1}{\mu_{L_1}} + 2 \frac{\alpha_1}{\mu_{L_3}} + 4 - 2 \frac{\mu_{T_1}}{\mu_{L_1}} + \frac{\beta_1}{\mu_{L_1}}.
\]

Transforming the above equation by using \(q = \frac{1}{p_0} = \frac{p_2}{p_3}\), we obtain

\[ g_4 q^4 + g_2 q^2 + g_0 = 0. \]  

From equation (3.47), we see that for a given value of \(p_0\) (i.e., \(\theta_0\)), there are two roots of \(c^2\) corresponding to the speeds of incident \(qP\)- and \(qSV\)-waves and from equation (3.49), for a given value of \(c\), there are two positive roots of \(q\) corresponding to the angles of propagation of reflected \(qP\)- and \(qSV\)-waves. The smaller positive root will correspond to reflected \(qSV\)-wave and larger positive root will correspond to reflected \(qP\)-wave. Thus, the angles of reflection of \(qP\)- and \(qSV\)-waves respectively, are given by

\[
\theta = \tan^{-1}(q_{13}) \quad \text{and} \quad \phi = \tan^{-1}(q_{14}),
\]

where \(q_{13}\) and \(q_{14}\) are positive roots of equation (3.49) such that \(q_{13} > q_{14}\).

Similarly, equations can be set up for the medium \(H_2\), and the angles of propagation of transmitted \(qP\)- and \(qSV\)-waves can be obtained as follows

\[
\delta = \tan^{-1}(q_{23}) \quad \text{and} \quad \gamma = \tan^{-1}(q_{24}).
\]

This is how one can obtain the directions of propagation of regularly reflected and transmitted waves. After computing the directions of propagation of regularly reflected and transmitted waves, the directions of propagation of irregularly reflected
and transmitted waves can be computed from Spectrum theorem given by (3.16). In order to perform the numerical computation, we have taken the following values of relevant parameters in the fibre reinforced half-spaces.

For the half-space $H_1$:
\[ \lambda_1 = 7.65 \times 10^{11} \text{ N/m}^2, \; \mu_{L1} = 5.26 \times 10^{11} \text{ N/m}^2, \; \mu_T = 2.46 \times 10^{11} \text{ N/m}^2, \; \alpha_1 = -3.3 \times 10^{11} \text{ N/m}^2, \; \beta_1 = 220 \times 10^{11} \text{ N/m}^2, \; \rho_1 = 7.8 \times 10^3 \text{ Kg/m}^3 \]
and for the half-space $H_2$:
\[ \lambda_2 = 5.8 \times 10^{11} \text{ N/m}^2, \; \mu_{L2} = 5.5 \times 10^{11} \text{ N/m}^2, \; \mu_T = 2.25 \times 10^{11} \text{ N/m}^2, \; \alpha_2 = -1.72 \times 10^{11} \text{ N/m}^2, \; \beta_2 = 195 \times 10^{11} \text{ N/m}^2, \; \rho_2 = 7.2 \times 10^3 \text{ Kg/m}^3. \]

Since the Rayleigh’s method of approximation is applicable only when the amplitude and slope of the corrugated interface are very small, therefore, we take the values of the corrugation parameter as $pd = 0.000125$. The frequency parameter is taken as $\omega/pc_1 = 150$, when $qP$- wave is incident and $\omega/pc_2 = 150$, when $qSV$ - wave is incident, wherever not mentioned. We have computed the modulus of reflection and transmission coefficients of regularly and irregularly reflected and transmitted waves. In all the figures, the values of the modulus of reflection and transmission coefficients of the reflected and transmitted $qP -$ and $qSV -$ waves are drawn after magnifying their original values. Some coefficients are found to suffer critical angles.

