Chapter 5

Fitted mesh finite difference scheme for singularly perturbed differential difference turning point problems with small delay as well as advance

5.1 Introduction

In this chapter, we design a robust fitted mesh finite difference scheme for the numerical solution of singularly perturbed differential-difference turning point problems with delay as well as advance and exhibiting interior layer. Singularly perturbed problems with or without turning points arise very frequently in applications and have been extensively studied in recent years. They occur in various fields of applied mathematics and engineering applications, for instance, Michaelis-Menten theory for enzyme reaction [163], simulation of oil extraction from underground reservoirs [57], the Navier Stokes equation of fluid flow at high Reynolds number [185], control theory [171], electrical networks [185] and many other physical models. It is a well-known fact that the solution of the singularly perturbed boundary-value problem exhibit a multiscale character, i.e., there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. So, the numerical treatment of singularly perturbed differential equations give
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major computational difficulties which need to be addressed.


Singularly perturbed differential-difference equations provide more realistic models in comparison to conventional singularly perturbed ordinary differential equations for phenomenon in many areas of science that display lag. The numerical treatment of singularly perturbed differential-difference turning point problems present some major computational difficulties due to presence of the perturbation parameter, turning point and the shift terms. During last decade many researchers [7, 99, 103, 104, 106, 108, 109, 115, 128, 129, 130, 134, 177, 187, 189, 197] to name a few have worked on singularly perturbed differential-difference equations but their work is limited to the non-turning point case only. Lange and Miura [134] remarked that the solution exhibit more complicated turning point behavior when the coefficient of the convection term changes sign inside the domain.

There are two classes of $\varepsilon$-uniformly convergent finite difference methods namely, “fitted operator methods” where given the mesh, an appropriate finite difference operator is constructed and “special mesh” methods where given the finite difference operator an appropriate mesh is constructed. An early important contribution towards the construction of numerical methods by means of special mesh was made by Bakhvalov [16] in 1969. In early 1990s Shishkin [155, 199] proposed piecewise-equidistant meshes. Simpler structure of Shishkin meshes made them easier to analyze than other non-uniform meshes, although numerical approximations given by them are
5.2 Problem formulation

We consider the following singularly perturbed differential-difference equation with delay as well as advance on $\Omega = [-1, 1]$

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) + c(x)y(x - \delta) + d(x)y(x + \eta) = f(x), \quad x \in (-1, 1) = \Omega \quad (5.2.1)$$

under interval conditions

$$y(x) = \phi(x), \quad -1 - \delta \leq x \leq -1, \quad y(x) = \gamma(x), \quad 1 \leq x \leq 1 + \eta, \quad (5.2.2)$$

where $0 < \varepsilon \ll 1$ is a small parameter, $\delta, \eta$ are delay and advance arguments of $o(\varepsilon)$.

Taylor series approximation of the delay/advance arguments gives

$$y(x - \delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \quad (5.2.3)$$

$$y(x + \eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x) \quad (5.2.4)$$

and substituting (5.2.3), (5.2.4) in the differential equation (5.2.1) we get

$$L_\varepsilon y(x) = C_\varepsilon y''(x) + A_\varepsilon(x)y'(x) + B_\varepsilon(x)y(x) = f(x) \quad (5.2.5)$$

$$y(-1) = \phi(-1), \quad y(1) = \gamma(1).$$

where $C_\varepsilon = \left( \varepsilon + \frac{\delta^2}{2} c(x) + \frac{\eta^2}{2} d(x) \right)$, $A_\varepsilon(x) = (a(x) - \delta c(x) + \eta d(x))$, and $B_\varepsilon(x) = (-b(x) + c(x) + d(x))$. The solution of the problem (5.2.5) differs from the solution of the problem (5.2.1) by $O(\delta^3 y'''(x), \eta^3 y'''(x))$ which gives good approximation to the solution of the problem (5.2.1) provided the delay/advance argument is sufficiently small. Here we assume that the convection coefficient vanishes at $x = 0$ and we define $A_\varepsilon^0(x) = \lim_{\varepsilon \to 0} A_\varepsilon(x), \ x \in [-1.0]$ and $A_\varepsilon^0(x) =$
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\[ \lim_{\varepsilon \to 0} A_\varepsilon(x), \ x \in (0,1]. \] Also \( A_\varepsilon^+(0) = \lim_{\varepsilon \to 0^+} A_\varepsilon^+(x). \) The above problem is considered under the following assumptions

\begin{align*}
A_\varepsilon(0) &= 0, \quad A_\varepsilon'(0) > 0, \quad (5.2.6) \\
B_\varepsilon(x) &\leq -K < 0, \quad x \in (-1,1) \quad (5.2.7) \\
|A_\varepsilon(x)| &\geq \beta(x) = \theta \left( 1 - \exp \left( -\frac{r |x|}{\varepsilon} \right) \right), \quad r \geq 2 \theta > 0, \quad (5.2.8) \\
\int_{x=1}^{x} |A_\varepsilon'(t)| \, dt &\leq C, \quad x \in \Omega. \quad (5.2.9)
\end{align*}

We also define

\[ T^-(x) = (A_\varepsilon - A_\varepsilon^-)(x), \ x \in [-1,0], \] and \( T^+(x) = (A_\varepsilon - A_\varepsilon^+)(x), \ x \in [0,1] \) such that \[ |T^-(x)| \leq |T^+(0)| \exp \left( -\frac{\theta}{2C_\varepsilon |x|} \right), \ x \in [-1,1]. \] (5.2.10)

5.3 A priori estimates

**Lemma 5.3.1.** Let \( \Psi(x) \) be any smooth function satisfying \( \Psi(-1) \geq 0, \Psi(1) \geq 0. \) Then \( L_\varepsilon \Psi(x) \leq 0, \forall -1 < x < 1 \) implies that \( \Psi(x) \geq 0, \forall -1 \leq x \leq 1. \)

**Proof.** Let \( x^* \in \bar{\Omega} \) be such that \( \Psi(x^*) = \min_{-1 \leq x \leq 1} \Psi(x) \) and \( \Psi(x^*) < 0. \) Clearly \( x^* \notin \{-1,1\}. \) Also \( \Psi'(x^*) = 0 \) and \( \Psi''(x^*) \geq 0. \) Now we have

\[
L_\varepsilon \Psi(x^*) = C_\varepsilon \Psi''(x^*) + A_\varepsilon(x^*) \Psi'(x^*) + B_\varepsilon(x^*) \Psi(x^*)
\]
\[
> 0 \quad \text{[since \( B_\varepsilon(x) < 0, \forall x \in (-1,1). \)]}
\]

which contradicts the assumption \( L_\varepsilon \Psi(x) \leq 0, \forall x \in (-1,1). \) Therefore we must have \( \Psi(x^*) \geq 0 \) and thus \( \Psi(x) \geq 0, \forall x \in [-1,1]. \) \( \square \)

