4.1 Introduction

In this chapter a general class of selection procedures for location parameters based on sub-sample quantiles is proposed. The proposed class of selection procedures generalises the selection procedures earlier proposed by Kumar et al (1992) based on two sample U-statistics using extrema of sub-samples. The class of selection procedures proposed by Kumar et al (1992) was based on maxima and minima of sub-samples. The proposed selection procedures are also based on two sample U-statistics. The rank representation of these selection procedures is given and some properties of these selection procedures are also discussed.

Suppose there are k independent populations such that the distributions associated with them differ only in terms of location parameters and our aim is to select a subset of the given k populations which contains the population associated with the distribution having largest location parameter.

In section 4.2 the problem has been formulated and proposed selection procedures are given in section 4.3. In section 4.4 we have obtained the expressions for the infimum of the PCS and expected subset size. The asymptotic relative efficiency (ARE) of our procedures relative to the procedures of Hsu (1981a) is
derived in section 4.5 and it is seen that Hsu's procedure, specialised to Wilcoxon scores, is a member of our class. In section 4.6 it is shown that these procedures can be implemented approximately with the help of existing tables and a practical example has been considered to illustrate the proposed selection procedures. A modification of these procedures to select a subset containing the populations better than an unknown control population is given in section 4.7. In section 4.8 simulation study has been carried out to assess the asymptotic performance of these procedures in terms of estimated expected subset sizes for particular parametric configurations of underlying populations.

4.2 Statement of the Problem

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be $k$ independent populations and let $F_i(x) = F(x - \mu_i)$ be the absolutely continuous cumulative distribution function (cdf) of the $i$th population indexed by the location parameter $\mu_i$, $i = 1, \ldots, k$. Let $\Theta = \{\mu : \mu = (\mu_1, \mu_2, \ldots, \mu_k)^T, -\infty < \mu_i < \infty, i = 1, \ldots, k\}$ denote the parametric space.

For any two populations $\Pi_i$ and $\Pi_j$, population $\Pi_i$ is considered to be better than population $\Pi_j$ if $\mu_j > \mu_i$. Thus, the population with the largest $\mu$-value is labelled as the best. We assume that there is a unique best population. If more than one populations are tied for the best, then arbitrary one of them is chosen to be the best.

For any $\mu = (\mu_1, \mu_2, \ldots, \mu_k)^T$, we shall denote by $\mu_{[k]}$ the
unique component of $\mu$ corresponding to the best population. The goal is to select a subset of $k$ populations containing the best population, the one with the largest location parameter $\mu[k]$. Any such selection will be called as correct selection (CS). Then the problem is to find a rule $R$ such that for a preassigned probability $P(k-1 < p < 1)$, this satisfies the probability requirement:

$$P_{\mu}[\text{CS}\mid R] \geq P^* \text{ for all } \mu \in \Omega.$$ ...(4.2.1)

### 4.3 Proposed Class of Procedures

The proposed selection procedures are based on two sample U-statistics. Fix $i$ in the discussion below and define

$$H(x) = F_i(x) = F(x - \mu_i),$$

where $H(.)$ is any continuous distribution function. Now

$$F_j(x) = F(x - \mu_j) = H(x - \Delta_{ij}),$$

where $\Delta_{ij} = \mu_j - \mu_i$, $j \neq i, j = 1, \ldots, k$.

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be a random sample of size $n_i$ from the $i$th population, $i = 1, \ldots, k$. Let $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})^t$ be a vector of observations from the $i$th population and let $X = (X_{11}, \ldots, X_{1n_1}, \ldots, X_{kn_k})^t$ be the vector of all the observations. Denote $n_{[1]} \leq \ldots \leq n_{[k]}$ as the ordered values of $n_1, \ldots, n_k$.

