Chapter 3

Geometric constructions of two-point methods

3.1 Introduction

In this chapter, the most relevant Newton-type iterative methods free from second-order derivative used for solving nonlinear equations of the form (1.1.1) and new geometrically constructed family of Ostrowski’s method, Stirling’s method, straight line method, Steffensen’s method, parabolic method, ellipse method by the author of the thesis are presented.

In the past and recent years, many researchers developed modification of Newton’s method or Newton-type methods [Jar66, Kin73, Wu00, MKKS05, Chu07c, SS11a, GK11b] free from second-order derivative in a number of ways to improve the local order of convergence at the expense of additional evaluations of functions and/or derivatives mostly at the point iterated by Newton’s method. All these modifications are targeted at increasing the local order of convergence with a view of increasing their efficiency index. Ostrowski’s method [Tra64, Ost60] and Jarratt’s method [Jar66] are the most efficient fourth-order two-point methods known to date. Another well-known example of fourth-order two-point methods with the

The contents of this chapter are published in:
same number of function evaluations is the King’s family [Kin73], given by
\[
\begin{align*}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= y_n - \frac{f(y_n)/f'(y_n)}{f'(x_n)/f'(y_n) + (\gamma - 2)f'(y_n)} + \gamma f'(y_n), \quad \gamma \in \mathbb{R}. 
\end{align*}
\] (3.1.1)

The research of finding iterative methods with optimal fourth-order convergence, not requiring the computation of second-order derivative is very important and interesting from practical point of view.

However, as is well-known, there are two major difficulties in the application of Newton’s method and its variants. Firstly, this method requires the computation of first-order derivative at each step. Generally, there are many practical situations in which the calculation of derivatives are expensive and/or it requires a great deal of time for them to be given or calculated. Therefore, derivative free methods are needed. Secondly, regarding the selection of initial guess such that neither the guess is far from required zero nor the derivative is small in the vicinity of required root (or \( f'(x) \neq 0 \)). There is no fixed criterion for choosing initial guess. Therefore, more effective globally convergent algorithms are still needed.

In order to overcome these problems, author also proposes new two-point families of Steffensen’s method, straight line method and ellipse method respectively. The beauty of these families is that whenever finding the derivative of functions is very complicated, one can use families of Steffensen’s method [Ste45] otherwise families of straight line method and ellipse method can be used for solving nonlinear equations.

The aim of this chapter is to provide two schemes that can be applied to any iteration function of order two or a family of second-order methods to further develop several new interesting families of cubically or quartically convergent iterative methods. These schemes are obtained by introducing quadratically convergent methods, a secant line and a parabola while moving along the curve for solving nonlinear equations numerically and the approach of derivation of the formula is different one. Fourth-order optimal family of Ostrowski’s method, parabolic method and ellipse method are the main findings of this chapter. Further, author also de-
develops many other new interesting families of Newton-Secant method, Steffensen’s
method, straight line method etc. Performance of the proposed family of Newton-
Secant method, straight line method, Steffensen’s method. Ostrowski’s method and
eellipse method are compared with their closest competitors namely, Newton-Secant
method, Ostrowski’s method, Jarratt’s method, King’s family etc. in a series of
numerical experiments.

3.2 First scheme of two-point methods

Here, author intends to construct new families of iteration method from available
iteration functions of order two based on a geometric observation. This work is a
further modification of Chun’s basic tool [Chu07c] for constructing the families of
higher order two-point methods. In the sequel, whenever author mentions that an
iteration function/a family of iteration functions \( \phi \) is of order \( p \), it means that the
corresponding iterative method defined by \( x_{n+1} = \phi(x_n) \) is of convergence order \( p \),
that is, the error \( |r - x_{n+1}| \) is proportional to \( |r - x_n|^p \) as \( n \to \infty \). One shall indicate
here that \( \phi \) is an iteration function whose order is \( p \) by writing \( \phi \in I_p \).

Our proposed scheme to develop new families of iteration functions is constructed
g eo metrically as follows:

Let \( \beta \) be a fixed parameter with \( 0 \leq \beta \leq 1 \) and let \( \phi(x_0) \in I_2 \) be an iteration
function of order two.

Let

\[
y = f(x),
\]

(3.2.1)

represents the graph of function \( f(x) \). Assume that two points, namely \( \left( \frac{m+\phi(x_0)}{2}, \frac{f(x_0)}{2} \right) \) and \( (\phi(x_0), \beta f(\phi(x_0))) \) lie on the same graph of function \( y = f(x) \). Then the approximated line of function \( f(x) \) passing through the above mentioned points is
given by

\[
y - \beta f(\phi(x_0)) = \frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0} (x - \phi(x_0)).
\]

(3.2.2)
Draw a parabola with vertex at \((\phi(x_0), 0)\) and axis parallel to \(y - \text{axis}\) on the graph of same function (3.2.1). The equation of this parabola is given by

\[
y = \alpha (x - \phi(x_0))^2,
\]

where \(\alpha\) is the scaling parameter. The parabola (3.2.3) widens as \(\alpha\) approaches zero and narrows as \(|\alpha|\) becomes large. The intersection of a line (3.2.2) with the parabola (3.2.3) is obtained by setting them equal to each other since each equals \(y\). Therefore, one end up with a quadratic equation given by

\[
\alpha(x - \phi(x_0))^2 - \frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0} (x - \phi(x_0)) - \beta f(\phi(x_0)) = 0.
\]

Solving this quadratic equation for \((x - \phi(x_0))\) and after some simplification, one gets the first approximation to the required root as

\[
x = \phi(x_0) + \frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0} \pm \sqrt{\frac{(2\beta f(\phi(x_0)) - f(x_0))^2}{\phi(x_0) - x_0} + 4\alpha \beta f(\phi(x_0))}.
\]

This can further be rewritten in the equivalent form (by rationalizing the numerator) as

\[
x = \phi(x_0) - \frac{2\beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0}\right) \pm \sqrt{\left(\frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0}\right)^2 + 4\alpha \beta f(\phi(x_0))}}.
\]

in which sign should be chosen so as to make the denominator largest in magnitude.

Now, consider the factor

\[
\frac{4\alpha \beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0}\right)^2}, \quad 0 \leq \beta \leq 1.
\]

Since the scaling parameter \(\alpha\) appears in the numerator of (3.2.7), it is clear that there exists some real values of \(\alpha\) such that

\[
\left|\frac{4\alpha \beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0}\right)^2}\right| < 1,
\]

(3.2.8)
3. Geometric constructions of two-point methods

holds.

