Chapter 6

Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

6.1 Introduction and problem formulation

Differential-difference equations govern a variety of physical processes, for instance, hydrodynamics of liquid helium [95], thermoelasticity [58], study of variational problems in control theory where problems are complicated by the effect of time delay in signal transmission [54], diffusion in polymers [146], study of bistable devices [44], evolutionary biology [223], micro scale heat transfer, description of human pupil light reflex [147], a variety of models of physiological processes or diseases [150, 223]. They are also satisfied by the moments of the time of first exit of temporally homogeneous Markov processes [211] governing such phenomena as the time between impulses of a nerve cell and the persistence time of populations with large random fluctuations.

In this chapter, we extend the numerical study of boundary value problems for second order singularly perturbed differential-difference equations with turning point which was initiated in Chapter 4 of this thesis. In Chapter 4 and 5 we considered the case when delay/advance arguments

165
were $o(\varepsilon)$ and used Taylor’s series to tackle the delay/advance argument. When delay/advance arguments are $O(\varepsilon)$ use of Taylor’s expansion may lead to bad approximation. To resolve this problem in this chapter, we construct a fitted operator finite difference scheme for the numerical approximation of the given class of problems without using any a-priori estimation for the retarded arguments, i.e., without using Taylor series expansion of the retarded terms. In this chapter, the difference approximation is obtained by constructing a special type of mesh so that the terms containing delay/advance arguments and the turning point lie at the nodal points after discretization.

We consider the following SPDDE on $x \in \Omega = (-1, 1)$ with delay as well as advance in the reaction term

$$L^{\delta, \eta}y \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x - \delta) + d(x)y(x + \eta) = f(x),$$

$$y(x) = \phi(x), \quad -1 - \delta \leq x \leq -1,$n(x) = \gamma(x), \quad 1 \leq x \leq 1 + \eta,$$ (6.1.1)

where $\delta, \eta$ are delay and advance arguments respectively, $a(x), b(x), f(x), c(x), d(x), \gamma(x), \phi(x)$ are sufficiently smooth functions. When delay/advance arguments are zero (i.e., $\delta = 0, \eta = 0$), the solution of the corresponding ordinary differential equation exhibit layer behavior or turning point behavior depending upon the coefficient of the convection term. The layer will be on the left or the right end of the domain depending upon the sign of the coefficient of convection term, i.e., according to $a(x) < 0$ or $a(x) > 0$ on $\Omega = [-1, 1]$. The point of the domain where $a(x) = 0$ is known as turning point. Presence of the turning point results into boundary or interior layer in the solution of the problem and are more difficult to handle as compared to the non-turning point case. In this chapter, we consider the case where presence of the turning point results into interior layer in the solution of the problem.

Problem (6.1.1) is considered under following assumptions

$$a(0) = 0, \quad a'(0) > 0,$$ (6.1.2)

$$|a(x)| \geq a_0 > 0, \quad 0 < |x| \leq 1.$$ (6.1.3)
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed
differential-difference turning point problems with delay as well as advance

\begin{equation}
  \beta = \frac{b(0)}{\alpha'(0)}
\end{equation}

and there exist constants \( \beta_l, \beta_u \) such that

\begin{equation}
  \beta_l < 1 < \beta_u \quad \text{and} \quad |\beta| \leq \beta_u.
\end{equation}

To ensure that there is no other turning point in the region \([-1, 1]\) it is assumed that

\begin{equation}
  |\alpha'(x)| \geq \frac{\alpha'(0)}{2}, \quad x \in \bar{\Omega}.
\end{equation}

Under the above assumptions the given turning point problem exhibit an interior layer at \( x = 0 \)
whose nature depends upon the value of the parameter \( \beta \) and delay/advance argument.

### 6.2 Continuous problem

Equation (6.1.1) can be written as

\begin{equation}
  L^{\delta, \eta} y(x) = F(x)
\end{equation}

where

\begin{equation}
  L^{\delta, \eta} y(x) \equiv \begin{cases}
    \varepsilon y''(x) + \alpha(x)y'(x) - b(x)y(x) + d(x)y(x + \eta), & \text{if } -1 < x \leq -1 + \delta \\
    \varepsilon y''(x) + \alpha(x)y'(x) - b(x)y(x) + c(x)y(x - \delta) + d(x)y(x + \eta), & \text{if } -1 + \delta < x < 1 - \eta \\
    \varepsilon y''(x) + \alpha(x)y'(x) - b(x)y(x) + c(x)y(x - \delta), & \text{if } 1 - \eta \leq x < 1.
  \end{cases}
\end{equation}

and

\begin{equation}
  F(x) = \begin{cases}
    f(x) - c(x)\phi(x - \delta), & \text{if } -1 < x \leq -1 + \delta \\
    f(x), & \text{if } -1 + \delta < x < 1 - \eta \\
    f(x) - d(x)\gamma(x + \eta), & \text{if } 1 - \eta \leq x < 1.
  \end{cases}
\end{equation}

The operator \( L^{\delta, \eta} y \) satisfies the following minimum principle

**Lemma 6.2.1.** Let \( \Psi(x) \) be a smooth function satisfying \( \Psi(-1) \geq 0, \Psi(1) \geq 0 \). Then \( L^{\delta, \eta} \Psi(x) \leq 0 \). \( x \in \Omega \) implies \( \Psi(x) \geq 0, \forall x \in \bar{\Omega} \).
6.2. Continuous problem

**Proof.** If possible, suppose that there is a point \( x^* \in [-1,1] \) such that \( \Psi(x^*) = \min_{x \in \Omega} \Psi(x) \) and \( \Psi(x^*) < 0 \) which implies that \( x^* \notin \{-1,1\} \), \( \Psi'(x^*) = 0 \), and \( \Psi''(x^*) \geq 0 \).