**Lemma 5.3.2.** (Stability). The solution \( y(x) \) of the problem (5.2.5) is bounded and satisfies the bound

\[
||y||_0 \leq \frac{||f||_0}{K} + \max(||\phi(-1)||, ||\gamma(1)||), \quad x \in [-1,1]
\]

where \( K \) is a positive constant such that \( B_\varepsilon(x) < -K < 0. \)
Proof. Consider the barrier function $\Psi^\pm(x) = K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm g(x)$. Therefore we have

$$
\Psi^\pm(-1) = K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm g(-1)
= K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm \phi(-1)
\geq 0,
$$

$$
\Psi^\pm(1) = K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm g(1)
= K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm \gamma(1)
\geq 0.
$$

Also

$$
L_x \Psi^\pm(x) = C_x (\Psi^\pm)'(x) + A_x (\Psi^\pm)'(x) + B_x (\Psi^\pm)(x)
= B_x (K^{-1}||f||_0 + \max(|\phi(-1)|, |\gamma(1)|)) \pm L_x g(x)
\leq 0.
$$

An application of Lemma 5.3.1 gives $\Psi^\pm(x) \geq 0$, $\forall x \in [-1, 1]$. Thus

$$
||y(x)||_0 \leq \frac{||f||_0}{K} + \max(|\phi(-1)|, |\gamma(1)|).
$$

Lemma 5.3.3. Assuming (5.2.6)-(5.2.10), the solution $y(x)$ of the problem (5.2.5) satisfy

$$
|y^{(k)}(x)| \leq CC^{-k}, \quad k = 0, 1, 2, \quad x \in \Omega.
$$

Proof. Integrating the differential equation (5.2.5) we get

$$
C_x y'(x) - C_x y'(-1) = \int_{-1}^{x} (f(t) - A_x y'(t) - B_x y(t)) dt.
$$
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Now
\[
\left| \int_{-1}^{x} A_{\varepsilon}(t)y'(t)dt \right| \leq A_{\varepsilon}(t)y(t)^{2}_{-1} + \|y\|_{0} \int_{-1}^{x} A'_{\varepsilon}(t)dt \quad (5.3.3)
\]
and
\[
\int_{-1}^{x} (f(t) - B_{\varepsilon}(t)y(t))dt \leq \|B_{\varepsilon}\|_{0}\|y\|_{0} + \|f\|_{0}. \quad (5.3.4)
\]

By mean value theorem there exist a point \( z \in (-1, -1 - C_{\varepsilon}) \) such that
\[
y'(z) = \frac{y(-1 + C_{\varepsilon}) - y(-1)}{C_{\varepsilon}} \quad (5.3.5)
\]
with \( x = z \) in (5.3.2) and using (5.3.3), (5.3.4) we obtain
\[
C_{\varepsilon}y'(-1) \leq 2\|y\|_{0} + \|B_{\varepsilon}\|_{0}\|y\|_{0} + \|f\|_{0} + 2\|A_{\varepsilon}\|_{0}\|y\|_{0} \int_{-1}^{x} A'_{\varepsilon}(t)dt \quad (5.3.6)
\]
\[
\leq \|f\|_{0} + \|B_{\varepsilon}\|_{0} + 2\|A_{\varepsilon}\|_{0} + \int_{-1}^{x} A'_{\varepsilon}(t)dt \leq C. \quad (5.3.7)
\]

Then for any \( x \in (-1, 1) \) we have
\[
C_{\varepsilon}|y'(x)| \leq 2\|f\|_{0} + \left[ 2 + 2 \left( \|B_{\varepsilon}\|_{0} + \|A_{\varepsilon}\|_{0} + \int_{-1}^{x} A'_{\varepsilon}(t)dt \right) \right] \|y\|_{0} \quad (5.3.8)
\]
\[
\Rightarrow |y'(x)| \leq CC_{\varepsilon}^{-1}. \quad (5.3.9)
\]

The result for higher order derivatives can be obtained by successively differentiating (5.2.1). □

To derive \( \varepsilon \)-uniform error estimates we need sharper bounds on the derivatives of the solution. For this, we divide our problem into two boundary value problems one each in the domain \([-1, 0]\) and \([0, 1]\). Let \( y_{1}(x), y_{2}(x) \) be the solution of the problems

\[
C_{\varepsilon}y''_{1}(x) + A_{\varepsilon}(x)y'_{1}(x) + B_{\varepsilon}y_{1}(x) = f(x), \quad x \in (-1, 0) \quad (5.3.8)
\]
\[
y_{1}(-1) = \phi(-1), \quad y_{1}(0) = y_{0}, \quad x \in (-1, 0)
\]

\[
y_{2}(x) = \int_{-1}^{x} y'_{1}(t)dt + y_{1}(0) - y_{0}, \quad x \in (0, 1)
\]

\[
y_{2}(1) = y_{0} - y_{0} = 0, \quad x \in (0, 1)
\]

\[
y_{2}(x) = \int_{-1}^{x} \left( A_{\varepsilon}(t)y'_{2}(t) + B_{\varepsilon}y_{2}(t) \right)dt + y_{1}(0) - y_{0}, \quad x \in (-1, 0)
\]

\[
y_{2}(x) = \int_{-1}^{x} \left( A_{\varepsilon}(t)y'_{2}(t) + B_{\varepsilon}y_{2}(t) \right)dt + y_{1}(0) - y_{0}, \quad x \in (0, 1)
\]
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\[ C_x y_2'(x) + A_x(x)y_2(x) + B_x y_2(x) = f(x), \quad x \in (0, 1) \]  
\[ y_2(0) = y_0, \quad y_2(1) = \gamma(1). \]  

We decompose the solution \( y_1, y_2 \) into regular components \( v_1, v_2 \) and singular components \( w_1, w_2 \) respectively. The regular component \( v_1 = v_{1,0} + C_x v_{1,1} + C_x^2 v_{1,2} \) of \( y_1 \) and \( v_2 = v_{2,0} + C_x v_{2,1} + C_x^2 v_{2,2} \) of \( y_2 \) satisfies

\[
L^1_v v_1(x) = C_x v_1''(x) + A_0 v_1'(x) + B_x v_1(x) = f(x), \quad x \in (-1, 0) \\
\quad v_1(-1) = \phi(-1), \quad v_1(0) = v_0 \tag{5.3.10}
\]

where \( v_{1,0}(x), v_{1,1}(x), \) and \( v_{1,2}(x) \) satisfy

\[
A_0 v_{1,0}'(x) + B_x v_{1,0}(x) = f(x), \quad v_{1,0}(-1) = \phi(-1) \\
A_0 v_{1,1}'(x) + B_x v_{1,1}(x) = -v_{1,0}'(x), \quad v_{1,1}(-1) = 0 \\
L^1_v v_{1,2}(x) = -v_{1,1}'(x), \quad v_{1,2}(-1)(x) = v_{1,2}(0) = 0, \tag{5.3.11}
\]

and

\[
L^2_v v_2(x) = C_x v_2''(x) + A_0^2 v_2'(x) + B_x v_2(x) = f(x), \quad x \in (0, 1) \\
\quad v_2(0) = v_0, \quad v_2(1) = \gamma(1), \tag{5.3.12}
\]

where \( v_{2,0}(x), v_{2,1}(x), \) and \( v_{2,2}(x) \) satisfy

\[
A_0^2 v_{2,0}'(x) + B_x v_{2,0}(x) = f(x), \quad v_{2,0}(1) = \gamma(1) \\
A_0^2 v_{2,1}'(x) + B_x v_{2,1}(x) = -v_{2,0}'(x), \quad v_{2,1}(1) = 0 \\
L^2 v_{2,2}(x) = -v_{2,1}'(x), \quad v_{2,2}(0) = v_{2,2}(1) = 0. \tag{5.3.13}
\]