With this assumption, the binomial theorem is applicable in equation (3.2.6) and one can get the following formula free from square root term as

\[
x = \phi(x_0) - \frac{\beta f(\phi(x_0)) (x_0 - \phi(x_0)) [f(x_0) - 2\beta f(\phi(x_0))]}{[f(x_0) - 2\beta f(\phi(x_0))]^2 + \alpha f(\phi(x_0)) (x_0 - \phi(x_0))^2}.
\]

(3.2.9)

Now repeating this process until the parabola becomes \(x - \text{axis}\), the general formula for successive approximation is given by

\[
x_{n+1} = \phi(x_n) - \frac{\beta f(\phi(x_n)) (x_n - \phi(x_n)) [f(x_n) - 2\beta f(\phi(x_n))]}{[f(x_n) - 2\beta f(\phi(x_n))]^2 + \alpha f(\phi(x_n)) (x_n - \phi(x_n))^2} + \alpha f(\phi(x_n)) (x_n - \phi(x_n))^2.
\]

(3.2.10)

The above family of iteration functions constructed in this manner again has order of convergence equal to at least three when \(\beta = 1\). This is proved by the next Theorem 3.2.1.

3.2.1 Convergence analysis

**Theorem 3.2.1** Let \(r\) be a simple zero of \(f(x)\) and \(\phi(x)\) be an iteration function with \(\phi \in I_2\), such that \(\phi''(r)\) is continuous in a neighborhood of \(r\). Let

\[
x_{n+1} = \phi(x_n) - \frac{\beta f(\phi(x_n)) (x_n - \phi(x_n)) [f(x_n) - 2\beta f(\phi(x_n))]}{[f(x_n) - 2\beta f(\phi(x_n))]^2 + \alpha f(\phi(x_n)) (x_n - \phi(x_n))^2}.
\]

(3.2.11)

Then the above family of iteration functions has at least third-order of convergence when \(\beta = 1\). Furthermore, if \(\phi''(r) = \frac{f''(r)}{f'(r)}\), then family (3.2.11) has fourth-order convergence.

**Proof** Let \(r\) be a simple zero of \(f(x)\). Since \(\phi(x)\) is an iteration function of order two, then one has \(\phi(r) = r\) and \(\phi'(r) = 0\). Expanding \(f(x_n)\) and \(\phi(x_n)\) about \(r\) by a Taylor series expansion, one has

\[
f(x_n) = f'(r)(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4),
\]

(3.2.12)

and

\[
\phi(x_n) = r + \frac{1}{2} \phi''(r)e_n^2 + \frac{1}{6} \phi'''(r)e_n^3 + O(e_n^4),
\]

(3.2.13)
respectively.
Furthermore
\[ x_n - \phi(x_n) = e_n - \frac{1}{2} \phi''(r)e_n^2 - \frac{1}{6} \phi'''(r)e_n^3 + O(e_n^4), \quad (3.2.14) \]
and in combination of the Taylor series expansion of \( f(\phi(x_n)) \) about \( r \), one has
\[ f(\phi(x_n)) = f'(r) \left\{ \frac{1}{2} \phi''(r)e_n^2 + \frac{1}{6} \phi'''(r)e_n^3 \right\} + O(e_n^4), \quad (3.2.15) \]
and
\[ f(x_n) = 2\beta f(\phi(x_n)) = f'(r) \left\{ e_n + (c_2 - \beta \phi''(r))e_n^2 + \left( c_3 - \frac{\beta}{3} \phi'''(r) \right)e_n^3 \right\} + O(e_n^4). \quad (3.2.16) \]
Using (3.2.13)-(3.2.16), one gets
\[
\frac{\beta f(\phi(x_n))(x_n - \phi(x_n))}{[f(x_n) - 2\beta f(\phi(x_n))]^2} + \alpha f(\phi(x_n))(x_n - \phi(x_n)) = \frac{\beta \phi''(r)}{2} e_n^2 + \frac{\beta}{12} \left\{ 6\beta - 3 \right\} \phi'''(r)^2 e_n^4
+ 2\phi'''(r) - 6\phi''(r)c_2^2 e_n + \frac{1}{12 \phi''(r)} \left\{ \beta \phi''(r) \left\{ 3\alpha \beta \phi''(r) - (2\beta - 1) f'(r) (3\beta \phi''(r))^2 + 2\phi'''(r) \right\} + f'(r)c_2 \left\{ 3(4\beta - 1)(\phi''(r))^2 + 2\phi'''(r) - 6\phi''(r)c_2 \right\} + 6f'(r)\phi''(r)c_3 \right\} e_n^4,
+ O(e_n^4),
\]
(3.2.17)
Thus, using (3.2.13) and (3.2.17) in (3.2.11), one gets
\[
x_{n+1} = r + \frac{1 - \beta}{2} \phi''(r)e_n^2 + \frac{1}{12} \left\{ 2\phi'''(r) - \beta \left\{ 6\beta - 3 \right\} \phi'''(r)^2 + 2\phi'''(r) - 6\phi''(r)c_2 \right\} e_n^3
+ \frac{1}{12 \phi''(r)} \left\{ \beta \phi''(r) \left\{ 3\alpha \beta \phi''(r) - (2\beta - 1) f'(r) (3\beta \phi''(r))^2 + 2\phi'''(r) \right\} + f'(r)c_2 \left\{ 3(4\beta - 1)(\phi''(r))^2 + 2\phi'''(r) - 6\phi''(r)c_2 \right\} + 6f'(r)\phi''(r)c_3 \right\} e_n^4 + O(e_n^4),
\quad (3.2.18) \]
which further implies that
\[
e_{n+1} = \frac{1 - \beta}{2} \phi''(r)e_n^2 + \frac{1}{12} \left\{ 2\phi'''(r) - \beta \left\{ 6\beta - 3 \right\} \phi'''(r)^2 + 2\phi'''(r) - 6\phi''(r)c_2 \right\} e_n^3
+ \frac{1}{12 \phi''(r)} \left\{ \beta \phi''(r) \left\{ 3\alpha \beta \phi''(r) - (2\beta - 1) f'(r) (3\beta \phi''(r))^2 + 2\phi'''(r) \right\} + f'(r)c_2 \left\{ 3(4\beta - 1)(\phi''(r))^2 + 2\phi'''(r) - 6\phi''(r)c_2 \right\} + 6f'(r)\phi''(r)c_3 \right\} e_n^4 + O(e_n^4),
\quad (3.2.19) \]
Therefore, when $\beta = 1$, one has the following error equation:

$$
\epsilon_{n+1} = \frac{1}{4} \phi''(r) (2c_2 - \phi''(r)) \epsilon_n^3 + \frac{1}{12} f'(r) \left[ \phi''(r) \left\{ 3\alpha \phi''(r) - f'(r) \left( 3\{\phi''(r)\}^2 + 2\phi''(r) \right) \right\} 
+ f'(r) c_2 \left( \phi''(r) \right)^2 + 2\phi''(r) - 6\phi''(r) c_2 \right] \epsilon_n^4 + O(\epsilon_n^5) \quad (3.2.20)
$$

This means that the family of iteration functions (3.2.11) is of order at least three. By numerical experiments with the help of Wolfram Mathematica 9, it turns out from the error equation (3.2.19) with $\beta$ very close to 1, for instance, $\beta = 1 - 10^{-30}$, that in most cases the computer outputs three as the order of corresponding iterative methods seemingly due to round-off of computer computation. Furthermore, when $\phi''(r) = f'(r)$, one can observe from (3.2.20) that the family (3.2.11) is of order at least four. It should be noted that the result of Theorem 3.2.1 is independent of the structure of iteration function of order two involved. This completes the proof of Theorem (3.2.1). $\square$