Now we will prove that \( L^{\delta,\eta} \Psi(x) \geq 0 \).

**Case 1:** \(-1 < x^* \leq -1 + \delta \), in this case we have,
\[
L^{\delta,\eta} \Psi(x^*) = \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - b(x^*) \Psi(x^*) + d(x^*) \Psi(x^*) + \eta
\]
\[
= \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - \Psi(x^*) + d(x^*) \Psi(x^*) + \eta
\]
\[
> 0
\]

**Case 2:** \(-1 + \delta < x^* < 1 - \eta \)
\[
L^{\delta,\eta} \Psi(x^*) = \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - b(x^*) \Psi(x^*) + c(x^*) \Psi(x^* - \delta) + d(x^*) \Psi(x^* + \eta)
\]
\[
= \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - b(x^*) \Psi(x^*) + c(x^*) \Psi(x^* - \delta) + d(x^*) \Psi(x^* + \eta)
\]
\[
> 0
\]

**Case 3:** \(1 - \eta \leq x^* < 1 \)
\[
L^{\delta,\eta} \Psi(x^*) = \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - b(x^*) \Psi(x^*) + c(x^*) \Psi(x^* - \delta)
\]
\[
= \varepsilon \Psi''(x^*) + a(x^*) \Psi'(x^*) - b(x^*) \Psi(x^*) + c(x^*) \Psi(x^* - \delta)
\]
\[
> 0
\]

Combining the above three cases we get a contradiction to the assumption \( L^{\delta,\eta} \Psi(x) \leq 0 \). Hence \( \Psi(x) \geq 0 \), \( \forall x \in \Omega \). \( \square \)

**Lemma 6.2.2.** Let \( y(x) \) be the solution of the boundary value problem (6.1.1) and \( b(x) - c(x) - d(x) \geq K > 0 \), \( \forall x \in \Omega \). Then we have
\[
\| y(x) \| \leq \| f \|_{0} K^{-1} + C \max \{ \| \phi \|_{0}, \| \gamma \|_{0} \}.
\]

(6.2.4)

**Proof.** We consider the barrier function \( \Psi^\pm(x) \) defined by
\[
\Psi^\pm(x) = \| f \|_{0} K^{-1} + C \max \{ \| \phi \|_{0}, \| \gamma \|_{0} \} \pm y(x).
\]

We have
\[
\Psi^\pm(-1) = \| f \|_{0} K^{-1} + C \max \{ \| \phi \|_{0}, \| \gamma \|_{0} \} \pm y(-1)
\]
\[
= \| f \|_{0} K^{-1} + C \max \{ \| \phi \|_{0}, \| \gamma \|_{0} \} \pm \phi(-1)
\]
\[
\geq 0.
\]

168
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

\[ \Psi^\pm(1) = \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \pm y(1) \]

\[ = \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \pm \gamma(1) \]

\[ \geq 0. \]

Next we have

Case 1: \(-1 < x < -1 + \delta\)

\[ L^\Delta \Phi^\pm(x) = -b(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] + d(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] \]

\[ \pm \left( f(x) - c(x) \phi(x - \delta) \right) \]

\[ = - (b(x) - d(x)) \|f\|_0 K^{-1} \pm f(x) \] - \( (b(x) - d(x)) C \max \{ ||\phi||_0, ||\gamma||_0 \} \pm c(x) \phi(x - \delta) \]

\[ < 0, \]

Case 2: \(-1 + \delta < x < 1 - \eta,\)

\[ L^\Delta \Phi^\pm(x) = -b(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] + d(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] \]

\[ \pm \left( f(x) - c(x) \phi(x - \delta) \right) \]

\[ = - (b(x) - c(x) - d(x)) \|f\|_0 K^{-1} \pm f(x) \] - \( (b(x) - c(x) - d(x)) C \max \{ ||\phi||_0, ||\gamma||_0 \} \]

\[ < 0, \]

Case 3: \(1 - \eta < x < 1,\)

\[ L^\Delta \Phi^\pm(x) = -b(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] + c(x) \left[ \|f\|_0 K^{-1} + C \max \{ ||\phi||_0, ||\gamma||_0 \} \right] \]

\[ \pm \left( f(x) - d(x) \gamma(x + \eta) \right) \]

\[ = - (b(x) - c(x)) \|f\|_0 K^{-1} \pm f(x) \] - \( (b(x) - c(x)) C \max \{ ||\phi||_0, ||\gamma||_0 \} \]

\[ < 0. \]

Therefore we get \( L^\Delta \Phi^\pm(x) < 0, \forall x \in \Omega. \) Now an application of Lemma 6.2.1 yields \( \Psi^\pm(x) > 0, \forall x \in \Omega \) and thus giving the desired estimate.  

Now suppose \([p, q]\) is a subinterval of \([-1, 1]\) which do not contain the turning point, then we have following bound on the solution and its derivatives.