The errors \( (L_1 - L^1_v) v_1(x), (L_2 - L^2_v) v_2(x) \) are incorporated into the singular component \( w_1(x), w_2(x) \) respectively. Thus \( w_1(x), w_2(x) \) are defined as the solution of

\[
L_x w_1(x) = T^-(x)v_1'(x), \quad w_1(-1) = 0, \quad w_1(0) = y_0 - v_1(0), \quad x \in (-1, 0) \tag{5.3.14}
\]
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and

\[ I_1, w_2(x) = T^+(x) v_2'(x), \quad w_2(1) = 0, \quad w_2(0) = y_0 - v_2(0), \quad x \in (0,1). \quad (5.3.15) \]

The following result gives bounds on the singular part \( w_1, \ w_2 \) and the regular part \( v_1, \ v_2 \) of the solution of the problems (5.3.8) and (5.3.9).

**Theorem 5.3.1.** If \( v_1(x), \ v_2(x), \ w_1(x), \ w_2(x) \) are the solution of (5.3.10), (5.3.12), (5.3.14), (5.3.15), respectively, then for \( k = 0, 1, 2 \) we have

\[
\begin{align*}
||v_1^{(k)}(x)||_0 &\leq C(1 + C_{\varepsilon}^{-(2-k)}), \quad x \in [-1,0] \\
||v_2^{(k)}(x)||_0 &\leq C(1 + C_{\varepsilon}^{-(2-k)}), \quad x \in [0,1] \\
|w_1^{(k)}(x)| &\leq C \varepsilon^{-k} \exp \left( -\frac{\theta}{2C_{\varepsilon}} |x| \right), \quad x \in [-1,0] \\
|w_2^{(k)}(x)| &\leq C \varepsilon^{-k} \exp \left( -\frac{\theta}{2C_{\varepsilon}} |x| \right), \quad x \in [0,1].
\end{align*}
\]

**Proof.** We observe that \( v_{1,0}, \ v_{1,1}, \ v_{2,0}, \ v_{2,1} \) are independent of \( C_{\varepsilon} \) and \( v_{1,2}, \ v_{2,2} \) are the solution of the problem similar to that defining \( y_1, \ y_2 \) respectively. Therefore, using Lemma 5.3.3, we have

\[
\begin{align*}
||v_1^{(k)}||_0 &\leq C(1 + C_{\varepsilon}^{2-k}) \\
||v_2^{(k)}||_0 &\leq C(1 + C_{\varepsilon}^{2-k}).
\end{align*}
\]

Now we derive bounds for the singular component \( w_2(x) \) and its derivatives in \([0,1]\). In a similar fashion, one can prove an analogous result for \( w_1(x) \) in \([-1,0]\). To obtain bounds on the singular component \( w_2(x) \) let us define two barrier functions

\[
\Psi^\pm(x) = M \exp \left( -\frac{1}{2C_{\varepsilon}} \int_0^x \beta(t)\,dt \right) \pm w_2(x), \quad x \in [0,1] \quad M = \max\{|w_2(0)|, \|A_{\varepsilon}\||v_2||_0\}.
\]

We have \( \Psi^\pm(0), \Psi^\pm(1) \geq 0 \) and for sufficiently small \( \varepsilon \)

\[
L^2_2 \Psi^\pm(x) = \frac{M}{2C_{\varepsilon}^2} \left( \frac{\beta^2(x)}{2} - C_{\varepsilon} \beta(x) - A_{\varepsilon}^0(x) \beta(x) + 2C_{\varepsilon} B_{\varepsilon} \right) \exp \left( -\frac{1}{2C_{\varepsilon}} \int_0^x \beta(t)\,dt \right) \pm T^+(x) v_2'(x)
\]
\[
\leq - \left( \frac{M}{2C_{\varepsilon}^2} \left( \frac{\beta^2(x)}{2} + C_{\varepsilon} \beta(x) \right) \right) \pm 2C_{\varepsilon} |T^+(0)||v_2'||_0 \exp \left( -\frac{\theta |x|}{2C_{\varepsilon}} \right)
\]
\[
\leq 0.
\]
Using Lemma 5.3.1 and (5.2.6)-(5.2.10) we get

\[ |w_2(x)| \leq C \exp \left( \frac{-\theta |x|}{2C_\varepsilon} \right). \tag{5.3.17} \]

Next in order to find out bounds on the derivatives of the singular component \( w_2(x) \) we construct a neighborhood \( N_\varepsilon = (x, x + C_\varepsilon) \). Therefore by mean value theorem, \( \exists \) a point \( z \in N_\varepsilon \) such that

\[ w_2'(z) = \frac{w_2(x + C_\varepsilon) - w_2(x)}{C_\varepsilon}, \tag{5.3.18} \]

\[ \Rightarrow C_\varepsilon |w_2'(z)| \leq 2||w||_0. \tag{5.3.19} \]

We have, \( \int_x^z w_2'(t)dt = w_2'(x) - w_2'(z) \), substituting this value of \( \int_x^z w_2'(t)dt \) in (5.3.15) we obtain

\[ w_2'(x) - w_2'(z) = \int_x^z -C_\varepsilon^{-1}[A_\varepsilon(t)w_2'(t) + B_\varepsilon(t)w_2(t)]dt \tag{5.3.20} \]

which implies

\[ |w_2'(x)| \leq |w_2'(z)| + C_\varepsilon^{-1} \int_x^z -[A_\varepsilon(t)w_2'(t) + B_\varepsilon(t)w_2(t)]dt \]

\[ \leq 2C_\varepsilon^{-1}||w_2||_0 + C_\varepsilon^{-1} \int_x^z -[A_\varepsilon(t)w_2'(t) + B_\varepsilon(t)w_2(t)]dt. \tag{5.3.21} \]

Integrating by parts we have

\[ \int_x^z A_\varepsilon(t)w_2'(t)dt = A_\varepsilon(t)w_2(t)|_x^z - \int_x^z A_\varepsilon'(t)w_2(t)dt \]

\[ \leq 2(||A_\varepsilon||_0 + ||A_\varepsilon'||_0)||w_2||_0. \tag{5.3.22} \]