For a fixed odd integer $c$, $1 \leq c \leq \min(n_1, \ldots, n_k)$, and a fixed $m(1 \leq m \leq c)$, define the following kernels $\phi^{(1)}(.)$ and $\phi^{(2)}(.)$ as:
\[ \phi^{(1)}(x_{i1}, \ldots, x_{ic}, x_{j1}, \ldots, x_{jc}) = \begin{cases} 1 & \text{if the } m^{\text{th}} \text{ order statistic (o.s.) of } (x_{i1}, \ldots, x_{ic}) < m^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \\ 0 & \text{otherwise} \end{cases} \] ...

\[ \phi^{(2)}(x_{i1}, \ldots, x_{ic}, x_{j1}, \ldots, x_{jc}) = \begin{cases} 1 & \text{if the } (c-m+1)^{\text{th}} \text{ o.s. of } (x_{i1}, \ldots, x_{ic}) < (c-m+1)^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \\ 0 & \text{otherwise} \end{cases} \] ...

and let

\[ \phi(x_{i1}, \ldots, x_{ic}; x_{j1}, \ldots, x_{jc}) = \phi^{(1)}(x_{i1}, \ldots, x_{ic}; x_{j1}, \ldots, x_{jc}) + \phi^{(2)}(x_{i1}, \ldots, x_{ic}; x_{j1}, \ldots, x_{jc}) \]

\[ = \begin{cases} 2 & \text{if the } m^{\text{th}} \text{ order statistic (o.s.) of } (x_{i1}, \ldots, x_{ic}) < m^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \text{ and } (c-m+1)^{\text{th}} \text{ o.s. of } (x_{i1}, \ldots, x_{ic}) < (c-m+1)^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \\ 1 & \text{if either } m^{\text{th}} \text{ order statistic (o.s.) of } (x_{i1}, \ldots, x_{ic}) < m^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \text{ or } (c-m+1)^{\text{th}} \text{ o.s. of } (x_{i1}, \ldots, x_{ic}) < (c-m+1)^{\text{th}} \text{ o.s. of } (x_{j1}, \ldots, x_{jc}) \text{ but not both} \\ 0 & \text{otherwise.} \end{cases} \] ...

The U-statistic based on this kernel is

\[ U_{ij}^{m,c} = \binom{n_i}{c} \binom{n_j}{c}^{-1} \sum_{i_1 \ldots i_c} \phi(x_{i_1}, \ldots, x_{i_c}; x_{j_1}, \ldots, x_{j_c}) \]
\[ \hat{U}_{ij}^{m,c} = \binom{n_i}{c} \binom{n_j}{c}^{-1} \sum \left[ \phi^{(1)}(x_{ij_1}, \ldots, x_{ij_c}; x_{js_1}, \ldots, x_{js_c}) \ight. \\
+ \left. \phi^{(2)}(x_{ij_1}, \ldots, x_{ij_c}; x_{js_1}, \ldots, x_{js_c}) \right], \quad \ldots(4.3.4) \]

where the summation is extended over all combinations of \( c \) integers \((r_1, \ldots, r_c)\) chosen without replacement from \((1, \ldots, n_i)\) and all combinations of \( c \) integers \((s_1, \ldots, s_c)\) chosen without replacement from \((1, \ldots, n_j)\).

The U-statistic \( \hat{U}_{ij}^{m,c} \) can also be expressed as

\[ \hat{U}_{ij}^{m,c} = \hat{U}_{ij}^{m,c(1)} + \hat{U}_{ij}^{m,c(2)}, \]

where \( \hat{U}_{ij}^{m,c(1)} \) and \( \hat{U}_{ij}^{m,c(2)} \) are the U-statistics associated with the kernels \( \phi^{(1)}(x_{ij_1}, \ldots, x_{ij_c}; x_{js_1}, \ldots, x_{js_c}) \) and \( \phi^{(2)}(x_{ij_1}, \ldots, x_{ij_c}; x_{js_1}, \ldots, x_{js_c}) \), respectively.