**Theorem 6.2.1.** Let \( a(x), b(x), c(x), d(x), f(x) \in C^m[-1, 1] \) where \( m \) is a positive integer, \( |a(x)| > \xi, ||a(x)||_0 = \kappa, x \in \Omega. \) Then there exist a positive constant \( C \) such that for \( a(x) < 0 \) on \([p, q]\), the solution \( y(x) \) of the problem (6.1.1) satisfies

\[ |y^{(k)}(x)| \leq C \left[ 1 + \varepsilon^{-k} \exp \left( -\frac{\xi(q - x)}{\varepsilon} \right) \right] \quad \text{for } k = 1, \ldots, m + 1 \]

169
6.2. Continuous problem

and for \( a(x) > 0 \) on \([p, q]\), we have

\[ |y^{(k)}(x)| \leq C \left[ 1 + \epsilon^{-k} \exp \left( \frac{-\xi(x-p)}{\epsilon} \right) \right] \quad \text{for } k = 1, \ldots, m + 1. \]

**Proof.** Let’s consider the case \( a(x) < 0 \) and the case \( a(x) > 0 \) can be proved analogously. Problem (6.1.1) can be written as

\[ \epsilon y''(x) - a_1(x)y'(x) = h(x) \quad (6.2.5) \]

where \( a_1(x) = -a(x), \ h(x) = f(x) + b(x)y(x) - c(x)y(x-\delta) - d(x)y(x+\eta). \)

The solution of the above equation is given by

\[ y(x) = y_u(x) + K_1 + K_2 \int_x^q \exp \left[ -\epsilon^{-1}(K(q)-K(t)) \right] dt, \quad (6.2.6) \]

where \( y_u(x) = -\int_x^q u(t)dt, \quad K(x) = \int a_1(x)dx \)

\[ u(t) = \int_x^q \epsilon^{-1}h(t)\exp \left[ -\epsilon^{-1}(K(t)-K(x)) \right] dt, \ \text{if } x \leq t. \quad (6.2.7) \]

Now we have

\[ K(t) - K(x) \geq \xi(t-x) \]

which implies

\[ \exp \left[ -\epsilon^{-1}(K(t)-K(x)) \right] \leq \exp \left[ -\epsilon^{-1}\xi(t-x) \right], \quad (6.2.8) \]

using (6.2.8) in (6.2.7) we get

\[ |u(x)| \leq C\epsilon^{-1} \int_x^q \exp \left[ -\epsilon^{-1}(K(t)-K(x)) \right] dt \leq C\epsilon^{-1}\xi \epsilon \leq C_1. \quad (6.2.9) \]

If \( y(p) = d_1, \ y(q) = d_2 \) are taken as the boundary conditions, then using this in (6.2.6) we get

\[ K_1 = d_2 \quad (6.2.10) \]

and

\[ K_2 \leq C\epsilon^{-1}. \quad (6.2.11) \]
Differentiating (6.2.6) once we get
\[ y'(x) = u(x) - K_2 \exp \left[ -\varepsilon^{-1}(K(q) - K(x)) \right]. \]  
(6.2.12)

Taking modulus on both sides and using triangle inequality we get
\[ |y'(x)| \leq |u(x)| + K_2 \exp \left[ -\varepsilon^{-1}(K(q) - K(x)) \right] \]
\[ \leq C \left[ 1 + \varepsilon^{-1} \exp \left( -\frac{\xi(x-x)}{\varepsilon} \right) \right] \]
(6.2.13)

which is the required estimate. The result for the case \( a(x) < 0 \) can be proved analogously.

Theorem 6.2.1 gives us bound on the derivatives of the solution of the problem (6.1.1) outside the turning point region. Next theorem gives us bounds on the derivatives of the solution in the neighborhood of the turning point.

**Theorem 6.2.2.** Let \( a(x), b(x), c(x), d(x), f(x) \in C^m[-1, 1], \beta \) being a non-integer such that \( 0 < \beta < 1 \) and conditions (6.1.3)-(6.1.7) holds good. Then there exist a positive constant \( C \) depending on \( S(m) = \{ ||a||_{2}, ||b||_{1}, ||c||_{1}, ||d||_{1}, ||f||_{1}, b_0, \beta, \beta, ||b||_{0}, ||c||_{0}, ||d||_{0}, ||f||_{0} \} \) such that for the solution \( y(x) \) of the problem (6.1.1) we have
\[ |y^{(k)}(x)| \leq C(|x| + \varepsilon^{1/2})^{\beta-k}, \quad k = 1, \ldots, m + 1. \]
(6.2.14)

**Proof.** Proof is similar to that of the Theorem 4.3.2. Proceeding similarly we obtain
\[ |y^{(k)}(x)| \leq C(||x|| + \varepsilon^{1/2})^{\beta-k} I(x, C, \beta) \quad \text{for} \quad -1 \leq x \leq 1, \quad k = 1, \ldots, m + 1. \]  
(6.2.15)

Now for \( 0 < \beta < 1 \)
\[ I(x, C, \beta) = \frac{2}{1-\beta} \left[ \frac{6^{(1-\beta)/2}}{2} - (x^2 + \varepsilon)^{(1-\beta)/2} \right] \]
\[ \leq C(\beta). \]
(6.2.16)

Substituting (6.2.16) in (6.2.15) we get (6.2.14).
6.3 Description of the numerical scheme

To construct the discrete counterpart of the problem (6.1.1) we consider an exponentially fitted finite difference scheme [62] on a specially designed uniform mesh. Presence of the delay/advance argument and the turning point make the problem (6.1.1) difficult to deal with. To deal with the delay/advance argument and the turning point, a mesh is designed in such a way that the term containing delay/advance argument and the turning point lies at the mesh point after the discretization. Also to deal with the turning point, forward difference is used in the first derivative term if \( a(x_t) > 0 \) whereas backward difference is used if \( a(x_t) < 0 \), \( x_t < x_2, \ldots, x_N = 1 \), \( x_t = -1 + ih \) where \( i = 0, \ldots, N \), \( h = 2/N \). Thus the difference scheme for the boundary value problem (6.1.1) is given by

\[
L^NY_i = \varepsilon \rho_i D_+ D_- Y_i + a_i D_i Y_i - b_i Y_i + d_i Y_{i+p} + c_i Y_{i-m} = f_i, \quad x \in \Omega^N
\]