Using (5.3.22) in (5.3.21) we get

\[ |w_2'(x)| \leq CC_\varepsilon^{-1}||w_2||_0 \]

\[ \leq C C_\varepsilon^{-1} \exp \left( \frac{-\theta |x|}{2C_\varepsilon} \right). \tag{5.3.23} \]

Using similar arguments we can obtain bounds for \( w_1(x) \) which is given by

\[ |w_1'(x)| \leq CC_\varepsilon^{-1} \exp \left( \frac{-\theta |x|}{2C_\varepsilon} \right), \quad x \in \bar{\Omega}. \tag{5.3.24} \]

The result for higher order derivatives of the singular component follows by induction process. \( \square \)
To discretize the boundary value problem (5.2.5) we use fitted mesh finite difference scheme consisting of standard finite difference operator on piecewise uniformly fitted mesh condensing in the interior layer region around $x = 0$. The piecewise uniformly fitted mesh $\Omega^N = \{ x_i : 0 \leq i \leq N \}$ on the interval $[-1, 1]$ is constructed by partitioning the interval into four subintervals $[-1, -\tau], [-\tau, 0], [0, \tau], [\tau, 1]$ where the transition parameter $\tau$ is chosen such that

$$\tau = \min \left\{ \frac{1}{2} \frac{2C^*}{\theta} \ln N \right\}, \quad C^* = \varepsilon + \frac{\sigma^2}{2} M_1 + \frac{\nu^2}{2} M_2$$

where $M_1 = ||\phi||_0, \ M_2 = ||\psi||_0$. Then piecewise uniform mesh is described by mesh points $x_i$ such that

$$x_i = \begin{cases} -1 + \frac{4(1-\tau)i}{N}, & 0 \leq i \leq \frac{N}{4} \\ -\tau + \frac{i - \frac{N}{4}}{\frac{4N}{4}}, & \frac{N}{4} < i \leq \frac{N}{2} \\ \frac{i - \frac{N}{2}}{\frac{4N}{4}}, & \frac{N}{2} < i \leq \frac{3N}{4} \\ \tau + \frac{i - \frac{3N}{4}}{\frac{4N}{4}}, & \frac{3N}{4} < i \leq N \\ \end{cases}$$

and $h_i = x_i - x_{i-1}$. Note that this mesh is uniform if $\tau = 1/2$. On the piecewise uniform mesh $\Omega^N$ we use standard finite difference operator. Denoting $Y(x_i) = Y_i$ for any mesh function, the fitted mesh method for the problem (5.2.5) is as follows. Find a mesh function $Y_i$ such that

$$L^N Y_i = C^* D^+ Y_i + A_c(x_i) D^+ Y_i + B_c(x_i) Y_i = f(x_i), \quad x_i \in \Omega^N = \{ x_i : 1 \leq i \leq N - 1 \}$$

$$Y(-1) = \phi(-1), \quad Y(1) = \gamma(1), \quad (5.4.1)$$

with $D^+ Y_i = \frac{Y_{i+1} - Y_i}{h_{i+1}}, \ D^- Y_i = \frac{Y_i - Y_{i-1}}{h_i}, \ D^+ = \frac{2(D^+ Y_i - D^- Y_i)}{h_i + h_{i+1}}$ and

$$D^+ Y_i = \begin{cases} D^+ Y_i & \text{if } A_c(x_i) > 0 \\ D^- Y_i & \text{if } A_c(x_i) < 0 \end{cases}$$

Lemma 5.4.1. Let $\Psi$ be a mesh function defined on $\hat{\Omega}$. If $\min \{ \Psi_0, \Psi_N \} \geq 0$ and $L^N \Psi_i \leq 0, \ 1 \leq i \leq N - 1$ implies $\Psi_i \geq 0, \ for \ all \ i, \ 0 \leq i \leq N$. 150
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Proof. Let \( \Psi_k = \min_{0 \leq i \leq N} \Psi_i \), and suppose \( \Psi_k < 0 \). It follows from the hypothesis that \( k \notin \{0, N\} \). Also we have \( \Psi_k - \Psi_{k-1} \leq 0 \), \( \Psi_{k+1} - \Psi_k \geq 0 \).

Now

\[
L^N \Psi_k = C_{ij} D^2 \Psi_k + A_i(x_k) D^1 \Psi_k + B_i(x_k) \Psi_k
\]

\[
> 0
\]

which contradicts the hypothesis \( L^N \Psi_i \leq 0 \) for all \( 1 \leq i \leq N - 1 \). Therefore \( \Psi_k \geq 0 \) and since \( k \) is chosen arbitrary, therefore \( \Psi_i \geq 0 \) for all \( 0 \leq i \leq N \). \( \square \)

Lemma 5.4.2. If \( Y \) is the solution of the problem (5.4.1), then

\[
|Y(x_i)| \leq K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|), \quad \forall x_i \in \Omega^N.
\]  

(5.4.2)

Proof. Considering the barrier function

\[
\Phi^k(x_i) = K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm Y(x_i)
\]

(5.4.3)

we have

\[
\Phi^\pm(-1) = K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm Y(-1)
\]

\[
= K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm \phi(-1)
\]

\[
\geq 0
\]

and

\[
\Phi^\pm(1) = K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm Y(1)
\]

\[
= K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \pm \gamma(1)
\]

\[
\geq 0.
\]  

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Also for \( x \in \Omega^N \) we have

\[
L^N \phi^\pm(x_i) = B_i(x_i) \left[ K^{-1} ||f||_0 + \max(|\phi(-1)|, |\gamma(1)|) \right] \pm L^N Y(x_i)
\]

\[
= [B_i(x_i) K^{-1} ||f||_0 \pm f(x_i)] + B_i(x_i) \max(|\phi(-1)|, |\gamma(1)|)
\]

\[
\leq 0 \quad \left( \cdot \frac{B_i(x_i)}{K} \leq -1 \right).
\]

Using Lemma 5.4.1 for the barrier function \( \Phi^\pm(x_i) \), \( \forall x_i \in \Omega^N \) we get the desired bound. □

Now we decompose the solution \( Y \) of the discrete problem into the regular components \( V_L(x_i), V_R(x_i) \) and the singular components \( W_L(x_i), W_R(x_i) \), respectively. The regular components \( V_L(x_i) \) and \( V_R(x_i) \) are defined as the solution of the following discrete problems

\[
L^N V_L(x_i) = (C \delta_x D^2 + A^-_0(x_i) D^- + B_i(x_i)) V_L(x_i) = f(x_i), \quad x_i \in \Omega^N \cap (-1,0)
\]

\[
V_L(-1) = v_1(-1), \quad V_L(0) = v_1(0)
\]