The expected value of \( \hat{U}_{ij}^{m,c} \) under \( \mu_i = \mu_j \) is

\[ E[\hat{U}_{ij}^{m,c}] = \binom{C}{m} \sum_{t=m}^{C} \binom{C}{t} \beta[m+t, 2c-m+1-t] \\
+ \sum_{t=c-m+1}^{c} \binom{C}{t} \beta[c-m+1+t, c+m-t] \\
= 1 \quad \text{for all } c \text{ and } m. \]

Define

\[ \hat{r}_{ij}^{m,c} = \hat{U}_{ij}^{m,c} - 1. \quad \ldots(4.3.5) \]
Remark 4.3.2: The statistic $U_{ij}^{m,c}$ can be written in terms of ranks as follows:

Let $X_{j(1)} < X_{j(2)} < \ldots < X_{j(n_j)}$ be the order statistics corresponding to the $j$th sample observations. Suppose $R_j(s)$ denote the rank of $X_{j(s)}$ in the joint ranking of $i$th and $j$th sample observations. For any fixed $s$, $X_{j(s)}$ can be chosen as the $m$th order statistic of the sub sample of size $c$ of $n_j$ observations in $\binom{n_j-s}{c-m}$ ways and the $j$th order statistic of the sub sample of size $c-m$ of $n_j$ observations can be chosen in $\binom{n_j}{j}$ ways. Therefore for every $s$, there are $n_j$ sub-sample pairs for which $\phi^{(1)}(\cdot)=1$.

Thus, we have

$$U_{ij}^{m,c} = \sum_{s=m}^{c-m} \sum_{j'=m}^{c-m} \binom{n_j}{j'} \binom{n_j}{j} \binom{n_j-s}{c-m} \binom{n_j}{j} \binom{n_j+s-R_j(s)}{c-m} \binom{n_j}{j'} \binom{n_j+s-R_j(s)}{c-m}$$

Similarly, we have

$$U_{ij}^{m,c} = \sum_{s=c-m+1}^{c} \sum_{j'=c-m+1}^{c} \binom{n_j}{j'} \binom{n_j}{j} \binom{n_j-s}{c-m} \binom{n_j}{j} \binom{n_j+s-R_j(s)}{c-m} \binom{n_j}{j'} \binom{n_j+s-R_j(s)}{c-m}$$

Therefore rank representation of statistic $U_{ij}^{m,c}$ can be given
The proposed class of selection procedures based on the statistic $T_{ij}^{m,c}$ is:

$$R_{m,c}': \text{Select population } \Pi_j \text{ in the subset if and only if}$$

$$T_{ij}^{m,c} = b \left( c \rho_{m,c} \left( n_j^{-1} + n_{ij}^{-1} \right) \right)^{1/2} \text{ for all } i, i \neq j.$$ 

Here $\rho_{m,c}$ is a constant which depends upon the choice of $m$ and $c$ (see expression (4.5.4)) and the constant $b$ is chosen such that

$$P \left( T_{ij}^{m,c} > b \left( c \rho_{m,c} \left( n_j^{-1} + n_{ij}^{-1} \right) \right)^{1/2} \right) \text{ for all } i, i \neq j \in \mathbb{P},$$

where $\mathbb{P}$ indicates that the probability is computed under $\mu_1 = \ldots = \mu_k$.

For fixed values of $m$ and $c$ in their domain of definition, the selection procedure $R_{m,c}'$ is a member of the proposed class. In the following section we obtain the expressions for PCS and expected subset size $E(S)$.

### 4.4 Probability of Correct Selection and Expected Subset Size

Let $A$ be the action space of the subset selection problem which is the set of all nonempty subsets of $\{1, \ldots, k\}$, where taking action $a \in A$ means the selection of those populations whose indices...
are in a. For any \( a \in A \), let
\[
CS(\mu, a) = \begin{cases} 
1 & \text{if } \mu[k] \in \{\mu_i, i \in a\} \\
0 & \text{otherwise}
\end{cases}
\]
and \(|a|\) denotes the number of elements in \( a \).

For any subset selection procedure \( R \), let \( Z(R, X, a) \) be the probability assigned to \( R \) having observed \( X \).