(6.3.1)

where, \( \rho_i = \rho(x_i) \), \( a_i = a(x_i) \), \( b_i = b(x_i) \), \( c_i = c(x_i) \), \( d_i = d(x_i) \), \( f_i = f(x_i) \), \( \rho_i = \frac{a_i b}{2 \pi} \coth \left( \frac{a_i h}{2 \pi} \right) \), \( D_+ D_- Y_i = \frac{(Y_{i+1} - 2Y_i + Y_{i-1})}{h^2} \), \( D_+ D_+ Y_i = \frac{(Y_{i+1} - Y_i)}{h} \), \( D_- D_- Y_i = \frac{(Y_i - Y_{i-1})}{h} \), \( x_{N_1} \) is the turning point and

\[
D_+ = \begin{cases} 
D_+ D_+ Y_i & \text{if } a_i > 0 \\
D_- Y_i & \text{if } a_i < 0.
\end{cases}
\]

On simplification the discrete problem (6.3.1) gives

\[
L^NY_i = F_i
\]

(6.3.2)

where

\[
L^NY_i = \begin{cases} 
\varepsilon \rho_i D_+ D_+ Y_i + a_i D_+ Y_i - b_i Y_i + d_i Y_{i+p} & \text{for } i = 0, \ldots, m \\
\varepsilon \rho_i D_+ D_- Y_i + a_i D_+ Y_i - b_i Y_i + c_i Y_{i-m} + d_i Y_{i+p} & \text{for } i = m + 1, \ldots, N_1 - 1 \\
\varepsilon \rho_i D_+ D_- Y_i + a_i D_+ Y_i - b_i Y_i + c_i Y_{i-m} + d_i Y_{i+p} & \text{for } i = N_1, \ldots, N - p - 1 \\
\varepsilon \rho_i D_- D_- Y_i + a_i D_- Y_i - b_i Y_i + c_i Y_{i-m} & \text{for } i = N - p, \ldots, N
\end{cases}
\]

(6.3.3)
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed
differential-difference turning point problems with delay as well as advance

\[ F_i = \begin{cases} 
    f_i - c_i \phi_{i-m} & \text{for } i = 0, 1, \ldots, m \\
    f_i & \text{for } i = m + 1, \ldots, N - p - 1 \\
    f_i - d_i \gamma_{i-p} & \text{for } i = N - p, \ldots, N 
\end{cases} \quad (6.3.4) \]

\[ Y_0 = \phi_0 \]
\[ Y_N = \gamma_N. \quad (6.3.5) \]

Corresponding to the continuous problem we have discrete minimum principle and bound on the
discrete solution.

**Lemma 6.3.1.** Suppose \( \Psi_0 \geq 0 \) and \( \Psi_N \geq 0 \). Then \( L^N \Psi_i \leq 0 \) for all \( i = 1, 2, \ldots, N - 1 \) implies
that \( \Psi_i \geq 0 \) for all \( i = 0, \ldots, N \).

**Proof.** Let \( k \in \{0, 1, \ldots, N\} \) such that \( \Psi_k = \min_{0 \leq i \leq N} \Psi_i \). Let us assume \( \Psi_k < 0 \), then \( k \not\in \{0, N\} \) and we have \( \Psi_{k+1} - \Psi_k \geq 0 \), \( \Psi_k - \Psi_{k-1} \leq 0 \). Now the following cases arise:

Case 1: \( k \in \{1, \ldots, m\} \). In this case, we have

\[
L^N \Psi_k = \varepsilon \rho_k \left[ \frac{(\Psi_{k+1} - \Psi_k) - (\Psi_{k+1} - \Psi_{k-1})}{h^2} \right] + a_k \left[ \frac{\Psi_{k+1} - \Psi_k - b_k \Psi_k + d_k \Psi_{k+p}}{h} \right] - b_k \Psi_k + c_k \Psi_{k-m} + d_k \Psi_{k+p} \\
> 0. 
\]

Case 2: \( k \in \{m + 1, \ldots, N_1 - 1\} \), here

\[
L^N \Psi_k = \varepsilon \rho_k \left[ \frac{(\Psi_{k+1} - \Psi_k) - (\Psi_{k+1} - \Psi_{k-1})}{h^2} \right] + a_k \left[ \frac{\Psi_{k+1} - \Psi_k - b_k \Psi_k + d_k \Psi_{k+p}}{h} \right] - b_k \Psi_k + c_k \Psi_{k-m} + d_k \Psi_{k+p} \\
> 0. 
\]

Case 3: \( k \in \{N_1, \ldots, N - p - 1\} \)

\[
L^N \Psi_k = \varepsilon \rho_k \left[ \frac{(\Psi_{k+1} - \Psi_k) - (\Psi_{k+1} - \Psi_{k-1})}{h^2} \right] + a_k \left[ \frac{\Psi_{k+1} - \Psi_k - b_k \Psi_k + d_k \Psi_{k+p}}{h} \right] - b_k \Psi_k + c_k \Psi_{k-m} + d_k \Psi_{k+p} \\
> 0. 
\]