(5.4.4)

and

\[
L^N V_R(x_i) = (C \delta_x D^2 + A^+_0(x_i) D^+ + B_i(x_i)) V_R(x_i) = f(x_i), \quad x_i \in \Omega^N \cap (0,1)
\]

\[
V_R(0) = v_2(0), \quad V_R(1) = v_2(1).
\]

(5.4.5)

The errors \( (L^N - L^N_k) V_L(x_i) \) and \( (L^N - L^N_k) V_R(x_i) \) are incorporated into the discrete components \( W_L(x_i) \) and \( W_R(x_i) \), respectively. Hence \( W_L(x_i) \) and \( W_R(x_i) \) are defined as the solution of the following discrete problems

\[
L^N W_L(x_i) = (L^N - L^N_k) V_L(x_i), \quad x_i \in \Omega^N \cap (-1,0)
\]

(5.4.6)

\[
W_L(-1) = 0, \quad W_L(0) = Y(0) - V_L(0)
\]

(5.4.7)

\[
L^N W_R(x_i) = (L^N - L^N_k) V_R(x_i), \quad x_i \in \Omega^N \cap (0,1)
\]

(5.4.8)

\[
W_R(1) = 0, \quad W_R(0) = Y(0) - V_R(0).
\]

(5.4.9)

Lemma 5.4.3. Let \( Z^1 \) and \( Z^2 \) be any two mesh functions such that \( Z^1(x_0) = Z^1(x_N/2) = 0 \) and \( Z^2(x_{N/2}) = Z^2(x_N) = 0 \), respectively. Then we have

\[
|Z^i(x_i)| \leq \frac{1}{K} \max_{1 \leq j \leq N/2 - 1} |L^N_k Z^j(x_j)|, \quad 0 \leq i \leq N/2.
\]

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Chapter 5: Fitted mesh finite difference scheme for singularly perturbed differential difference turning point problems with small delay as well as advance

\[ |Z^2(x_i)| \leq \frac{1}{K} \max_{N/2-1 \leq j \leq N-1} |L_N^2 Z^2(x_j)|, \quad N/2 \leq i \leq N. \]

**Proof.** Let us introduce two barrier functions \( \Psi^\pm \) defined by

\[
\Psi^\pm_i = M \pm Z_i
\]

where \( M = \frac{1}{K} \max_{1 \leq j \leq N/2-1} |L_N^1 Z^1_j|, \quad 0 \leq j \leq N/2 \) and \( M = \frac{1}{K} \max_{N/2+1 \leq j \leq N-1} |L_N^2 Z^2_j|, \quad N/2 \leq j \leq N \) then

\[
\Psi^\pm_0 = M \pm Z^1_0 \geq 0
\]

\[
\Psi^\pm_{N/2} = M \pm Z^1_{N/2} \geq 0
\]

and

\[
L_N^1 \Psi^\pm_i = MB(x_i) \pm L_N^1 Z^1(x_i)
\]

\[
\leq 0
\]

\( \forall 1 \leq i \leq N/2 - 1. \) Therefore, by discrete minimum principle (Lemma 5.4.1), we get \( \Psi^\pm_i \geq 0, \) for \( 0 \leq i \leq N/2. \) The result for \( Z^2(x_i) \) for \( N/2 \leq i \leq N \) can be proved analogously. \( \square \)

**Theorem 5.4.1.** The solutions \( y \) of (5.2.5) and \( Y \) of (5.4.1) satisfy the following \( \varepsilon \)-uniform error estimate

\[
||| (Y - y)(x_i) ||_0 \leq C \varepsilon^{-1} n_i N, \quad x_i \in \hat{\Omega}^N.
\]

**Proof.** The estimate on the smooth component is obtained by using the following stability and consistency argument. We consider the local truncation error

\[
L_N^1 (V_i - v_1)(x_i) = (L_N^1 - L_N^1) v_1, \quad 1 < i < N/2
\]

\[
= C_n \left( \frac{d^2}{dx^2} - D^2 \right) v_1(x_i) + A_n^1(x_i) \left( \frac{d}{dx} - D_+ \right) v_1(x_i), \quad (5.4.10)
\]
5.4. Fitted mesh finite difference scheme

\[ L^2_N(V_R - v_2)(x_i) = (L^2_L - L^2_N)v_2, \quad N/2 < i < N \]

\[ = C\left( \frac{\partial}{\partial r^2} - D^+ \right) v_2(x_i) + A^+_n(x_i) \left( \frac{d}{dr} - D^- \right) v_2(x_i). \]  \hspace{1cm} (5.4.11)

Then by using local truncation error estimates in (5.4.10) and (5.4.11), we obtain

\[ |L^1_N(V_L - v_1)(x_i)| \leq \frac{C}{3} (x_{i+1} - x_{i-1})[v_1^{(3)}] + \frac{A_n^-(x_i)}{2} (x_i - x_{i-1})[v_1^{(2)}], \quad 1 < i < N/2 \]  \hspace{1cm} (5.4.12)

and

\[ |L^2_N(V_R - v_2)(x_i)| \leq \frac{C}{3} (x_{i+1} - x_{i-1})[v_2^{(3)}] + \frac{A^+_n(x_i)}{2} (x_{i+1} - x_i)[v_2^{(2)}], \quad N/2 < i < N. \]  \hspace{1cm} (5.4.13)

Using Theorem 5.3.1 in (5.4.12) and (5.4.13) we get

\[ |L^1_N(V_L - v_1)(x_i)| \leq CN^{-1}, \quad x_i \in \bar{\Omega}^N \cap (-1, 0) \]  \hspace{1cm} (5.4.14)

and

\[ |L^2_N(V_R - v_2)(x_i)| \leq CN^{-1}, \quad x_i \in \bar{\Omega}^N \cap (0, 1). \]  \hspace{1cm} (5.4.15)

Applying Lemma 5.4.3 to the mesh functions \((V_L - v_1)(x_i)\) and \((V_R - v_2)(x_i)\) on the domain \(x_i \in \bar{\Omega}^N \cap [-1, 0]\), \(\bar{\Omega}^N \cap [0, 1]\) respectively, we obtain

\[ (V_L - v_1)(x_i) \leq CN^{-1}, \quad x_i \in \bar{\Omega}^N \cap [-1, 0] \]  \hspace{1cm} (5.4.16)

and

\[ (V_R - v_2)(x_i) \leq CN^{-1}, \quad x_i \in \bar{\Omega}^N \cap [0, 1]. \]  \hspace{1cm} (5.4.17)

Next we prove that \(W_R(x_i)\) and \(W_L(x_i)\) satisfy the following inequalities

\[ |W_R(x_i)| \leq |W_R(0)| \prod_{j=1}^i \left( 1 + \frac{\beta(x_j) h_j}{2C^2} \right)^{-1} + CN^{-1}(1 - x_i), \quad x_i \in \bar{\Omega}^N \cap [0, 1] \]  \hspace{1cm} (5.4.18)

and

\[ |W_L(x_i)| \leq |W_L(0)| \prod_{j=1}^i \left( 1 - \frac{\beta(x_j) h_j}{2C^2} \right)^{-1} + CN^{-1}(1 + x_i), \quad x_i \in \bar{\Omega}^N \cap [-1, 0]. \]  \hspace{1cm} (5.4.19)
We first prove (5.4.18). For this, we define and bound the function $\tilde{W}$ as follows

$$\tilde{W}(x_i) = \prod_{j=1}^{i} \left( 1 + \frac{\beta(x_j)h_j}{2C_\varepsilon} \right)^{-1} \geq \exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right)$$

(5.4.20)

where $h_i$ is such that $h_i = h_{i+1} = h_{i-1} = h$ and $h/C_\varepsilon \leq CN^{-1}\ln N$.