Figures 3.2 - 3.5 show the variation of the modulus of reflection and transmission coefficients of regularly and irregularly reflected and transmitted $qP -$ and $qSV -$ waves with the angle of incidence $\theta$, when a plane $qP -$ wave is incident. In Figure 3.2, Curve - I shows the reflection coefficient $R_{pp}$ of the regularly reflected $qP -$ wave, which is small throughout the range of angle of incidence, however it initially decreases to the value zero and then increases to a certain extent with the increase of the angle of incidence. Curve - II shows that the reflection coefficient $R_{pp}^{I}$ of the irregularly reflected $qP -$ wave at an angle $\theta$ starts from certain value at normal incidence and then decreases with $\theta$, goes to zero at $\theta = 12^0$. Thereafter, the curve is of almost parabolic nature in the range $12^0 \leq \theta \leq 72^0$, beyond which there is a sharp increase in its value and attains its maximum value at an angle $\theta = 83^0$. At this angle, the wave suffers critical reflection. Curve - III represents the variation of reflection coefficient $R_{pp}^{I}$ corresponding to the irregularly reflected $qP -$ wave at an angle $\theta^-$. Its value is equal to the value taken by $R_{pp}^{I}$ at $\theta = 1^0$. It decreases to the value zero at $\theta = 15^0$ and then it increases to a certain value and then decreases very slowly and slowly with
Figure 3.2: Variation of the reflection coefficients with angle of incidence, $\theta$ when $pd = 0.000125$ and $\frac{\rho c_1}{\rho c_1} = 150$.

Figure 3.3: Variation of the reflection coefficients with angle of incidence, $\theta$ when $pd = 0.000125$ and $\frac{\rho c_1}{\rho c_1} = 150$.

Figure 3.4: Variation of the transmission coefficients with angle of incidence, $\theta$ when $pd = 0.000125$ and $\frac{\rho c_1}{\rho c_1} = 150$.

Figure 3.5: Variation of the transmission coefficients with angle of incidence, $\theta$ when $pd = 0.000125$ and $\frac{\rho c_1}{\rho c_1} = 150$. 
increase of the angle of incidence. In Figure 3.3, Curves - I, II and III show the variation of the reflection coefficients $R_{ps}$ corresponding to the regularly reflected $qSV$– wave, $R_{ps}^{1}$, corresponding to the irregularly reflected $qSV$– wave at an angle $\phi_i^+$ and $R_{ps}^{1}$, corresponding to the irregularly reflected $qSV$– wave at an angle $\phi_i^-$ respectively. We note that all these coefficients behave alike with $\theta_0$. They have maxima at $\theta_0 = 10^0$ and the coefficient $R_{ps}^{1}$ suffers critical angle at $\theta_0 = 83^0$. In Figure 3.4, Curve - I shows the variation of transmission coefficient $T_{pp}$ corresponding to the regularly transmitted $qP$– wave. It starts from a certain value and almost maintains the same value upto $\theta_0 = 9^0$, thereafter it decreases with the increase of the angle of incidence. Curve - II shows the variation of the transmission coefficient $T_{ps}$, corresponding to the irregularly transmitted $qP$– wave at angle $\delta_i^+$. Its value decreases and approaches to zero as $\theta_0 \to 12^0$, it then increases, decreases and again increases with increase of the angle of incidence and finally achieving its maximum value at an angle $\theta = 83^0$. Curve - III shows the variation of transmission coefficient $T_{ps}$, corresponding to the irregularly transmitted $qP$– wave at angle $\delta_i^-$. Its value starts from a certain non-zero value and decreases with increase of the angle of incidence getting its minimum value at $\theta = 12^0$. It again increases to a certain value with the angle of incidence and then decreases further and vanishes at $\theta_0 = 85^0$, beyond which it again increases a little bit. In Figure 3.5, Curves - I, II and III show the variations of the transmission coefficients $T_{ps}$ corresponding to the regularly transmitted $qSV$– wave, $T_{ps}^{1}$, corresponding to the irregularly transmitted $qSV$– wave at an angle $\gamma_i^+$ and $T_{ps}^{1}$, corresponding to the irregularly transmitted $qSV$– wave at an angle $\gamma_i^-$ respectively. All these coefficients behave alike with angle of incidence. They have their maximum value at $\theta_0 = 9^0, 11^0$ and $10^0$. It is noted that there exists a critical angle at $\theta = 83^0$ for the waves corresponding to the coefficient $T_{ps}^{1}$. Thus we have seen that all the reflection and transmission coefficients are functions of the angle of incidence.