Let us define \( \tilde{A} = \{a \in A | CS(\mu, a) = 1\} \). The PCS is given by
\[
P_{\mu} [CS| R'] = P_{\mu} [\text{Population corresponding to } \mu[k] \text{ is in the selected subset } | R']
\]
\[
= P_{\mu} [\bigcup_{\mu} (X \text{ is observed and action } a \text{ is taken } | R')] \
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_{\mu} \left( \bigcup_{\mu} (\text{action } a \text{ is taken } | X, R') \right) \prod_{r=1}^{k} n_r dF(x)
\]
\[
= E \left[ \sum_{a \in A} P_{\mu} (\text{action } a \text{ is taken } | X, R') \right] \
= E \left[ \sum_{a \in A} CS(\mu, a) Z(R, X, a) \right].
\]

Let \( S \) denote the size of the selected subset. Then the expected subset size is seen to be
\[
E_{\mu} [S | R'] = E_{\mu} [\sum_{a \in A} |a| Z(R, X, a)].
\]

Let \( P^{*} > k^{-1} \) be specified. In order to show that procedures \( R_{m, c}' \) satisfy \( P^{*} \)-condition, we first prove that the family of distributions of statistics \( T_{ij}^{m, c} \) is stochastically increasing.

**Lemma 4.4.1:** The family of distributions of \( T_{ij}^{m, c} \) is stochastically increasing (SI).

**Proof:** For \( \mu_i < \mu_j \), we have
\[ F_i(x) \leq F_j(x) \leq F_j'(x), \text{ for all } x. \]

Here \( F_i(x) = F(x - \mu_i), i = 1, \ldots, k. \) Let \( G(x; \Delta_{ij}) \) be the cdf of \( T_{ij}^{m,c}. \)

Now \( \mu_i < \mu_j < \mu_i' \), \( \iff \Delta_{ij} < \Delta_{ij}' \), and we want to show that \( T_{ij}^{m,c} \)

\[ \leq T_{ij}'^{m,c} \], i.e., \( G(x; \Delta_{ij}) \leq G(x; \Delta_{ij}') \) for all \( x. \)

Consider the event \( m \) th order statistic (o.s.) of \( (X_{i1}, \ldots, X_{ic}) \) < \( m \) th o.s. of \( (X_{j1}, \ldots, X_{jc}) \)

\[ = m \text{ th o.s. of } (X_{i1}, \ldots, X_{ic}) < m \text{ th o.s. of } (F_{j1}^{-1}F_i(X_{i1}), \ldots, F_{jc}^{-1}F_i(X_{ic})) \]

\[ \implies m \text{ th o.s. of } (X_{i1}, \ldots, X_{ic}) < m \text{ th o.s. of } (F_{j1}^{-1}F_i(X_{i1}), \ldots, F_{jc}^{-1}F_i(X_{ic})) \]

(since \( F_{j1}^{-1},(x) \geq F_j^{-1}(x) \) for all \( x, 0 < x < 1). \)

Thus

\[ \phi_{ij}^{(1)}(X_{i1}, \ldots, X_{ic}, X_{j1}, \ldots, X_{jc}) = 1 \]

\[ \implies \phi_{ij}^{(1)}(X_{i1}, \ldots, X_{ic}; X_{j1}, \ldots, X_{jc}) = 1. \]

Similarly, consider the event \( (c-m+1) \) th order statistic (o.s.) of \( (X_{i1}, \ldots, X_{ic}) \)

\[ (c-m+1) \text{ th o.s. of } (X_{j1}, \ldots, X_{jc}) < (c-m+1) \text{ th o.s. of } (F_{j1}^{-1}F_i(X_{i1}), \ldots, F_{jc}^{-1}F_i(X_{ic})) \]

\[ (\text{since } F_{j1}^{-1},(x) \geq F_j^{-1}(x) \text{ for all } x, 0 < x < 1). \]
Thus
\[ \Phi_{ij}(x_{i1}, \ldots, x_{ic}; x_{j1}, \ldots, x_{jc}) = 1 \]
\[ \implies \Phi_{ij}(x_{i1}, \ldots, x_{ic}; x_{j1}, \ldots, x_{jc}) = 1. \]

Hence by using (4.3.4), we get
\[ U_{ij}^{*} \leq U_{ij}' \]
or
\[ T_{ij}^{m,c} \leq T_{ij}'^{m,c}. \]

Hence the lemma follows.