Case 4: \( k \in \{N - p, \ldots, N - 1\} \)

\[
L^N \Psi_k = \varepsilon \rho_k \left[ \frac{(\Psi_{k+1} - \Psi_k) - (\Psi_{k+1} - \Psi_{k-1})}{h^2} \right] + a_k \left[ \frac{\Psi_{k+1} - \Psi_k - b_k \Psi_k + d_k \Psi_{k+p}}{h} \right] - b_k \Psi_k + c_k \Psi_{k-m} \\
> 0. 
\]

Thus for \( k \in \{1, \ldots, N - 1\} \) the above four cases leads to a contradiction to the hypothe­
sis \( L^N \Psi_k \leq 0 \). therefore \( \Psi_k \geq 0 \). Since \( k \) was chosen arbitrary we have \( \Psi_i \geq 0 \) for all
\( i = 0, 1, \ldots, N \). \( \square \)
6.3. Description of the numerical scheme

Lemma 6.3.2. The solution $Y$ of the discrete problem (6.3.4) with the boundary conditions (6.3.5) satisfies

$$||Y||_0 \leq ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0)$$

where $K = \min_{0 \leq i \leq N} \{(b_i - c_i - d_i)\}$.

Proof. Let us consider the two barrier functions

$$\Psi_i^\pm = ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0) \pm Y_i$$

then

$$\Psi_i^+ = ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0) \pm Y_0$$
$$\geq 0$$

and

$$\Psi_i^- = ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0) \pm Y_N$$
$$\geq 0.$$ 

Now the following four cases arise

Case 1 : $1 < i < m$

$$L^N \Psi_i^+ = a_i D_\gamma \Psi_i^+ + a_i D_\phi \Psi_i^+ - b_i \Psi_i^+ + d_i \Psi_{i+1}^+$$
$$= -b_i \left[ ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0) \right] + d_i \left[ ||f||_0 K^{-1} + C \max(||\phi||_0, ||\gamma||_0) \right] \pm L^N Y_i$$
$$= \left[-(b_i - d_i) ||f||_0 K^{-1} \pm f_i\right] - (b_i - d_i) C \max(||\phi||_0, ||\gamma||_0) \pm c_i \phi_{i-m}$$
$$\leq 0,$$
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

Case 2: \( m + 1 \leq i \leq N_1 - 1 \)

\[
L^N \psi_i^\pm = \epsilon D_+ D_- \psi_i^\pm + a_i D_- \psi_i^\pm - b_i \psi_i^\pm + c_i \psi_{i-m}^\pm + d_i \psi_{i+p}^\pm
\]

\[
= -b_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right] + d_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right]
\]

\[
+ c_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right] \pm L^N y_i
\]

\[
= \left[ -(b_i - c_i - d_i) \|f\|_0 K^{-1} \pm f_i \right] - (b_i - c_i - d_i) C \max(\|\phi\|_0, \|\gamma\|_0)
\]

\[
\leq 0.
\]

Case 3: \( N_1 \leq i \leq N - p - 1 \)

\[
L^N \psi_i^\pm = \epsilon D_+ D_- \psi_i^\pm + a_i D_- \psi_i^\pm - b_i \psi_i^\pm + c_i \psi_{i-m}^\pm + d_i \psi_{i+p}^\pm
\]

\[
= -b_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right] + d_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right]
\]

\[
+ c_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right] \pm L^N y_i
\]

\[
= \left[ -(b_i - c_i - d_i) \|f\|_0 K^{-1} \pm f_i \right] - (b_i - c_i - d_i) C \max(\|\phi\|_0, \|\gamma\|_0)
\]

\[
\leq 0.
\]

Case 4: \( N - p \leq i < N \)

\[
L^N \psi_i^\pm = \epsilon D_+ D_- \psi_i^\pm + a_i D_- \psi_i^\pm - b_i \psi_i^\pm + c_i \psi_{i-m}^\pm
\]

\[
= -b_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right] + c_i \left[ \|f\|_0 K^{-1} + C \max(\|\phi\|_0, \|\gamma\|_0) \right]
\]

\[
\pm L^N y_i
\]

\[
= \left[ -(b_i - c_i) \|f\|_0 K^{-1} \pm f_i \right] - b_i C \max(\|\phi\|_0, \|\gamma\|_0) \pm d_i \psi_{i+p}
\]

\[
\leq 0.
\]

From the above four cases we obtain \( L^N \psi_i \leq 0 \) for \( i = 1, \ldots, N - 1 \) and using Lemma 6.3.1 this proves that \( \psi_i \geq 0 \) for \( i = 0, \ldots, N \) and hence gives the desired estimates.

Now the following theorem gives bound on the truncation error.
6.3. Description of the numerical scheme

**Theorem 6.3.1.** If $Y$ is the solution of the discrete problem (6.3.1) corresponding to the solution $y(x)$ of the problem (6.1.1) then the truncation error is estimated by

$$|\tau_i^N| = |(L_N y_i - L^x,y_i)| \leq C h_{\text{min}}^{(3.1)}, \quad 0 \leq i \leq N.$$  

**Proof.** We will prove the result for the case $a(x_i) \geq 0$ and the case $a(x_i) < 0$ can be proved analogously. We have

$$|\tau_i^N| = |L_N y_i - L^{x,y} y_i|$$

$$= |e_i y'' + a(x_i) y_i - a(x_i) y_i'| \quad \text{ (where } e_i^0 = \varepsilon p_i)$$

$$= |e_i y'' + a(x_i) y_i - a(x_i) y_i'|$$

$$\leq |e_i y''| + |a(x_i) y_i| + |a(x_i)| |y_i'|.$$  

(6.3.6)