### 3.2.2 Examples

**Example 3.2.2** Consider the well-known Newton’s scheme defined by $\phi(x_n) = y_n = x_n - \frac{f(x_n)}{f'(x_n)}$. In this case, $x_{n+1}$ defined by (3.2.11) becomes

$$
\begin{align*}
\begin{cases}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \left[ 1 + \frac{\{f'(x_n)\}^2 f(y_n) \{f(x_n) - 2f(y_n)\}}{\{f'(x_n)\}^2 \{f(x_n) - 2f(y_n)\}^2 + \alpha f(y_n) \{f(x_n)\}^2} \right].
\end{cases}
\end{align*}
$$

(3.2.21)

This is a new optimal fourth-order family of famous Ostrowski’s method [Tra64, Ost60, Kin73, McN07, PNPD12]. This exactly agrees with the result predicted by Theorem 3.2.1 since $\phi''(r) = \frac{f''(r)}{f'(r)}$ and the error equation of the above family is

$$
\epsilon_{n+1} = \left\{ c_2^2 \left( c_2 + \frac{\alpha}{f'(r)} \right) - c_2 c_3 \right\} \epsilon_n^4 + O(\epsilon_n^5).
$$
Example 3.2.3 Consider the second-order Stirling’s scheme [AS93] defined by $\phi(x_n) = g_n = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}$, in (3.2.11), one can obtain

$$
\begin{cases}
  g_n = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}, \\
  x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}
\end{cases}
$$

$$
\left[ 1 + \frac{f'(g_n) \{ f'(x_n - f(x_n)) \}^2 \{ f(x_n) - 2f(g_n) \}}{\{ f'(x_n - f(x_n)) \}^2 \{ f(x_n) - 2f(g_n) \}^2 + \alpha \{ f(x_n) \}^2 f'(g_n) \right].
$$

(3.2.22)

This is again a new cubically convergent family of Stirling’s method. It has the following error equation:

$$
\epsilon_{n+1} = c_2 f''(r) (1 - 2f'(r)) \epsilon_n^3 + O(\epsilon_n^4).
$$

Example 3.2.4 Consider the second-order scheme defined by $\phi(x_n) = u_n = x_n - \frac{f(x_n)}{f'(x_n - \lambda_1 f(x_n))}$, [Wu00, KT09, KKTS11] in (3.2.11), one can have

$$
\begin{cases}
  u_n = x_n - \frac{f(x_n)}{f'(x_n) - \lambda_1 f(x_n)}, \\
  x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda_1 f(x_n)}
\end{cases}, \quad \lambda_1 \in \mathbb{R},
$$

$$
\left[ 1 + \frac{f'(u_n) \{ f'(x_n) - \lambda_1 f(x_n) \}^2 \{ f(x_n) - 2f(u_n) \}}{\{ f'(x_n) - \lambda_1 f(x_n) \}^2 \{ f(x_n) - 2f(u_n) \}^2 + \alpha f(u_n) \{ f(x_n) \}^2} \right].
$$

(3.2.23)

This is another new cubically convergent family of straight line method and has the following error equation:

$$
\epsilon_{n+1} = \lambda_1 (c_2 - \frac{1}{c_2} \epsilon_n^3 + \left( \frac{\alpha (\lambda_1^3 - 2\lambda_1 c_2 + c_2^2)}{f''(r)} + \lambda^2 c_2^2 - 2\lambda_1 c_2^2 + c_3^2 + 3\lambda c_3 - c_2 c_3 \right) \epsilon_n^4
$$

+ O(\epsilon_n^5).

Note: Selection of parameter ‘$\lambda_1$’ in family (3.2.23).

The parameter ‘$\lambda_1$’ in family (3.2.23) is chosen so as to give the largest value of denominator. In order to make this happen, one can take

$$
\lambda_1 = \begin{cases}
  + \epsilon, & \text{if } f(x_n) f'(x_n) \leq 0, \\
  - \epsilon, & \text{if } f(x_n) f'(x_n) \geq 0.
\end{cases}
$$
Example 3.2.5 Consider the second-order Steffensen’s scheme [Ste45, OR70] defined by \( \phi(x_n) = w_n = x_n - \frac{\{f(x_n)\}^2}{f(x_n + f(x_n)) - f(x_n)} \), in (3.2.11), one obtains

\[
\begin{align*}
  w_n &= x_n - \frac{\{f(x_n)\}^2}{f(x_n + f(x_n)) - f(x_n)}, \\
  x_{n+1} &= x_n - \frac{\{f(x_n)\}^2}{f(x_n + f(x_n)) - f(x_n)} \\
  &\quad \times \left[ 1 + \frac{f(w_n)\{f(x_n) - 2f(w_n)\}\{f(x_n + f(x_n)) - f(x_n)\}^2}{\{f(x_n + f(x_n)) - f(x_n)\}^2\{f(x_n) - 2f(w_n)\}^2 + \alpha f(w_n)\{f(x_n)\}^2} \right].
\end{align*}
\]

(3.2.24)

This scheme is again cubically convergent and does not require the evaluation of any derivative. It satisfies the following error equation:

\[ e_{n+1} = -\frac{1}{4} f''(r)(2c_2 + f''(r))e_n^3 + O(e_n^4). \]

Example 3.2.6 Consider the quadratically convergent family of parabolic method [MKKS05] free from square root defined by \( \phi(x_n) = k_n = x_n - \frac{f(x_n)f'(x_n)}{(f(x_n))^2 + (f'(x_n))^2} \). In this case, one has \( \phi''(r) = f'(r)^2 \). Hence, by Theorem 3.2.1, the family of iteration functions (3.2.11) becomes

\[
\begin{align*}
  k_n &= x_n - \frac{f(x_n)f'(x_n)}{(f(x_n))^2 + (f'(x_n))^2}, \\
  x_{n+1} &= x_n - \frac{f(x_n)f'(x_n)}{(f(x_n))^2 + (f'(x_n))^2} \\
  &\quad \times \left[ 1 + \frac{f(k_n)\{f(x_n) - 2f(k_n)\}\{f(x_n)\}^2}{\{f(x_n)\}^2 + (f'(x_n))^2\{f(x_n) - 2f(k_n)\}^2 + \alpha f(k_n)\{f(x_n)\}^2} \right].
\end{align*}
\]

(3.2.25)

The above family of parabolic method has optimal fourth-order convergence and satisfies the following error equation:

\[ e_{n+1} = c_2 \left( c_2^2 - c_3 + \frac{\alpha c_2}{f'(r)} - 1 \right) e_n^4 + O(e_n^5). \]