**Theorem 4.4.1:** For every choice of \( m \) and \( c \) the selection procedure \( R' \) satisfies the \( P^* \)-condition.

**Proof:** Assume without loss of generality that \( \Pi_j \) is the best population. Now for any fixed \( j \), \( \min_{i \neq j} T_{ij}^{m,c} \) is nondecreasing in \( X_{j1}, \ldots, X_{jn_j} \) and nonincreasing in the other components of \( X \).

Furthermore the family of distributions of \( T_{ij}^{m,c} \) is SI. Define the indicator function \( I_3(\cdot) \) as
\[ I_3(T_{ij}^{m,c}, T_{2j}^{m,c}, \ldots, T_{j-1,j}^{m,c}, T_{j+1,j}^{m,c}, \ldots, T_{kj}^{m,c}) \]
\[ = \begin{cases} 
1 & \text{if } T_{ij}^{m,c} \geq b \left[ c^2 \rho_{m,c} \left( n_j^{-1} + n_{ij}^{-1} \right) \right]^{1/2} \text{ for all } i, i \neq j \\
0 & \text{otherwise.} 
\end{cases} \]

By writing the expected value of the indicator function \( I_3(\cdot) \) as the expectation of the conditional expectation and using lemma 2.4.2, we have for any \( \mu \in \Omega \),
\[ P^* \leq P_0 \left[ T_{ij}^{m,c} \geq b \left[ c^2 \rho_{m,c} \left( n_j^{-1} + n_{ij}^{-1} \right) \right]^{1/2} \text{ for all } i, i \neq j \right] \]
$P_{\mu} \left[ \sum_{i=1}^{m,c} T_{1j} \geq b \left[ c^2 \rho_{m,c} (n^{-1}_{1j} + n^{-1}_{1i}) \right]^{1/2} \right]$ for all $i, i \neq j$

$= P_{\mu} \left[ \sum_{i=1}^{m,c} T_{1j} \leq b \left( c^2 \rho_{m,c} (n^{-1}_{1j} + n^{-1}_{1i}) \right)^{1/2} \right]$.

Hence the theorem.

With the help of the following theorem, we can establish the strong monotonicity of procedure $R'_{m,c}$ which in turn establishes its monotonicity and unbiasedness.

**Theorem 4.4.2:** For every $m$ and $c$ the selection procedure $R'_{m,c}$ is strongly monotone.

**Proof:** From theorem 4.4.1 we have seen that the indicator function $I_{3}^{m,c} (T_{1j}, T_{2j}, \ldots, T_{j-i, j}, T_{j+i, j}, \ldots, T_{k_j})$ is nondecreasing function of observations from the $j$th population and nonincreasing function of observations from the $i$th population ($i=1,2,\ldots,k, i \neq j$). By writing the expectation as expectation of conditional expectation and then using lemma 2.4.2, we get

$P_{\mu} (j) = E_{\mu} \left[ I_{3}^{m,c} (T_{1j}, T_{2j}, \ldots, T_{j-i, j}, T_{j+i, j}, \ldots, T_{k_j}) \right]$

$= E_{\mu} \left[ I_{3}^{m,c} (T_{1j}, T_{2j}, \ldots, T_{j-i, j}, T_{j+i, j}, \ldots, T_{k_j}) \right]$

$= P_{\mu} (j)$,

where $\mu = (\mu_1, \ldots, \mu_{j-1}, \mu_j, \mu_{j+1}, \ldots, \mu_k)^T$ and

$\mu^* = (\mu_1, \ldots, \mu_{j-1}, \mu_j^*, \mu_{j+1}, \ldots, \mu_k)^T$ with $\mu_j \leq \mu_j^*$. This proves the theorem.

Since strong monotonicity implies monotonicity which in turn implies unbiasedness (see Santner (1975)), we have the following
Corollary 4.4.1: The selection procedures $R_{m,c}$ are monotone and unbiased.