For Il’in-Allen-Southwell scheme [62] the following result holds good

$$|e_i^0 - \varepsilon| \leq C h(|a(x_i)| + h) \quad \text{(6.3.7)}$$

and using mean value theorem in the turning point region we get

$$|a(x_i)| \leq C h \quad \text{(6.3.8)}$$

which when used in (6.3.7) gives

$$|e_i^0 - \varepsilon| \leq C h^2. \quad \text{(6.3.9)}$$

Using (6.3.8) and Taylor series expansion we get

$$|a(x_i)||D_x y_i - y_i'| \leq C h^2 |y''_i| + O(h^3)$$

$$\leq C h^2 (|x_i| + \varepsilon^{1/2})^{\beta - 2}. \quad \text{(6.3.10)}$$

Now for $\beta > 2$ we have

$$(|x_i| + \varepsilon^{1/2})^{\beta - 2} < C \quad \text{(6.3.11)}$$

and $\beta < 2$ gives

$$(|x_i| + \varepsilon^{1/2})^{\beta - 2} < h^{\beta - 2}. \quad \text{(6.3.12)}$$
Using (6.3.11) and (6.3.12) in (6.3.10) we get

\[ |a(x_i)|D_\pm y_i - y'_i| \leq C h_{min(\beta, 2)}. \]  

(6.3.13)

Now (6.3.9) gives us

\[ |\varepsilon_i^0 - \varepsilon_i|y''_i| \leq Ch^2(\varepsilon^{1/2})^{\beta-2}. \]  

(6.3.14)

If \( \beta < 2 \)

\[(|x| + \varepsilon^{1/2})^{\beta-2} < h^{\beta-2}.\]  

(6.3.15)

whereas if \( \beta > 2 \),

\[(|x| + \varepsilon^{1/2})^{\beta-2} < C\]  

(6.3.16)

using (6.3.15) and (6.3.16) in (6.3.14) we get

\[ |\varepsilon_i^0 - \varepsilon_i|y''_i| \leq C h_{min(\beta, 2)}. \]  

(6.3.17)

Further we have

\[ \left| \varepsilon_i^0 \right|D_\pm D_\pm y_i - y''_i| \leq C(\varepsilon^{1/2} + h)^2 h^2|y''_i| \leq C(\varepsilon^{1/2} + h)^2 h^2(|x_i| + \varepsilon^{1/2})^{\beta-4}. \]

Here the following three cases arise:

Case 1: \( |x| > h. \)

If \( \beta > 4 \) then

\[(|x_i| + \varepsilon^{1/2})^{\beta-4} \leq C.\]  

(6.3.18)

which implies

\[ |\varepsilon_i^0|D^-D^- y_i - y''_i| \leq C h^2. \]  

(6.3.19)

If \( \beta < 4 \) then

\[ \frac{1}{|x_i| + \varepsilon^{1/2}} < \frac{1}{h + \varepsilon^{1/2}} < \frac{1}{h}. \]  

(6.3.20)
6.3. Description of the numerical scheme

which gives

\[ |\varepsilon^0_1||D^+D^-y_i - y_i''| \leq C(\varepsilon^{1/2} + h)^2 h^2(|x_i| + \varepsilon^{1/2})^\beta - 4 \]
\[ \leq (\varepsilon^{1/2} + h)^2 h^2 \frac{1}{(\varepsilon^{1/2} + h)^{\beta - 4}} \]
\[ \leq Ch^2 \frac{1}{h^{\beta - 2}} \]
\[ \leq Ch^\beta. \quad (6.3.21) \]

Combining (6.3.19) and (6.3.21) we have

\[ |\varepsilon^0_1||D^+D^-y_i - y_i''| \leq C h^{\min(\beta, 2)}. \quad (6.3.22) \]

Case 2: \( |x| = h \), in this case we have

\[ |\varepsilon^0_1||D^+D^-y_i - y_i''| \leq C(\varepsilon^{1/2} + h)^2 h^2 (h + \varepsilon^{1/2})^{\beta - 4} \]
\[ \leq Ch^2 (h + \varepsilon^{1/2})^{\beta - 2} \]
\[ \leq Ch^\beta. \quad (6.3.23) \]

Case 3: \( x = 0 \).

Here \( a(x) = 0 \) therefore we have \( \rho \to 0 \) and hence \( \varepsilon^0 \to \varepsilon \). In this case our difference scheme is converted to the classical finite difference scheme which is convergent of \( O(h) \) for \( h < \varepsilon^{1/2} \).

Next if \( h > \varepsilon^{1/2} \) then

\[ \varepsilon + h^2 < Ch^2 \quad (6.3.24) \]

which gives

\[ |\varepsilon^0_1||D^+D^-y_i - y_i''| \leq Ch^2 \varepsilon^{\beta/2} \]
\[ \leq C\varepsilon^{\beta/2} \]
\[ \leq Ch^\beta. \quad (6.3.25) \]

Combining the above three cases we have

\[ |\varepsilon^0_1||D^+D^-y_i - y_i''| \leq Ch^{\min(\beta, 2)}. \quad (6.3.26) \]
Thus combining (6.3.13), (6.3.17) and (6.3.26) we get
\[ |\tau| \leq C h^{\min(\beta, 2)} \]  
(6.3.27)

in the turning point region. Also, away from the turning point region our solution is smooth and we have
\[ |y^{(k)}(x)| \leq C, \quad k = 0, 1, 2, \ldots, \]  
(6.3.28)

using (6.3.28) we have
\[ |a(x_i)||D_x y_i - y_i'| \leq C h \]  
(6.3.29)
\[ |\varepsilon_{i}^{0} - \varepsilon||y_{i}''| \leq C h (|a(x_i)| + h) \leq C h \]  
(6.3.30)
\[ |\varepsilon_{i}^{0}||D_x D_y y_i - y_i''| \leq C (\varepsilon + h) h^2 |y''(x)| \leq C h^2 (\varepsilon + h) \leq C h^2. \]  
(6.3.31)