Now consider the barrier function

$$\Psi^\pm(x_i) = |W_R(0)|\tilde{W}(x_i) + \frac{2}{\theta} |T^+(0)||1 + \xi R\||aN^{-1}(1 - x_i) \pm W_R(x_i).$$

(5.4.21)

we have $\Psi^\pm(0), \Psi^\pm(1) \geq 0$.

For mesh points $x_i \in [x_{N/2+1}, x_{K-1}]$ where $K = N$ if $\tau = 1/2$ and $K = 3N/4$ if $\tau < 1/2$ we have $h_{i-1} = h_i = h_{i+1} = h$ and $h/C_\varepsilon \leq CN^{-1}\ln N$. Thus for sufficiently large $N$ using the inequality

$$\frac{\beta(x_i)\beta(x_{i+1})}{2} + C_\varepsilon D^+ \beta(x_i) \geq \theta^2/2$$

we have

$$L^N \Psi^\pm(x_i) \leq |W_R(0)|L^N \tilde{W}(x_i) + \frac{2}{\theta} |T^+(0)||1 + \xi R\||aN^{-1}L^N(1 - x_i) \pm L^N W_R(x_i)

\leq -\frac{W(x_{i+1})W_R(0)}{2C_\varepsilon} \left( \frac{\beta(x_i)\beta(x_{i+1})}{2} + C_\varepsilon D^+ \beta(x_i) \right) + B_i(x_i)W(x_i)

- \frac{2}{\theta} |T^+(0)||1 + \xi R\||aN^{-1}A_i(x_i) \pm |T^+(0)||1 + \xi R\||a\exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right)

\leq -\frac{1}{C_\varepsilon} \left( \frac{|W_R(0)|\theta^2}{8} \pm |T^+(0)||1 + \xi R\||a\exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right)

\leq 0.$$

For all other mesh points i.e., $x_i \in [x_{N}, x_{N-1}]$ where $K = 3N/4$ using inequalities $|T^+(x_i)| \leq |T^+(0)|\exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right) \leq |T^+(0)||N^{-1}$ and $\beta(x_i) \geq \theta \left( 1 - \exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right) \right) \geq \theta/2$ we have

$$L^N \Psi^\pm(x_i) \leq -\frac{2}{\theta} |T^+(0)||1 + \xi R\||aN^{-1}A_i(x_i) \pm |T^+(0)||1 + \xi R\||a\exp \left( -\frac{\theta x_i}{2C_\varepsilon} \right)

\leq 0.$$

Using Lemma 5.4.1 for the barrier function $\Psi^\pm(x_i)$ we get (5.4.18).
5.5. Numerical results

Similarly for proving (5.4.19) we define and bound $W$ as

$$W(x_i) = \prod_{j=1}^{i} \left( 1 - \frac{\beta(x_j)h_j}{2C_\varepsilon} \right)^{-1} \geq \exp \left( -\frac{\theta|x|}{2C_\varepsilon} \right)$$

where $h_i$ is such that $h_i = h_{i+1} = h_{i-1} = h$ and $h/C_\varepsilon \leq C \varepsilon^{-1} \ln N$ and consider the barrier function

$$\Psi^\pm(x_i) = |W_L(0)| W(x_i) + \frac{2}{\theta} |T^-(0)|||D^-V_L||_0 N^{-1}(1 + x_i) \pm W_L(x_i)$$

which are non-negative at $x = -1, x = 0$. Then in this case also we get for $x_i \in [x_0, x_{N/2-1}]$

$$L^N \Psi^\pm(x_i) = |W_L(0)| L^N W(x_i) + \frac{2}{\theta} |T^-(0)|||D^-V_L||_0 N^{-1} L^N (1 + x_i) \pm L^N W_L(x_i)$$

$$\leq - \frac{W(x_{i+1}) W_L(0)}{2C_\varepsilon} \left( \frac{\beta(x_i) \beta(x_{i+1})}{2} + C_\varepsilon D^+ \beta(x_i) \right) + B_i(x_i) W(x_i)$$

$$+ \frac{2}{\theta} |T^-(0)|||D^-V_L||_0 N^{-1} A_i(x_i) W(x_i) \pm |T^-(0)|||D^-V_L||_0 \exp \left( -\frac{\theta|x|}{2C_\varepsilon} \right)$$

$$\leq 0.$$ 

Using Lemma 5.4.1 we get (5.4.19).

Now using analysis analogous to Riordon ([175], Lemma 2.4) on both side of the turning point we get the desired result.

5.5 Numerical results

In this section, we shall present some numerical experiments and compare the actual performance with the theoretical results for the scheme discussed above. The exact solutions of these problems are not available, therefore we estimate the accuracy of the numerical solution by comparing it with the numerical solution on finer mesh [46].

Example 1:

$$\varepsilon y''(x) + 2\tanh \frac{4x}{\varepsilon} y'(x) - y(x) - 0.5 y(x-\delta) - 0.25 y(x+\eta) = \frac{1}{2} - x$$
subject to conditions

\[ y(x) = 1, \quad -1 - \delta \leq x \leq -1 \quad y(x) = 1, \quad 1 \leq x \leq 1 + \eta. \]

**Example 2:**

\[ \varepsilon y''(x) + 2xy'(x) - y(x) - y(x - \delta) + y(x + \eta) = 0. \quad x \in (-1, 1) \]

subject to conditions

with \[ y(x) = 1, \quad -1 - \delta \leq x \leq -1. \quad y(x) = 1, \quad 1 \leq x \leq 1 + \eta. \]

### 5.6 Discussion

This chapter is concerned with the numerical analysis of a class of singularly perturbed differential-difference equations with delay as well as advance and interior turning point. We obtained some
5.6. Discussion

### Fitted mesh finite difference scheme

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### El-Mistikawy Wele exponential finite difference scheme

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Table 5.1: The maximum pointwise error and the rate of convergence when applied to example 1 for various values of $\varepsilon$ and $N$, when $\delta = \eta = 0$. 