4.5 Asymptotic Relative Efficiency:

We shall compare our procedures $R_{m,c}$ with the procedures $R_a$ proposed by Hsu (1981a) in the sense of Pitman ARE where the Hsu's procedure based on two-sample linear rank statistics is as follows:

Let $X_{i1}, \ldots, X_{in_i}$ be the $n_i$ observations taken from the $i$th population under $R_a$, $i=1,\ldots,k$. For the $i$th and $j$th samples, let $R_{ij}^{(i)}$ be the rank of $X_{ij}^{(i)}$, $j=1,\ldots,n_j'$, in the combined sample $X_{i1},\ldots,X_{in_i}'; X_{j1},\ldots,X_{jn_j}'. Let$

S_{ij}^{(i)} = \left[ \frac{1}{n_{i1}} + \frac{1}{n_{j1}} \right] \left[ \sum_{\beta=1}^{n_j'} a_{\beta}^{(i)} (R_{ij}^{(i)}) - n_j' \tilde{a}_n^{(i)} \right],

where $n' = n_{i1}' + n_{j1}'$, $\tilde{a}_m = \frac{1}{m} \sum_{\beta=1}^{m} a_\beta^{(i)}$ and $a_\beta^{(i)}$ are some given scores.

The selection procedures $R_a$ of Hsu(1981a) are:

$R_a :$ Select $\Pi_j$ in the subset if and only if

$S_{ij}^{(i)} \geq C_j^{(i)} (n', P^*)$ for all $i, i \neq j$,

where $n' = (n_1',\ldots,n_k')$ and the constants $C_j^{(i)} (n', P^*)$ are chosen such that for a pre-assigned probability $P^*$, we have

$P_0 [S_{ij}^{(i)} \geq C_j^{(i)} (n', P^*)$ for all $i, i \neq j] \geq P^*.$

Here $P_0$ indicates that the probability is computed under the configuration $\mu_1 = \ldots = \mu_k$. Hsu (1981a) has shown that
Now consider the vector
$$T = (T_{m1}^m, c, \ldots, T_{m1}^m, c, T_{m2}^m, c, \ldots, T_{m2}^m, c, \ldots, T_{1k}^m, c, \ldots, T_{1k}^m, c) \.Transpose.$$

In order to find the asymptotic distribution of $T$ we first prove the following lemma and then obtain the mean and variance-covariance structure of $T$.

**Lemma 4.5.1:** \( \mathbb{E}[T_{m1}^m] = \frac{c!}{(m-1)!(c-m)!} \sum_{t=m}^{c} \binom{c}{t} (t)_{-m} \int_{-\infty}^{\infty} (H(x))^t (1-H(x))^{c-t} (G(x))(m-1) (1-G(x))(c-m) \ dG(x) + \frac{c!}{(m-1)!(c-m)!} \sum_{t=c-m+1}^{c} \binom{c}{t} (t)_{c-m} \int_{-\infty}^{\infty} (H(x))^t (1-H(x))^{c-t} (G(x))(m-1) (1-G(x))(c-m) \ dG(x) - 1. \)

where \( G(x) = F_{j} (x) = H(x-A_{1j}), H(x) = F_{i} (x) \) for all \( x \).

**Proof:**

$$E[T_{m1}^m] = E[U_{m1}^m] - 1 = P[ m \text{ o.s. } (X_{i1}, \ldots, X_{ic}) < m \text{ o.s. } (X_{j1}, \ldots, X_{jc}) ] + P[ (c-m+1) \text{ o.s. } (X_{i1}, \ldots, X_{ic}) < (c-m+1) \text{ o.s. } (X_{j1}, \ldots, X_{jc}) ] - 1.$$

This proves the lemma.
Further $T_{ij}^{m,c} = U_{ij}^{m,c} - 1$, $i \neq j$, and $U_{ij}^{m,c}$ being two sample U-statistic, the joint limiting normality of $\{T_{ij}^{m,c}\}$ as stated in lemma below, follows immediately from a result of Lehmann (1963a).