Combining (6.3.29), (6.3.30), and (6.3.31) we have
\[ |\tau| \leq C h. \]  
(6.3.32)

Finally combining the truncation error (6.3.27) and (6.3.32) we get
\[ |\tau_i^N| \leq C h^{\min(\beta, 1)}, \quad i = 0, 1, \ldots, N. \]  
(6.3.33)

**Remark:** Theorem 6.3.1 together with the uniform stability result (Lemma 6.3.2) proves uniform convergence of the proposed numerical scheme.

### 6.4 Test examples and numerical results

In this section, we apply the proposed numerical scheme on some test problems. Since exact solutions for the considered problems are not available, the maximum pointwise errors \( E_i^N \) are
6.4. Test examples and numerical results

evaluated using the double mesh principle for the proposed numerical scheme

\[ E^N = \max_{x \in \Omega} |Y^N_j - Y^{2N}_j| \]

where \( Y^N_j \) and \( Y^{2N}_j \) are the solutions computed by taking \( N \) and \( 2N \) points, respectively. The numerical rate of convergence is computed using the formula

\[ r_N = \log_2\left( \frac{E^N}{E^{2N}} \right). \]

**Example 1:** Consider the problem on \( x \in (0,1) \) with the turning point at \( x = 0.5 \)

\[
\varepsilon y''(x) + 2(x - 0.5)[1 + 0.3121(x - 0.5)]y'(x) - [4/3 + 0.2764(x - 0.5)]y(x) + 0.2y(x - \delta) + 0.125y(x + \eta) = x \\
y(x) = 0, \ -\delta \leq x \leq 0, \quad y(x) = 0, \ 1 \leq x \leq 1 + \eta.
\]

**Example 2:** Consider the problem on \( x \in (0,1) \) with the turning point at \( x = 0.5 \)

\[
\varepsilon y''(x) + (x - 0.5)[3 + 4(x - 0.5)]y'(x) - 2y(x) + 4(x - 0.5)^2y(x - \delta) + y(x + \eta) = 1 \\
y(x) = 0, \ -\delta \leq x \leq 0, \quad y(x) = 1, \ 1 \leq x \leq 1 + \eta.
\]

**Example 3:** Consider the problem on \( x \in (-1,1) \) with the turning point at \( x = 0 \)

\[
\varepsilon y''(x) + 2xy'(x) - y(x) - y(x - \delta) + y(x + \eta) = 0. \\
\text{with } \ y(x) = 1 \text{ on } -1 - \delta \leq x \leq -1, \quad y(x) = 1 \text{ on } 1 \leq x \leq 1 + \eta.
\]
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

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Table 6.1: The maximum pointwise error ($E^N$) and rate of convergence for the example 1 for $\delta = 0.2$, $\eta = 0.1$.

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Table 6.2: The maximum pointwise error ($E^N$) and rate of convergence for the example 2 for $\delta = 0.2$, $\eta = 0.1$. 

181
### 6.4. Test examples and numerical results

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Table 6.3: The maximum pointwise error \( (E^N) \) and rate of convergence for the example 1 for \( \delta = 0.2, \eta = 0.2 \).

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Table 6.4: The maximum pointwise error \( (E^N) \) and rate of convergence for the example 2 for \( \delta = 0.2, \eta = 0.2 \).
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance.

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Table 6.5: The maximum pointwise error ($E^N$) and rate of convergence for the example 1 for $\delta = 0.1$, $\eta = 0.2$.

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<td>1.03</td>
<td>1.01</td>
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<tr>
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<td>$4.501E-03$</td>
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</tr>
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<td>$7.097E-03$</td>
<td>$4.322E-03$</td>
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<td>$7.097E-03$</td>
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</tr>
</tbody>
</table>

Table 6.6: The maximum pointwise error ($E^N$) for the example 2 for $\delta = 0.1$, $\eta = 0.2$. 183
6.4. Test examples and numerical results

Table 6.7: The maximum pointwise error ($E_{\varepsilon}^N$) and rate of convergence for the example 1 for $\delta = 0.4, \eta = 0.2$.

<table>
<thead>
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<td>$4.093E-03$</td>
<td>$2.591E-03$</td>
<td>$1.533E-03$</td>
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<td>$4.093E-03$</td>
<td>$2.591E-03$</td>
<td>$1.533E-03$</td>
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Table 6.8: The maximum pointwise error ($E_{\varepsilon}^N$) and rate of convergence for the example 2 for $\delta = 0.4, \eta = 0.2$.