Example 3.2.7 Consider the quadratically convergent family of ellipse method [GKK09] defined by \( \phi(x_n) = h_n = x_n \pm \frac{f(x_n)}{\sqrt{(f'(x_n))^2 + \lambda_2^2(f(x_n))^2}} \). In this case, \( x_{n+1} \)
defined by (3.2.11) becomes

\[
\begin{aligned}
    h_n &= x_n \pm \frac{f(x_n)}{\sqrt{f'(x_n)^2 + \lambda_2^2 f(x_n)^2}}, \quad \lambda_2 \neq 0 \in \mathbb{R}, \\
    x_{n+1} &= h_n \pm \frac{f(x_n)f(h_n)(f(x_n) - 2f(h_n))\sqrt{f'(x_n)^2 + \lambda_2^2 f(x_n)^2}}{(f(x_n) - 2f(h_n))^2(f'(x_n)^2 + \lambda_2^2 f(x_n)^2 + \alpha f(h_n)f(x_n)^2)}.
\end{aligned}
\] (3.2.26)

One can take positive sign if \( x_0 \leq r \) and negative sign if \( x_0 \geq r \). Geometrically, one can say if the slope of curve \( f'(x_0) \) at the point \((x_0, f(x_0))\) is negative, then take positive sign otherwise, negative. The above new family of ellipse method has optimal fourth-order convergence and satisfies the following error equation:

\[
e_{n+1} = \left( \frac{\alpha^2}{f'(r)} + \frac{1}{2} + \frac{1}{2} \lambda_2^2 + 2 \alpha \right) e_n^4 + O(e_n^5).
\]

### 3.3 Second scheme of multi-point methods

On similar lines, if one takes two points, namely \( \phi(x_0), f(\phi(x_0)) \) and \( \left( \phi(x_0) + \frac{x_0}{2}, f(\phi(x_0)) + \frac{x_0}{2} \right) \) on the graph of function \( y = f(x) \) and adopting the same procedure as in the development of first family, one gets another family of methods as follows:

\[
x_{n+1} = \phi(x_n) - \frac{f(\phi(x_n))(x_n - \phi(x_n))[f(x_n) - f(\phi(x_n))]}{[f(x_n) - f(\phi(x_n))]^2 + \alpha f(\phi(x_n))(x_n - \phi(x_n))^2}. \quad (3.3.1)
\]

### 3.3.1 Convergence analysis

**Theorem 3.3.1** Let \( r \) be a simple zero of \( f(x) \) and \( \phi(x) \) be an iteration function with \( \phi' \in I_2 \), such that \( \phi''(r) \) is continuous in a neighborhood of \( r \). Let

\[
x_{n+1} = \phi(x_n) - \frac{f(\phi(x_n))(x_n - \phi(x_n))[f(x_n) - f(\phi(x_n))]}{[f(x_n) - f(\phi(x_n))]^2 + \alpha f(\phi(x_n))(x_n - \phi(x_n))^2}. \quad (3.3.2)
\]

Then the family of iteration functions defined by (3.3.2) has at least third-order convergence.
Proof From the equations (3.2.12), (3.2.13), (3.2.15), one has
\[
f(x_n) - f(\phi(x_n)) = f'(r) \left\{ e_n + \frac{1}{2} (2c_2 - \phi''(r)) e_n^2 + \left( c_3 - \frac{1}{6} \phi'''(r) \right) e_n^3 \right\} + O(e_n^4).
\] (3.3.3)

Using (3.2.13), (3.2.15) and (3.3.3), one gets
\[
\frac{f(\phi(x_n))(x_n - \phi(x_n))}{[f(x_n) - f(\phi(x_n))]^2 + \alpha f(\phi(x_n))(x_n - \phi(x_n))^2} = \frac{\phi''(r)}{2} e_n^2 + \frac{1}{6} (\phi'''(r) - 3 \phi''(r) c_2) e_n^3
\]
\[\quad - \frac{1}{12} \left[ \left( \frac{3 \alpha}{f'(r)} + 3 c_2 \right) \left( \phi''(r) \right)^2 + 2 \phi'''(r) c_2 - 6 \phi''(r) (c_2^2 - c_3) \right] e_n^4 + O(e_n^5).
\] (3.3.4)

Thus, using (3.2.13) and (3.3.4) in (3.3.2), one has
\[
x_{n+1} = x_n + \frac{1}{2} \phi''(r) c_2 e_n^3 - \frac{1}{12} \left[ \left( \frac{3 \alpha}{f'(r)} + 3 c_2 \right) \left( \phi''(r) \right)^2 + 2 \phi'''(r) c_2 - 6 \phi''(r) (c_2^2 - c_3) \right] e_n^4
\]
\[\quad + O(e_n^5).
\] (3.3.5)

which further implies that
\[
e_{n+1} = \frac{1}{2} \phi''(r) c_2 e_n^3 - \frac{1}{12} \left[ \left( \frac{3 \alpha}{f'(r)} + 3 c_2 \right) \left( \phi''(r) \right)^2 + 2 \phi'''(r) c_2 - 6 \phi''(r) (c_2^2 - c_3) \right] e_n^4
\]
\[\quad + O(e_n^5).
\] (3.3.6)

This means that the family of iteration functions (3.3.1) has at least third-order of convergence. This completes the proof of Theorem 3.3.1. □

3.3.2 Examples

Example 3.3.2 Consider the well-known Newton’s method defined by \( \phi(x_n) = y_n = x_n - \frac{f(x_n)}{f'(x_n)} \), in (3.3.1), one obtains
\[
\left\{ \begin{array}{l}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} = y_n - \frac{f(y_n)(f(x_n) - f(y_n))}{f'(x_n)} \left\{ \frac{f'(x_n)^2 (f(x_n) - f(y_n))^2 + \alpha f(y_n) f'(x_n)^2}{f(x_n)} \right\}.
\end{array} \right.
\] (3.3.7)
This is a new cubically convergent family of Newton-Secant method [Tra64]. The error equation of the above family is

\[ e_{n+1} = c_2^3 e_n^3 + c_2 \left( \frac{\alpha c_2}{f'(r)} - 3c_2^2 + 3c_3 \right) e_n^4 + O(e_n^5). \]

**Example 3.3.3** Consider the Stirling’s scheme of order two defined by \( \phi(x_n) = gn = x_n - \frac{f(x_n)}{f'(x_n)} \). In this case, \( x_{n+1} \) defined by (3.3.1) becomes

\[
\begin{align*}
  g_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
  x_{n+1} &= g_n - \frac{f'(x_n - f(x_n))f(x_n)(f(x_n) - f(g_n))f(g_n)}{(f'(x_n) - f(x_n))^2 (f(x_n) - f(g_n))^2 + \alpha f(x_n)^2 f(g_n)}.
\end{align*}
\]

This is again a new cubically convergent family of Stirling’s method. It has the following error equation:

\[
e_{n+1} = \left( 1 - 2f'(r) \right) c_2^3 e_n^3 + \frac{c_2}{f'(r)} \left[ \alpha (1 - 2f'(r))^2 c_2 - f'(r) (3 - 6f'(r) + 4f'(r)^2) c_2 ^2 \\
+ f'(r) (3 - 8f'(r) + 3f'(r)^2) c_2 ^3 \right] e_n^4 + O(e_n^5).
\]