**Lemma 4.5.2:** The asymptotic distribution of $N^{1/2}[T - E(T)]$ as minimum $(n_1, \ldots, n_k) \to \infty$, in such a way that $(n_i/N) \to p_i$, $0 < p_i < 1$, for $i = 1, \ldots, k$, is multivariate normal with mean vector $0$ and dispersion matrix $\Sigma = \{\sigma_{ij}\}$, when $N = n_1 + \ldots + n_k$ and

$$\sigma_{ij} = N \text{cov}(T_{ij}^{m,c}, T_{ij}^{m,c})$$

for $i = 1, \ldots, k$.

$$\sigma_{ij} = N \left[ E(T_{ij}^{m,c} - E(T_{ij}^{m,c})E(T_{ij}^{m,c})) \right]$$

$$= N \left[ E(T_{ij}^{m,c} - E(T_{ij}^{m,c}))E(T_{ij}^{m,c}) \right]$$

$$= \begin{bmatrix}
(c^2/p_i) f_{i,j;i,j}^{(i)} + (c^2/p_j) f_{i,j;i,j}^{(j)} & \text{if } i = h, j = e \\
(c^2/p_i) f_{i,j;i,e}^{(i)} & \text{if } i = h, j = e \\
(c^2/p_j) f_{i,j;i,h}^{(j)} & \text{if } i = h, j = e \\
(c^2/p_i) f_{i,j;j,h}^{(i)} & \text{if } i = e, j = h \\
(c^2/p_j) f_{i,j;j,h}^{(j)} & \text{if } i = e, j = h \\
0 & \text{otherwise,} \\
0 & \text{otherwise,} \\
\end{bmatrix}$$

$$\text{...(4.5.2)}$$

where

$$f_{i,j;i,j}^{(i)} = E[\psi_{ij}^{(i)}(X)^2],$$

$$f_{i,j;i,j}^{(j)} = E[\psi_{ij}^{(j)}(X)^2],$$

$$f_{i,j;i,e}^{(i)} = E[\psi_{ij}^{(i)}(X) \psi_{i,e}^{(i)}(X)].$$
In the following theorem we obtain the asymptotic mean and
dispersion matrix of the standardised vector $T$ under the
configuration $\mu_1=\ldots=\mu_k$.

**Theorem 4.5.1:** Under the configuration $\mu_1=\ldots=\mu_k$, and as $\min n_i \to \infty$ in such a way that $(n_i/N) \to p_i$, $0 < p_i < 1$, for $i=1,\ldots,k$,
where $N = n_1 + \ldots + n_k$, the vector $(N/c^2 \rho_{m,c}^{-1/2})^{-1}T$ is asymptotically
normal with mean vector 0 and variance-covariance matrix as

$$(N/c^2 \rho_{m,c}^{-1/2}) \eta(T_{ij}^{m,c_{m,c}}) = \begin{cases} 
1/p_i + 1/p_j & \text{if } i=h, j=e \\
1/p_i & \text{if } i=h, j\neq e \\
1/p_j & \text{if } i\neq h, j=e \\
-1/p_i & \text{if } i=e, j\neq h \\
-1/p_j & \text{if } i\neq e, j=h \\
0 & \text{otherwise. ... (4.5.3)}
\end{cases}$$

Where $\rho_{m,c} = \left(\frac{c}{c^2-m-1}\right)^2 \sum_{u=0}^{2(c-m)} \sum_{u'=0}^{2(c-m)} (-1)^{u+u'} \binom{2(c-m)}{u} \binom{2(c-m)}{u'} \frac{1}{(2m+u-1)} \\
\times \left[ \frac{1}{(2m+u')^2} - \frac{1}{(2m+u)(4m-1+u+u')} \right] \\
+ \left[ \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v] \right]^2 \left[ \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c+s-m+1, c+m-s-1] \right]^2
+ \left( \begin{array}{c}
\binom{c-1}{m} \left( \begin{array}{c}
2m-2 \\
z=0
\end{array} \right)
\right) \left( \begin{array}{c}
2m-2 \\
z'=0
\end{array} \right) \frac{1}{(2c+z-2m+1)}
\right)

\times \left[ \frac{1}{(2c+z'-2m+2)} - \frac{1}{(2c+z-2m+2)(4c+z+z'-4m+3)} \right]