Figures 3.6 - 3.9 show the variation of the modulus of reflection and transmission coefficients of the regularly and irregularly reflected and transmitted waves with the corrugation parameter $pd$, when a $qP$– wave is incident at an angle $\theta = 10^0$. It is noted from these figures that reflection and transmission coefficients of the regularly reflected and transmitted waves are independent of corrugation parameter as was expected before and all the other coefficients increase with the increase of corrugation
Figure 3.6: Variation of the reflection coefficients with corrugation parameter, $pd$ when $\frac{\omega}{pc_1} = 150$, $\theta = 10^0$.

Figure 3.7: Variation of the reflection coefficients with corrugation parameter, $pd$ when $\theta = 10^0$, $\frac{\omega}{pc_1} = 150$.

Figure 3.8: Variation of the transmitted coefficients with corrugation parameter, $pd$ when $\theta = 10^0$, $\frac{\omega}{pc_1} = 150$.

Figure 3.9: Variation of the transmitted coefficients with corrugation parameter, $pd$ when $\theta = 10^0$, $\frac{\omega}{pc_1} = 150$. 
Figure 3.10: Variation of the reflection coefficients with frequency parameter, when $\theta = 10^\circ$ and $pd = 1.25 \times 10^{-4}$.

Figure 3.11: Variation of the reflection coefficients with frequency parameter, when $\theta = 10^\circ$ and $pd = 1.25 \times 10^{-4}$.

Figure 3.12: Variation of the transmission coefficients with frequency parameter, when $\theta = 10^\circ$ and $pd = 1.25 \times 10^{-4}$.

Figure 3.13: Variation of the transmission coefficients with frequency parameter, when $\theta = 10^\circ$ and $pd = 1.25 \times 10^{-4}$.
parameter. However the rate of increase is different for different coefficient. Thus, the reflection and transmission coefficients corresponding to the irregularly reflected and irregularly transmitted waves are strongly influenced by the amplitude of the corrugation of the interface. Figures 3.10 - 3.13 show the variation of the modulus of the reflection and transmission coefficients corresponding to the regular and irregular waves with the frequency parameter $\frac{\omega}{pc_{1}}$, when a $qP$ wave is incident at an angle $\theta = 10^\circ$. It can be noticed from these figures that the behavior of these coefficients with the frequency parameter is similar to their behavior with corrugation parameter that the coefficients corresponding to the irregular waves increase with the increase of the $pc_{1}$.

Figures 3.14 - 3.17 show the variation of the modulus of reflection and transmission coefficients corresponding to the regularly and irregularly reflected waves with the angle of incidence $\phi$, when a plane $qSV$ wave is incident. From these figures, we note that there exist a critical angle at $\phi = 12^\circ$. In Figure 3.14, Curve - I shows that the variation of the reflection coefficient $R_{sp}$ increases with the increase of the angle of incident, $\phi$ and attains its maximum value at $\phi = 10^\circ$, thereafter it decreases with the further increase of the angle of incidence. Curve - II shows that the reflection coefficient $R_{tp}$ increases gently with the increase of the angle of incident $\phi$ and it has sharp increase in its value after $\phi = 11^\circ$. Curve - III shows that the reflection coefficient $R_{tp}^1$ decreases gently with the increase of the angle of incidence $\phi$ upto $10^\circ$ and thereafter, it increases with angle of incidence, achieving its maximum value at $\phi = 12^\circ$. In Figure 3.15, Curve - I shows that the reflection coefficient $R_{sp}$ decreases slowly with the increase of the angle of incidence $\phi$, attaining a minimum value at $\phi = 8^\circ$, thereafter it starts increasing with the increase of the angle of incidence till $\phi = 11^\circ$ and finally decreases as $\phi \rightarrow 12^\circ$. Curves - II and III show that the reflection coefficients $R_{sp}^1$ and $R_{sp}^2$ increase with the increase of the angle of incident $\phi$, but in a different fashion. In Figure 3.16, Curve - I shows that the transmission coefficient $T_{sp}$ remains near to zero throughout, however, it increases a little bit when approaching to the critical angle. Curves - II and III show that the transmission coefficients, $T_{sp}^1$, and $T_{sp}^2$ increase slowly with $\phi$ upto certain value and then there is sharp increase near $\phi = 11^\circ$. These coefficients possess similar behavior with $\phi$, however the values of $T_{sp}^1$ remain less than the values of $T_{sp}$ at each angle $\phi$. In Figure 3.17, Curve - I shows
Figure 3.14: Variation of the reflection coefficients with angle of incidence, \( \phi \) when \( pd = 0.000125 \) and \( \frac{\phi}{\rho c_2} = 150 \).