**Example 3.3.4** Consider the second-order scheme \( \phi(x_n) = u_n = x_n - \frac{f(x_n)}{f'(x_n) - \lambda_1 f(x_n)} \), in (3.3.1), one has

\[
\begin{align*}
  u_n &= x_n - \frac{f(x_n)}{f'(x_n) - \lambda_1 f(x_n)}, \quad \lambda_1 \in \mathbb{R}, \\
  x_{n+1} &= u_n - \frac{f(x_n)f(u_n)(f(x_n) - f(u_n))(f(x_n) - \lambda_1 f(x_n))}{(f(x_n) - f(u_n))^2 (f'(x_n) - \lambda_1 f(x_n))^2 + \alpha f(x_n)^2 f(u_n)}.
\end{align*}
\]

This is another new cubically convergent family of straight line method. It has the following error equation:

\[
e_{n+1} = c_2 (c_2 - \lambda_1) e_n^3 + \left( \frac{\alpha (\lambda_1^2 - 2\lambda_1 c_2 + c_2^2)}{f'(r)} - \lambda_1^2 c_2 + \lambda_1 (3c_2^2 - c_3) - 3c_2^3 + 3c_2 c_3 \right) e_n^4 + O(e_n^5).
\]
Example 3.3.5 Consider the second-order Steffensen’s method defined by \( \phi(x_n) = w_n = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n) - f(x_n))} \), in (3.3.1), one can have

\[
\begin{align*}
    w_n &= x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n) - f(x_n))}, \\
    x_{n+1} &= w_n - \frac{f(x_n)^2 f(w_n) (f(x_n) - f(w_n)) (f(x_n + f(x_n) - f(x_n)) - f(x_n))}{(f(x_n) - f(w_n))^2 (f(x_n + f(x_n) - f(x_n)) - f(x_n))^2 + \alpha (f(x_n))^2 f(w_n)}.
\end{align*}
\]

This scheme is again cubically convergent and does not require the evaluation of any derivative. It has the following error equation:

\[
\epsilon_{n+1} = (1 + f'(r)) c_n^3 + \frac{c_2}{f'(r)} [\alpha (1 + f'(r))^2 c_2 - f'(r) (3 + 3 f'(r) + \{f'(r)\}^2) c_2^2 + f'(r) (3 + 4 f'(r) + \{f'(r)\}^2) c_3] \epsilon_n^4 + O(\epsilon_n^6).
\]

Example 3.3.6 Consider the quadratically convergent family of parabolic method free from square-root defined by \( \phi(x_n) = h_n = x_n - \frac{f(x_n) f'(x_n)}{(f'(x_n))^2 + \frac{c_2}{f'(r)}} \), in (3.3.1) one has

\[
\begin{align*}
    h_n &= x_n - \frac{f(x_n) f'(x_n)}{(f'(x_n))^2 + \frac{c_2}{f'(r)}}, \\
    x_{n+1} &= h_n - \frac{h_n (f(x_n) - f(h_n)) [f(x_n) - f(h_n)] [f'(x_n)]^2 + \alpha (f(x_n))^2 f'(x_n) f(h_n)}{(f(x_n) - f(h_n))^2 [f'(x_n)]^2 + \alpha (f(x_n))^2 f'(x_n) f(h_n)}.
\end{align*}
\]

This is again a new cubically convergent family and has the following error equation:

\[
\epsilon_{n+1} = c_n^3 + \left(3 c_3 - 3 c_2 + \frac{\alpha c_2}{f'(r)}\right) \epsilon_n^4 + O(\epsilon_n^6).
\]

Example 3.3.7 Consider the quadratically convergent family of ellipse method defined by \( \phi(x_n) = h_n = x_n - \frac{f(x_n)}{[f'(x_n)]^2 + \lambda_2^2 (f(x_n))^2} \), in (3.3.1) one can obtain

\[
\begin{align*}
    h_n &= x_n \pm \frac{f(x_n)}{\sqrt{[f'(x_n)]^2 + \lambda_2^2 (f(x_n))^2}}, \\
    x_{n+1} &= h_n \pm \frac{h_n (f(x_n) - f(h_n)) [f(x_n) - f(h_n)] \sqrt{[f'(x_n)]^2 + \lambda_2^2 (f(x_n))^2}}{(f(x_n) - f(h_n))^2 [f'(x_n)]^2 + \lambda_2^2 (f(x_n))^2 + \alpha f(h_n) (f(x_n))^2}.
\end{align*}
\]
This is another new cubically convergent family. One can take positive sign if \( x_0 < r \) and negative sign if \( x_0 \geq r \). It has the following error equation:

\[
e_{n+1} = c_2^2 e_n^3 + \left( 3c_2 e_2 - 3c_2^2 + \frac{p^2 c_2^2}{2} - \frac{\alpha c_2^2}{f'(c)} \right) e_n^4 + O(e_n^5).
\]

It is straightforward to see that per step all the proposed families of method require three evaluations of function viz. two evaluations of \( f(x) \) and one of \( f'(x) \) or three evaluations of \( f(x) \) and none of \( f'(x) \). In order to obtain an assessment of the efficiency of our methods, we shall make use of the efficiency index defined by equation (1.2.9). For our proposed iteration schemes, namely (3.2.22) to (3.2.24), (3.3.7) to (3.3.12), we find \( p = 3 \) and \( d = 3 \), yielding \( *EFF = \sqrt[3]{3} \approx 1.442 \), which is better than \( *EFF = \sqrt[3]{2} \approx 1.414 \), the efficiency index of Newton’s method. For the family of methods namely, (3.2.21), (3.2.25) and (3.2.26), we find \( p = 4 \) and \( d = 3 \), yielding \( *EFF = \sqrt[3]{4} \approx 1.587 \), which is better than those of most third order methods \( *EFF \approx 1.442 \) and Newton’s method \( *EFF \approx 1.414 \).

### 3.4 Numerical experiments

This section is devoted to determine the numerical stability, usefulness and computational efficiency of newly proposed methods namely, modified family of Newton-Secant method (3.3.7) for \( \alpha = 0.5, \alpha = 1 \) \( (MNSM_3) \), modified family of straight line method (3.3.9) for \( (\alpha = 0.5, |\alpha_1| = 1), (\alpha = 1, |\alpha_1| = 1) \) \( (MKM_3) \), modified family of Steffensen’s method (3.3.10) for \( \alpha = 0.5 \) \( (MSM_3) \), modified family of Ostrowski’s method (3.2.21) for \( \alpha = 0.5, \alpha = 1 \) \( (MOM_4) \) and modified family of ellipse method (3.2.26) for \( (\alpha = 0.5, |\alpha_2| = 1), (\alpha = 1, |\alpha_2| = 1) \) \( (MEM_4) \) respectively to solve nonlinear equations given in Table 3.1. These proposed methods were compared with the existing methods namely, Newton’s method \( (NM_2) \), Homeier’s method (28) \( (HM_3) \) [Hom05], Weerakoon and Fernando method (3.8) \( (WF_0) \), Frontini and Sormani method \( (FM_3) \) [FS03b], Potra-Ptak method \( (PM_3) \) [PP84], Newton-Secant method \( (NSM_3) \) [Tra64], Jarratt’s method \( (JM_4) \) [Jar69], King’s method (3.1.1) for \( \gamma = 1 \) \( (KM_4) \), Ostrowski’s method \( (OM_4) \) [Ost60], Chun’s method (25) \( (CM_4) \)
and method (27) CM₁ [Chu07e], Cordero et al. method (9)(PM₄) [CHMTIOb] respectively. The test functions and root r correct up to 35 decimal places are displayed in Table 3.1. All computations have been performed using the programming package Mathematica 9 with multi-precision arithmetic.