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Chapter 5: Fitted mesh finite difference scheme for singularly perturbed differential difference turning point problems with small delay as well as advance

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Table 5.2: The maximum pointwise error and rate of convergence when applied to example 1 for various values of $\varepsilon$ and $N$, when $\delta = 0.5 \varepsilon$, $\eta = 0$.

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<td>$1.212 \times 10^{-3}$</td>
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<td>$3.072 \times 10^{-4}$</td>
<td>$1.539 \times 10^{-4}$</td>
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<td>$1.564 \times 10^{-2}$</td>
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<td>$5.309 \times 10^{-3}$</td>
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<td>$1.491 \times 10^{-3}$</td>
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<td>$4.406 \times 10^{-2}$</td>
<td>$2.486 \times 10^{-2}$</td>
<td>$1.379 \times 10^{-2}$</td>
<td>$7.509 \times 10^{-3}$</td>
<td>$4.147 \times 10^{-3}$</td>
<td>$2.238 \times 10^{-3}$</td>
<td>$1.189 \times 10^{-3}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$4.610 \times 10^{-2}$</td>
<td>$2.628 \times 10^{-2}$</td>
<td>$1.465 \times 10^{-2}$</td>
<td>$8.094 \times 10^{-3}$</td>
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<td>$2.383 \times 10^{-3}$</td>
<td>$1.270 \times 10^{-3}$</td>
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<td>$10^{-4}$</td>
<td>$4.633 \times 10^{-2}$</td>
<td>$2.643 \times 10^{-2}$</td>
<td>$1.474 \times 10^{-2}$</td>
<td>$8.148 \times 10^{-3}$</td>
<td>$4.454 \times 10^{-3}$</td>
<td>$2.399 \times 10^{-3}$</td>
<td>$1.279 \times 10^{-3}$</td>
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<tr>
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<td>$4.633 \times 10^{-2}$</td>
<td>$2.644 \times 10^{-2}$</td>
<td>$1.475 \times 10^{-2}$</td>
<td>$8.153 \times 10^{-3}$</td>
<td>$4.457 \times 10^{-3}$</td>
<td>$2.402 \times 10^{-3}$</td>
<td>$1.280 \times 10^{-3}$</td>
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<td>$10^{-6}$</td>
<td>$4.633 \times 10^{-2}$</td>
<td>$2.645 \times 10^{-2}$</td>
<td>$1.475 \times 10^{-2}$</td>
<td>$8.153 \times 10^{-3}$</td>
<td>$4.457 \times 10^{-3}$</td>
<td>$2.402 \times 10^{-3}$</td>
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<td>$4.457 \times 10^{-3}$</td>
<td>$2.402 \times 10^{-3}$</td>
<td>$1.280 \times 10^{-3}$</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>$4.633 \times 10^{-2}$</td>
<td>$2.645 \times 10^{-2}$</td>
<td>$1.475 \times 10^{-2}$</td>
<td>$8.153 \times 10^{-3}$</td>
<td>$4.457 \times 10^{-3}$</td>
<td>$2.402 \times 10^{-3}$</td>
<td>$1.280 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5.3: The maximum pointwise error and rate of convergence when applied to example 1 for various values of $\varepsilon$ and $N$, when $\delta = 0$, $\eta = 0.5 \varepsilon$. 

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5.6. Discussion

<table>
<thead>
<tr>
<th>ε</th>
<th>N = 16</th>
<th>N = 32</th>
<th>N = 64</th>
<th>N = 128</th>
<th>N = 256</th>
<th>N = 512</th>
<th>N = 1024</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>9.511E - 03</td>
<td>5.036E - 03</td>
<td>2.611E - 03</td>
<td>1.329E - 03</td>
<td>6.706E - 04</td>
<td>3.369E - 04</td>
<td>1.688E - 04</td>
</tr>
<tr>
<td>10⁻¹</td>
<td>2.872E - 02</td>
<td>1.624E - 02</td>
<td>9.464E - 03</td>
<td>5.530E - 03</td>
<td>2.995E - 03</td>
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<td>7.908E - 04</td>
</tr>
<tr>
<td>10⁻²</td>
<td>4.199E - 02</td>
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<td>1.384E - 02</td>
<td>7.629E - 03</td>
<td>4.166E - 03</td>
<td>2.242E - 03</td>
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</tr>
<tr>
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<td>9.089E - 03</td>
<td>5.174E - 03</td>
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<tr>
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<td>2.644E - 02</td>
<td>1.475E - 02</td>
<td>8.153E - 03</td>
<td>4.575E - 03</td>
<td>2.402E - 03</td>
<td>1.280E - 03</td>
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Table 5.4: The maximum pointwise error and rate of convergence applied to example 1 for various values of ε and N, when δ = 0.9ε, η = 0.5ε.

<table>
<thead>
<tr>
<th>ε</th>
<th>N = 16</th>
<th>N = 32</th>
<th>N = 64</th>
<th>N = 128</th>
<th>N = 256</th>
<th>N = 512</th>
<th>N = 1024</th>
</tr>
</thead>
<tbody>
<tr>
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<td>9.130E - 03</td>
<td>5.001E - 03</td>
<td>2.606E - 03</td>
<td>1.329E - 03</td>
<td>6.707E - 04</td>
<td>3.373E - 04</td>
<td>1.689E - 04</td>
</tr>
<tr>
<td>10⁻¹</td>
<td>2.802E - 02</td>
<td>1.583E - 02</td>
<td>9.241E - 03</td>
<td>5.414E - 03</td>
<td>2.923E - 03</td>
<td>1.516E - 03</td>
<td>7.721E - 04</td>
</tr>
<tr>
<td>10⁻²</td>
<td>4.410E - 02</td>
<td>2.489E - 02</td>
<td>1.380E - 02</td>
<td>7.611E - 03</td>
<td>4.155E - 03</td>
<td>2.236E - 03</td>
<td>1.191E - 03</td>
</tr>
<tr>
<td>10⁻³</td>
<td>4.611E - 02</td>
<td>2.629E - 02</td>
<td>1.466E - 02</td>
<td>8.096E - 03</td>
<td>4.425E - 03</td>
<td>2.384E - 03</td>
<td>1.271E - 03</td>
</tr>
<tr>
<td>10⁻⁴</td>
<td>4.631E - 02</td>
<td>2.643E - 02</td>
<td>1.474E - 02</td>
<td>8.135E - 03</td>
<td>4.575E - 03</td>
<td>2.402E - 03</td>
<td>1.280E - 03</td>
</tr>
<tr>
<td>10⁻⁵</td>
<td>4.633E - 02</td>
<td>2.644E - 02</td>
<td>1.475E - 02</td>
<td>8.153E - 03</td>
<td>4.575E - 03</td>
<td>2.402E - 03</td>
<td>1.280E - 03</td>
</tr>
<tr>
<td>10⁻⁶</td>
<td>4.633E - 02</td>
<td>2.645E - 02</td>
<td>1.475E - 02</td>
<td>8.153E - 03</td>
<td>4.575E - 03</td>
<td>2.402E - 03</td>
<td>1.280E - 03</td>
</tr>
</tbody>
</table>

Table 5.5: The maximum pointwise error and rate of convergence when applied to example 1 for various values of ε and N, when δ = 0.5ε, η = 0.9ε.
## Chapter 5: Fitted mesh finite difference scheme for singularly perturbed differential difference turning point problems with small delay as well as advance

Table 5.6: Maximum pointwise errors and rate of convergence when applied to Example 2 for various values of $\varepsilon$ and $N$, when $\delta = 0.6\varepsilon$, $\eta = 0.8\varepsilon$.