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</tr>
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<td>$1.098E-02$</td>
<td>$6.713E-03$</td>
<td>$4.11E-03$</td>
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<td>$1.098E-02$</td>
<td>$6.713E-03$</td>
<td>$4.11E-03$</td>
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<td>$1.747E-02$</td>
<td>$1.098E-02$</td>
<td>$6.713E-03$</td>
<td>$4.11E-03$</td>
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<td>$1.098E-02$</td>
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<td>0.70</td>
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<td>$1.098E-02$</td>
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<td>0.70</td>
<td>0.70</td>
<td>0.70</td>
<td>0.69</td>
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</table>
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

Table 6.9: The maximum pointwise error ($E^N_\tau$) and rate of convergence for the example 3 for $\delta = 0.2$, $\eta = 0.1$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
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<th>$N = 400$</th>
<th>$N = 800$</th>
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</tr>
<tr>
<td>10^{-2}</td>
<td>$7.785E - 03$</td>
<td>$3.841E - 03$</td>
<td>$1.902E - 03$</td>
<td>$9.469E - 04$</td>
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<td>10^{-6}</td>
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<td>$1.868E - 02$</td>
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<td>$2.695E - 02$</td>
<td>$1.866E - 02$</td>
<td>$1.397E - 02$</td>
<td>$1.033E - 02$</td>
</tr>
<tr>
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<tr>
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<td>$1.397E - 02$</td>
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Table 6.10: The maximum pointwise error ($E^N_\tau$) and rate of convergence for the example 3 for $\delta = 0.1$, $\eta = 0.2$.

<table>
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<td>$1.215E - 02$</td>
<td>$8.512E - 03$</td>
<td>$5.933E - 03$</td>
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<tr>
<td>10^{-12}</td>
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<td>$1.215E - 02$</td>
<td>$8.512E - 03$</td>
<td>$5.933E - 03$</td>
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<tr>
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<td>$5.933E - 03$</td>
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<tr>
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<td>$1.215E - 02$</td>
<td>$8.512E - 03$</td>
<td>$5.933E - 03$</td>
</tr>
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Singularly perturbed differential-difference equation exhibiting turning point behavior and having delay as well as advance in the reaction term is considered. If in -Allen- Southwell Fitted operator finite difference scheme is constructed on a specially designed uniform mesh. Numerical experiments are carried out to support the theoretical estimates and illustrate the effect of delay/advance arguments on the layer behavior of the solution. Table 1 – 11 gives maximum pointwise error for the considered examples for various values of $\delta$ and $\eta$ and it is seen that the rate of convergence is independent of the value of the delay/advance argument. It is observed that interior layer is maintained but layer get shifted as delay/advance argument changes. Shifts in the layer depends upon the size of the delay/advance argument as well as on the value of the coefficients of the terms containing delay/advance.

Figure 1 – 9 depicts effect of the delay/advance argument on the interior layer. Comparative
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$N = 100$</th>
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<th>$N = 400$</th>
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<td>$0.99$</td>
<td>$0.99$</td>
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<td>$10^{-4}$</td>
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<td>$0.99$</td>
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<td>$1.0$</td>
</tr>
<tr>
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<td>$1.614E-02$</td>
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<td>$0.75$</td>
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</table>

Table 6.11: The maximum pointwise error ($E_\infty^N$) and rate of convergence for the example 3 for $\delta = 0.2$, $\eta = 0.2$.

Figure 6.2: The numerical solution for example 1 when $\eta = 0$ ($\epsilon = 10^{-4}$).
6.5. Discussion

El-Mistikawy Wele exponential finite difference scheme

<table>
<thead>
<tr>
<th>$\delta$</th>
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<th>$E_f$</th>
<th>$E_r$</th>
<th>$E_r$</th>
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<td>1.99</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
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<td>1.859E-05</td>
<td>4.659E-06</td>
<td>1.168E-06</td>
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</table>

Fitted mesh finite difference scheme

<table>
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<th>$E_r$</th>
<th>$E_r$</th>
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<td>0.98</td>
<td>0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>$10^{-2}$</td>
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<td>4.049E-04</td>
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</table>

Il'in-Allen-Southwell fitted operator finite difference scheme

<table>
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<th>$E_r$</th>
<th>$E_r$</th>
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<tbody>
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<td>0.97</td>
<td>0.98</td>
<td>0.99</td>
<td>0.99</td>
</tr>
<tr>
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Table 6.12: The maximum pointwise error ($E_f^\infty$) and rate of convergence for the example 3 for $\delta = 0$, $\eta = 0$ using various techniques.
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance

Figure 6.3: The numerical solution for example 1 when \( \delta \) as well as \( \eta \neq 0 \) \((\epsilon = 10^{-3})\).

Figure 6.4: The numerical solution for example 2 when \( \delta \) as well as \( \eta \neq 0 \) \((\epsilon = 10^{-3})\).
6.5. Discussion

Figure 6.5: The numerical solution for example 2 when $\eta = 0$ ($\varepsilon = 10^{-3}$).

Figure 6.6: The numerical solution for example 2 when $\delta = 0$ ($\varepsilon = 10^{-3}$).
Chapter 6: Exponentially fitted finite difference scheme for singularly perturbed differential-difference turning point problems with delay as well as advance.

Figure 6.7: The numerical solution for example 3 when $\delta = 0 (\varepsilon = 10^{-2})$.

Figure 6.8: The numerical solution for example 3 when $\eta = 0 (\varepsilon = 10^{-2})$.
6.5. Discussion

Figure 6.9: The numerical solution for example 3 for various values of $S, r_j (s = 10^{-2})$.

study for various scheme for the case when $\delta = \eta = 0$ is given (Table 12). It is observed that for $\delta = \eta = 0$, El-Mistikawy Werle exponential finite difference scheme and Fitted mesh finite difference scheme are better than the II’in Allen Southwell Fitted operator finite difference scheme in terms of rate of convergence. The advantage of the II’in Allen Southwell fitted operator finite difference scheme proposed in this chapter over the schemes given in chapter 4 and 5 is that whereas those scheme work only for sufficiently smaller value of the delay/advance argument which put restriction on the size of the delay/advance argument while this scheme also works for the bigger values of the delay/advance parameter, i.e., it works irrespective of the size of the delay/advance argument.