Table 3.2 and Table 3.5 show the comparison of different methods of order three and four with respect to the number of iterations respectively. Here, the stopping criterion is described as the distance between two consecutive approximations for the required root is less than the precision of 10⁻³⁴ as described in (1.6.19). In the implementation of iterative methods, the good choice of initial guess is very important for all the variants of Newton’s method otherwise, convergence is not guaranteed. However, the family of straight line method and ellipse method will converge to root even though the guess is far away from the required root.

In order to verify theoretical order of convergence, Table 3.3 and Table 3.6 displayed the values of computational order of convergence (ρ) calculated by using formula (1.2.11). Table 3.4 and Table 3.7, display absolute values of the function (|f(x_n)|) based on the same total number of functional evaluations (TNFE = 12) required by each method. Also, it can be observed from Table 3.4 and Table 3.7, that in majority of the problems tested here, the proposed methods namely, modified family of straight line method (3.3.9) for (α = 0.5, |λ₁| = 1) (MKM₃), modified family of Newton-Secant method (3.3.7) for α = 0.5, α = 1 (MNSM₃), modified family of Steffensen’s method (3.3.10) for α = 0.5, modified family of Ostrowski’s method (3.2.21) for α = 0.5, α = 1 (MOM₄) and modified family of ellipse method for (3.2.26) for (α = 0.5, |λ₂| = 1) , (α = 1, |λ₂| = 1) (MEM₄) are efficient and show better performance than Newton’s method and other existing methods when the accuracy is tested in multi-precision digits.

**Example 3.4.1** \( \sin x = 0 \).

This equation has an infinite number of roots but our desired root is \( r = 0 \), which is correct up to 35 digits. It can be seen that Newton’s method and its variants do not necessarily converge to the root that is nearest to starting value. For example, \( NM₂, PM₃, NMSM₃ \) for \( \alpha = 0.5, \alpha = 1 \), \( PM₄, CM₁ \),
Numerical experiments

CM4, KM4, OM4, MOM4 (for $\alpha = 0.5, \alpha = 1$) with initial guess $x_0 = -1.50$ converge to $4\pi$ respectively while $HM_3$ converges to $2\pi$. Similarly, $NM_2, HM_3, WM_3, PM_3, NM_3, MNSM_3$ (for $\alpha = 0.5, \alpha = 1$), $JM_4, PM_4, CM_4, CM_2, KM_4, OM_4, MOM_4$ with initial guess $x_0 = 1.50$ converge to $-\pi$ respectively, while $HM_3$ converges to $-2\pi$ and so on. However, some of our newly proposed methods namely, (3.2.26), (3.3.9) and (3.3.10) do not exhibit this type of behavior.

Example 3.4.2 $e^{-x} + \sin x = 0$.

This equation has an infinite number of roots but our desired root is $r = 3.1\ldots$. It can be seen that Newton’s method and its variants do not necessarily converge to the root that is nearest to starting value. For example, $NM_2, HM_3, NM_3, MNSM_3$ (for $\alpha = 0.5, \alpha = 1$), $JM_4, CM_4, CM_2, KM_4, OM_4, MOM_4$ (for $\alpha = 0.5, \alpha = 1$) with initial guess $x_0 = 1.4$ converge to $21.9\ldots, 6.2\ldots, 9.4\ldots, 9.4\ldots, 12.5\ldots, 47.1\ldots, 34.5\ldots, 12.5\ldots, 18.8\ldots, 18.8\ldots$ while $PM_3$ and $PM_4$ are divergent. Similarly, $NM_2, HM_3, NM_3, MNSM_3$ (for $\alpha = 0.5$), $JM_4, KM_4, MOM_4$ (for $\alpha = 0.5, \alpha = 1$) with initial guess $x_0 = 4.5$ converge to $6.2\ldots, 6.2\ldots, 254.4\ldots, 138.2\ldots, 659.7\ldots, 6.2\ldots, 6.2\ldots$ respectively, while $WM_3, PM_3, MNSM_3$ (for $\alpha = 1$) and $PM_4$ are divergent. However, some of our newly proposed methods namely, (3.2.26) (3.3.9) and (3.3.10) do not exhibit this type of behavior.

Example 3.4.3 $x^3 + 4x^2 - 10 = 0$.

Clearly, this equation has three number of roots but our desired root is $r = 1.3\ldots$ which is correct up to 35 digits. It can be seen that all the mentioned variants of Newton’s method fail with initial guess $x_0 = 0$. However, some of our newly proposed methods namely, (3.2.26) (3.3.9) and (3.3.10) do not exhibit this type of behavior.

Example 3.4.4 $x^3 - \cos x + 2 = 0$.

This equation has finite number of roots but our desired root is $r = -1.1\ldots$ which is correct up to 35 digits. It can be seen that all the mentioned methods fail with initial guess $x_0 = 0.0$. However, some of our newly proposed methods namely, (3.2.26) (3.3.9) and (3.3.10) do not exhibit this type of behavior.
3. Geometric constructions of two-point methods

Table 3.1: (Test functions)

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>([a, b])</th>
<th>( \text{root}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = \sin x )</td>
<td>([-1.5, 1.5])</td>
<td>0.00000000000000000000000000000000000000</td>
</tr>
<tr>
<td>( f_2(x) = e^{-x} + \sin x )</td>
<td>([1.4, 4.5])</td>
<td>3.1830630119333635919391869956363946</td>
</tr>
<tr>
<td>( f_3(x) = x^3 + 4x^2 - 10 )</td>
<td>([0, 2])</td>
<td>1.3632901341406684575676086828816661</td>
</tr>
<tr>
<td>( f_4(x) = x^3 - \cos x + 2 )</td>
<td>([-1.5, 0])</td>
<td>-1.1725779647530700126733327148688486</td>
</tr>
<tr>
<td>( f_5(x) = x^4 - e^x - 3x + 2 )</td>
<td>([0, 1])</td>
<td>0.2575392845398076045553673043724178</td>
</tr>
</tbody>
</table>

Table 3.2: (Comparison of different third-order two-point methods and Newton’s method with respect to number of iterations)