Figure 3.15: Variation of the reflection coefficients with angle of incidence, \( \phi \) when \( pd = 0.000125 \) and \( \frac{\phi}{\rho c_2} = 150 \).

Figure 3.16: Variation of transmission coefficients with angle of incidence, \( \phi \) when \( pd = 0.000125 \) and \( \frac{\phi}{\rho c_2} = 150 \).

Figure 3.17: Variation of transmission coefficients with angle of incidence, \( \phi \) when \( pd = 0.000125 \) and \( \frac{\phi}{\rho c_2} = 150 \).
Figure 3.18: Variation of reflection coefficients with corrugation parameter, $pd$ when $\phi = 10^6, \frac{\omega}{pc_2} = 150$.

Figure 3.19: Variation of reflection coefficients with corrugation parameter, $pd$ when $\phi = 10^6, \frac{\omega}{pc_2} = 150$.

Figure 3.20: Variation of transmitted coefficients with corrugation parameter, $pd$ when $\phi = 10^6, \frac{\omega}{pc_2} = 150$.

Figure 3.21: Variation of transmitted coefficients with corrugation parameter, $pd$ when $\phi = 10^6, \frac{\omega}{pc_2} = 150$. 
Figure 3.22: Variation of reflection coefficients with frequency parameter, \( \phi \) when \( \phi = 10^0 \) and \( pd = 1.25 \times 10^{-4} \).

Figure 3.23: Variation of reflection coefficients with frequency parameter, \( \phi \) when \( \phi = 10^0 \) and \( pd = 1.25 \times 10^{-4} \).

Figure 3.24: Variation of transmission coefficients with frequency parameter, \( \phi \) when \( \phi = 10^0 \) and \( pd = 1.25 \times 10^{-4} \).

Figure 3.25: Variation of transmission coefficients with frequency parameter, \( \phi \) when \( \phi = 10^0 \) and \( pd = 1.25 \times 10^{-4} \).
that the transmission coefficient $T_{ss}$ increases very gently with the increase of the angle of incidence $\phi$, in the range $1^\circ \leq \phi \leq 8^\circ$, but as $\phi$ approaches to $12^\circ$ further, the rate of increase becomes fast. Curve - II shows the variation of the transmission coefficient $T_{ss}^*$ corresponding to the irregularly transmitted $qSV-$ wave at an angle $\gamma_1^-$, which first increases a little bit and then decreases till $\phi = 7^\circ$, thereafter its value increases with appreciable rate and goes to maximum value as $\phi$ approaches to $12^\circ$. The Curve - III shows the variation of the coefficient $T_{ss}^1$ corresponding to the irregularly transmitted $qSV-$ wave at an angle $\gamma_1^+$, which first increases slowly and then sharply, with the angle of incidence. These coefficients with angle of incident $\phi$ have similar trend near critical angle. It can be clearly seen from Figures 3.18 - 3.25 that the variations of the modulus of reflection and transmission coefficients corresponding to the irregular waves, in case of incident $qSV-$ wave, increase linearly with the increase of the corrugation parameter $(pd)$ and with the increase of the frequency parameter $(\frac{\omega}{\rho c})$, but at different rates. The coefficients corresponding to the regular waves are found to be independent of corrugation and frequency parameters.