<table>
<thead>
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<th>$N = 256$</th>
<th>$N = 512$</th>
<th>$1024$</th>
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</thead>
<tbody>
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<td>1.004$E - 03$</td>
<td>5.136$E - 04$</td>
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<td>1.306$E - 04$</td>
<td>6.551$E - 05$</td>
</tr>
<tr>
<td>2$^{-2}$</td>
<td>1.820$E - 03$</td>
<td>1.037$E - 03$</td>
<td>5.491$E - 04$</td>
<td>2.823$E - 04$</td>
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<td>5.862$E - 03$</td>
<td>3.165$E - 03$</td>
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<td>8.578$E - 04$</td>
<td>4.349$E - 04$</td>
<td>2.189$E - 04$</td>
</tr>
<tr>
<td>2$^{-8}$</td>
<td>1.381$E - 02$</td>
<td>6.546$E - 03$</td>
<td>2.932$E - 03$</td>
<td>1.298$E - 03$</td>
<td>5.988$E - 04$</td>
<td>2.889$E - 04$</td>
</tr>
<tr>
<td>2$^{-10}$</td>
<td>1.973$E - 02$</td>
<td>1.247$E - 02$</td>
<td>6.657$E - 03$</td>
<td>3.375$E - 03$</td>
<td>1.689$E - 03$</td>
<td>8.153$E - 04$</td>
</tr>
<tr>
<td>2$^{-12}$</td>
<td>1.804$E - 02$</td>
<td>9.195$E - 03$</td>
<td>4.290$E - 03$</td>
<td>1.961$E - 03$</td>
<td>9.841$E - 04$</td>
<td>4.985$E - 04$</td>
</tr>
<tr>
<td>2$^{-14}$</td>
<td>1.809$E - 02$</td>
<td>9.225$E - 03$</td>
<td>4.303$E - 03$</td>
<td>1.965$E - 03$</td>
<td>9.853$E - 04$</td>
<td>4.999$E - 04$</td>
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<td>2$^{-16}$</td>
<td>2.089$E - 02$</td>
<td>1.301$E - 02$</td>
<td>6.672$E - 03$</td>
<td>3.187$E - 03$</td>
<td>1.495$E - 03$</td>
<td>7.001$E - 04$</td>
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Table 5.7: Maximum pointwise error and rate of convergence when applied to Example 2 for various values of $\varepsilon$ and $N$, when $\delta = 0.5\varepsilon$, $\eta = 0.9\varepsilon$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
<th>$N = 128$</th>
<th>$N = 256$</th>
<th>$N = 512$</th>
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<tbody>
<tr>
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<td>2.487$E - 03$</td>
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<td>6.748$E - 04$</td>
<td>3.419$E - 04$</td>
<td>1.721$E - 04$</td>
<td>8.631$E - 05$</td>
</tr>
<tr>
<td>2$^{-2}$</td>
<td>1.756$E - 03$</td>
<td>1.012$E - 03$</td>
<td>5.378$E - 04$</td>
<td>2.774$E - 04$</td>
<td>1.408$E - 04$</td>
<td>7.002$E - 05$</td>
</tr>
<tr>
<td>2$^{-4}$</td>
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<td>1.714$E - 03$</td>
<td>8.930$E - 04$</td>
<td>4.555$E - 04$</td>
<td>2.301$E - 04$</td>
<td>1.157$E - 04$</td>
</tr>
<tr>
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<td>1.674$E - 03$</td>
<td>8.599$E - 04$</td>
<td>4.359$E - 04$</td>
<td>2.195$E - 04$</td>
</tr>
<tr>
<td>2$^{-8}$</td>
<td>1.386$E - 02$</td>
<td>6.546$E - 03$</td>
<td>2.932$E - 03$</td>
<td>1.298$E - 03$</td>
<td>5.988$E - 04$</td>
<td>2.890$E - 04$</td>
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<tr>
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<td>1.973$E - 02$</td>
<td>1.247$E - 02$</td>
<td>6.657$E - 03$</td>
<td>3.375$E - 03$</td>
<td>1.689$E - 03$</td>
<td>8.153$E - 04$</td>
</tr>
<tr>
<td>2$^{-12}$</td>
<td>1.804$E - 02$</td>
<td>9.195$E - 03$</td>
<td>4.290$E - 03$</td>
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<tr>
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<td>6.672$E - 03$</td>
<td>3.187$E - 03$</td>
<td>1.495$E - 03$</td>
<td>7.001$E - 04$</td>
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</tbody>
</table>

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Figure 5.2: The numerical solution for example 1 when $\delta = 0$ for $\varepsilon = 0.1$.

Figure 5.3: The numerical solution for example 1 for various values of $\varepsilon$ when $\delta = 0.5\varepsilon$, $\eta = 0.9\varepsilon$. 
Chapter 5: Fitted mesh finite difference scheme for singularly perturbed differential difference turning point problems with small delay as well as advance

a priori estimates on the solution and its derivatives for the above class of problem. To obtain the approximate solution of such type of problem a numerical approach based on fitted mesh finite difference scheme is presented. Table 1 – 7 display the results of our numerical experiments. From the tables we observe that the maximum pointwise error $E_{N,e}$ decreases as $N$ increases for each value of $\varepsilon$ which shows that the proposed method is parameter uniform with almost first order of convergence.

To study the effect of delay/advance argument on the interior layer behavior of the solution, graphs are plotted for the considered examples in Figure 1–3. We observe that when the coefficient of the terms containing delay/advance argument are negative (Example 1) steepness of the interior layer decreases as $\delta$ increases for $\eta = 0$ whereas, for $\delta = 0$ the steepness increases with increase in $\eta$. For $\delta$, $\eta$ both non-zero steepness of the solution depend upon the relative value of the coefficients of the shift terms. Comparing the numerical results obtained for example 2 with those of chapter 4 for various values of $\delta$ and $\eta$ it is observed that for smaller value of $N$ fitted mesh scheme is better than the El-Mistikawy Werle exponential finite difference scheme but the second one converges faster for large values of $N$. For example 1 we observe that El-Mistikawy Werle scheme is unstable (Table 1). So it is observed that when interior layer is of cusp type both scheme work but when the convection coefficient has exponential behavior fitted mesh scheme is better than El-Mistikawy Werle scheme.