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( x_0 )</th>
<th>( \text{NM}_3 )</th>
<th>( \text{HM}_3 )</th>
<th>( \text{WM}_3 )</th>
<th>( \text{FM}_3 )</th>
<th>( \text{PM}_3 )</th>
<th>( \text{NSM}_3 )</th>
<th>( \text{MNSM}_3 )</th>
<th>( \text{MNSM}_3 )</th>
<th>( \text{MKM}_3 )</th>
<th>( \text{MSM}_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1(x) = \sin x )</td>
<td>([-1.5, 1.5])</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
<td>3.05</td>
</tr>
<tr>
<td>( f_2(x) = e^{-x} + \sin x )</td>
<td>([1.4, 4.5])</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>( f_3(x) = x^3 + 4x^2 - 10 )</td>
<td>([0, 2])</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
</tr>
<tr>
<td>( f_4(x) = x^3 - \cos x + 2 )</td>
<td>([-1.5, 0])</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
<td>-1.17</td>
</tr>
<tr>
<td>( f_5(x) = x^4 - e^x - 3x + 2 )</td>
<td>([0, 1])</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
</tr>
</tbody>
</table>

\( D \): stands for divergent, \( C \): stands for convergence to undesired root, \( F \): stands for failure.
### Numerical experiments

Table 3.3: (Computational order of convergence of different third-order two-point methods and Newton's method)

<table>
<thead>
<tr>
<th>Method</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newton</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>BSR</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Note: Not applicable in the case of divergent.
Table 3.4: (Comparison of different third-order two-point methods and Newton’s method with the same total number of functional evaluations (TNFE=12))

<table>
<thead>
<tr>
<th>f(x)</th>
<th>x₀</th>
<th>N₅₂</th>
<th>H₅₃</th>
<th>W₅₃</th>
<th>F₅₃</th>
<th>P₅₃</th>
<th>N₅₅₃</th>
<th>M₅₅₃</th>
<th>M₅₅₃</th>
<th>M₅₅₃</th>
<th>M₅₅₃</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>-1.5</td>
<td>C</td>
<td>C</td>
<td>1.6e-26</td>
<td>4.4e-40</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>2.1e-64</td>
<td>1.56e-718</td>
</tr>
<tr>
<td></td>
<td>-1.0</td>
<td>1.93e-115</td>
<td>1.17e-29</td>
<td>6.93e-33</td>
<td>8.63e-42</td>
<td>4.14e-70</td>
<td>1.59e-209</td>
<td>1.22e-158</td>
<td>4.66e-123</td>
<td>8.40e-24</td>
<td>1.01e-90</td>
</tr>
<tr>
<td>1.0</td>
<td>1.93e-115</td>
<td>1.17e-29</td>
<td>6.93e-33</td>
<td>8.63e-42</td>
<td>4.14e-70</td>
<td>1.59e-209</td>
<td>8.77e-92</td>
<td>C</td>
<td>5.72e-132</td>
<td>7.62e-862</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>C</td>
<td>C</td>
<td>1.61e-26</td>
<td>4.4e-40</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>6.45e-96</td>
<td>8.06e-236</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>1.4</td>
<td>C</td>
<td>C</td>
<td>2.07e+6</td>
<td>3.6e-25</td>
<td>D</td>
<td>C</td>
<td>C</td>
<td>2.14e-33</td>
<td>7.20e-65</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>C</td>
<td>C</td>
<td>4.91e-25</td>
<td>3.61e-8</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>1.44e-36</td>
<td>2.45e-93</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>2.52e-74</td>
<td>2.63e-55</td>
<td>4.56e-57</td>
<td>4.30e-66</td>
<td>3.30e-83</td>
<td>3.78e-105</td>
<td>4.72e-95</td>
<td>1.95e-89</td>
<td>2.53e-68</td>
<td>4.98e-73</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>1.68e-60</td>
<td>3.75e-43</td>
<td>8.81e-50</td>
<td>1.87e-50</td>
<td>2.34e-49</td>
<td>7.81e-77</td>
<td>1.33e-87</td>
<td>3.02e-75</td>
<td>4.20e-38</td>
<td>2.00e-122</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>C</td>
<td>C</td>
<td>D</td>
<td>2.18e-48</td>
<td>5.87e-31</td>
<td>C</td>
<td>D</td>
<td>2.90e-20</td>
<td>3.28e-89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>0.0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>1.70e-8</td>
<td>1.70e-8</td>
</tr>
<tr>
<td>1.0</td>
<td>3.98e-43</td>
<td>1.44e-99</td>
<td>1.50e-52</td>
<td>5.32e-56</td>
<td>3.00e-38</td>
<td>9.11e-55</td>
<td>4.90e-55</td>
<td>2.61e-55</td>
<td>6.02e-60</td>
<td>6.62e-60</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>-1.5</td>
<td>3.05e-36</td>
<td>1.48e-66</td>
<td>1.24e-43</td>
<td>5.43e-47</td>
<td>2.71e-37</td>
<td>8.65e-46</td>
<td>5.38e-46</td>
<td>3.31e-46</td>
<td>2.80e-67</td>
<td>7.81e-15</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.04e-44</td>
<td>1.95e-9</td>
<td>6.69e-1</td>
<td>8.23e-2</td>
<td>2.15e-5</td>
<td>2.74e-1</td>
<td>1.20e00</td>
<td>1.77e00</td>
<td>8.38e-39</td>
<td>2.55e-9</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>5.66e-24</td>
<td>4.32e-5</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>0.0</td>
<td>5.99e-100</td>
<td>4.26e-112</td>
<td>7.80e-106</td>
<td>3.44e-168</td>
<td>2.00e-114</td>
<td>1.31e-126</td>
<td>3.23e-126</td>
<td>7.72e-126</td>
<td>2.23e-92</td>
<td>1.11e-216</td>
</tr>
<tr>
<td>1.0</td>
<td>2.61e-94</td>
<td>2.58e-68</td>
<td>8.15e-67</td>
<td>7.83e-85</td>
<td>1.44e-124</td>
<td>6.62e-120</td>
<td>2.65e-119</td>
<td>9.98e-119</td>
<td>3.17e-60</td>
<td>1.60e-78</td>
<td></td>
</tr>
</tbody>
</table>

D: stands for divergent, C: stands for convergence to undesired root, F: stands for failure.
### 3.4 Numerical experiments

Table 3.5: (Comparison of different optimal fourth-order two-point methods with respect to number of iterations)

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( x_0 )</th>
<th>( J_{M1} )</th>
<th>( P_{M1} )</th>
<th>( C_{M1} )</th>
<th>( R_{M1} )</th>
<th>( OM_{1} )</th>
<th>( MOM_{1} )</th>
<th>( MEM_{1} )</th>
<th>( MEM_{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 0.5 )</td>
<td>( \alpha = 0.5 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 0.5,</td>
<td>\lambda_1</td>
<td>= 1 )</td>
<td>( \alpha = 1,</td>
<td>\lambda_2</td>
<td>= 1 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1.5</td>
<td>5</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>5</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>1.4</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>1.5</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>4.5</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>12</td>
<td>C</td>
<td>C</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2.0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>-1.5</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>-0.5</td>
<td>4</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>0.0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1.0</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

D: stands for divergent, C: stands for convergence to undesired root, F: stands for failure.

Table 3.6: (Computational order of convergence of different optimal fourth-order two-point methods)

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( x_0 )</th>
<th>( J_{M1} )</th>
<th>( P_{M1} )</th>
<th>( C_{M1} )</th>
<th>( R_{M1} )</th>
<th>( OM_{1} )</th>
<th>( MOM_{1} )</th>
<th>( MEM_{1} )</th>
<th>( MEM_{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 0.5 )</td>
<td>( \alpha = 0.5 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 0.5,</td>
<td>\lambda_1</td>
<td>= 1 )</td>
<td>( \alpha = 1,</td>
<td>\lambda_2</td>
<td>= 1 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1.5</td>
<td>5.0001</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>-1.0</td>
<td>5.0010</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>5.0010</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0018</td>
<td>5.0000</td>
<td>5.0000</td>
<td>5.0000</td>
</tr>
<tr>
<td>1.5</td>
<td>5.0001</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
</tr>
<tr>
<td>2</td>
<td>1.4</td>
<td>na</td>
<td>nd</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>4.0000</td>
</tr>
<tr>
<td>1.5</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>na</td>
<td>4.0000</td>
</tr>
<tr>
<td>4.0</td>
<td>3.8972</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>3.9761</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0093</td>
</tr>
<tr>
<td>4.5</td>
<td>na</td>
<td>nd</td>
<td>na</td>
<td>4.0000</td>
<td>4.0000</td>
<td>3.7046</td>
<td>na</td>
<td>na</td>
<td>4.0061</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>1.0</td>
<td>3.9996</td>
<td>4.0000</td>
<td>3.9992</td>
<td>3.9984</td>
<td>3.9990</td>
<td>3.9998</td>
<td>3.9998</td>
<td>3.9998</td>
<td>4.0000</td>
</tr>
<tr>
<td>2.0</td>
<td>3.9995</td>
<td>4.0000</td>
<td>3.9985</td>
<td>4.0000</td>
<td>3.9985</td>
<td>3.9995</td>
<td>3.9994</td>
<td>3.9995</td>
<td>4.0000</td>
</tr>
<tr>
<td>4.0</td>
<td>3.9994</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>3.9994</td>
<td>3.9996</td>
<td>3.9997</td>
<td>4.0000</td>
</tr>
<tr>
<td>3.9997</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>4.0024</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>4.0000</td>
<td>3.9999</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
<td>4.0000</td>
</tr>
</tbody>
</table>

nd: not defined in the case of divergent, *: in the case of failure, na: not applicable in the case of convergence to undesired root.
Table 3.7: (Comparison of different optimal fourth-order two-point methods with the same total number of functional evaluations (TNFE=12))

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$x_0$</th>
<th>$JM_1$</th>
<th>$PM_1$</th>
<th>$CM_{H1}$</th>
<th>$CM_{C1}$</th>
<th>$KM_1$</th>
<th>$OM_{H1}$</th>
<th>$MOM_{H1}$</th>
<th>$MEM_{H1}$</th>
<th>$MEM_{C1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>-1.5</td>
<td>6.56e-44</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>9.97e-67</td>
<td>5.27e-69</td>
</tr>
<tr>
<td></td>
<td>-0.1</td>
<td>9.94e-155</td>
<td>1.90e-81</td>
<td>9.65e-130</td>
<td>9.29e-117</td>
<td>3.46e-123</td>
<td>5.58e-153</td>
<td>5.52e-132</td>
<td>5.05e-124</td>
<td>4.41e-140</td>
</tr>
<tr>
<td>2.</td>
<td>0.0</td>
<td>1.4</td>
<td>C</td>
<td>D</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>1.97e-29</td>
<td>2.38e-41</td>
</tr>
<tr>
<td>3.</td>
<td>0.0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>1.42e-45</td>
<td>3.08e-45</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>-0.5</td>
<td>5.71e-15</td>
<td>4.20e+15</td>
<td>6.92e-4</td>
<td>5.42e+2</td>
<td>6.64e+2</td>
<td>5.77e-15</td>
<td>1.47e-16</td>
<td>1.48e-18</td>
<td>1.87e-99</td>
</tr>
<tr>
<td>5.</td>
<td>0.0</td>
<td>2.92e-342</td>
<td>1.07e-461</td>
<td>1.03e-366</td>
<td>3.26e-391</td>
<td>1.69e-366</td>
<td>1.90e-352</td>
<td>3.22e-370</td>
<td>6.57e-408</td>
<td>5.40e-239</td>
</tr>
<tr>
<td>6.</td>
<td>1.0</td>
<td>1.31e-285</td>
<td>1.67e-265</td>
<td>7.00e-260</td>
<td>2.57e-262</td>
<td>7.02e-260</td>
<td>6.63e-258</td>
<td>1.28e-259</td>
<td>1.63e-262</td>
<td>9.92e-100</td>
</tr>
</tbody>
</table>

D: stands for divergent, C: stands for convergence to undesired root, F: stands for failure.
3.5 Conclusions

In this chapter, author has modified Chun’s scheme for constructing the iterative methods of order three or higher to solve nonlinear equations numerically. Now, author has obtained a wide class of general methods, which are without memory and include three functional evaluations per iteration. Fourth-order family of Ostrowski’s method (3.2.21), parabolic method (3.2.25), ellipse method (3.2.26) are the main findings of the present contribution in terms of speed and efficiency index. According to the Kung-Traub conjecture, these fourth-order families have the maximal efficiency index because only three function values are needed per step. The numerical results presented in Table 3.4 and Table 3.7 overwhelmingly support that the new modified families of Newton-Secant method, straight line method, Steffensen’s method, Ostrowski’s method and ellipse method are equally competent to Newton’s method, Homeier’s method (28) [Hom05], Weerakoon and Fernando method (3.8) [WF00], Frontini and Sormani method [FS03b], Potra-Pták method [PP84], Newton-Secant method [Tra64], Jarratt’s method [Jar69], King’s method (3.1.1) for $\gamma = 1$, Ostrowski’s method [Ost60], Chun’s method (25) and method (27) [Chu07e], Cordero et al. method (9) [CHMT10]. The beauty of optimal family of ellipse method (3.2.26) is that these methods will converge to the required root even though $f'(x) = 0$ in the neighborhood of required root. Further, author also proposes third-order derivative free families of Steffensen’s method and important in the sense that in many practical situations finding derivative of functions is not an easy task. Furthermore, it is also investigated that modified families (3.3.7), (3.3.9), (3.3.10), (3.2.21) and (3.2.26) give very good approximation to the required root when $|\alpha|$ (the scaling parameter) is small. This is because that for small values of $|\alpha|$, the parabola widens along the horizontal direction. This means that our next approximation will move faster towards the desired root. For large values of $|\alpha|$ (provided that the inequality (3.2.8) holds), the formulas still work but take more number of iterations as compared to the smaller values of $|\alpha|$. This idea can further be extended for the case of multiple roots.