+ 2 \left( \begin{array}{c}
\binom{c-1}{m-1} \left( \begin{array}{c}
c-1 \\
v=m
\end{array} \right) \beta[m+v, 2c-m-v]
\end{array} \right) \frac{2(c-m)}{u=0} \left( \begin{array}{c}
1 \left( \begin{array}{c}
2m-2 \\
z'
\end{array} \right)
\end{array} \right) \frac{1}{(2m+u)}

+ 2 \left( \begin{array}{c}
\binom{c-1}{m-1} \left( \begin{array}{c}
c-1 \\
v=m
\end{array} \right) \beta[m+v, 2c-m-v]
\end{array} \right) \frac{2(c-m)}{u=0} \left( \begin{array}{c}
1 \left( \begin{array}{c}
2m-2 \\
z
\end{array} \right)
\end{array} \right) \frac{1}{(2m+u)}

\times \left[ \frac{1}{(2m+u)} - \frac{1}{(2c+u+z+1)(2c+z-2m+2)} \right]

+ 2 \left( \begin{array}{c}
\binom{c-1}{m-1} \left( \begin{array}{c}
c-1 \\
v=m
\end{array} \right) \beta[m+v, 2c-m-v]
\end{array} \right) \frac{2(c-m)}{u=0} \left( \begin{array}{c}
1 \left( \begin{array}{c}
2m-2 \\
z
\end{array} \right)
\end{array} \right) \frac{1}{(2c+z-2m+2)}

\times \left[ \frac{1}{(2c+z-2m+2)} - \frac{1}{(2c+z-2m+2)(4c+z+z'-4m+3)} \right]

+ 2 \left( \begin{array}{c}
\binom{c-1}{m-1} \left( \begin{array}{c}
c-1 \\
v=m
\end{array} \right) \beta[m+v, 2c-m-v]
\end{array} \right) \frac{2(c-m)}{s=c-m+1} \left( \begin{array}{c}
1 \left( \begin{array}{c}
s-2m-2 \\
z
\end{array} \right)
\end{array} \right) \frac{1}{(2c+z-2m+2)}

\times \left[ \frac{1}{(2c+z-2m+2)} - \frac{1}{(2c+z-2m+2)(4c+z+z'-4m+3)} \right]

\right) - 1.

\ldots (4.5.4)
Proof: First we shall show that under the configuration

\[ \mu_1 = \ldots = \mu_k, \quad (N/c^2 \rho_{m,c})^{1/2} \mathbb{E}[T] = 0. \]

Using the lemma 4.5.1, we have

\[
\mathbb{E}[T^m_{i,j}] = \binom{c}{m} \left[ \sum_{t=m}^{c} \binom{c}{t} \int_{-\infty}^{\infty} (H(x))^t (1-H(x))^{c-t} \right. \\
\left. \times (G(x))^{m-1} (1-G(x))^{c-m} \, dG(x) \right] - 1.
\]

If \( \mu_1 = \ldots = \mu_k \), then

\[
\mathbb{E}[T^m_{i,j}] = \binom{c}{m} \left[ \sum_{t=m}^{c} \binom{c}{t} \beta_{m+t,2c-m+1-t} + \int_{c-m+1}^{c} \beta_{c-m+1+t, c+m-t} \right] - 1
\]

= 0 for all \( i \neq j; i, j=1, \ldots, k \).

Since this holds for all \( i \neq j \), therefore \( \mathbb{E}[T] = 0 \), and thus

\[(N/c^2 \rho_{m,c})^{1/2} \mathbb{E}[T] = 0. \]

The asymptotic normality of \( (N/c^2 \rho_{m,c})^{1/2} \mathbb{E}[T] \) follows from lemma 4.5.2 and its variance-covariance structure (4.5.3) will be established using (4.5.2) if under the configuration \( \mu_1 = \ldots = \mu_k \) we can prove the followings:
(i) \( \Sigma_{i,j}^{(i)} = \rho_{m,c} \)

(ii) \( \Sigma_{i}^{(j)} = \rho_{m,c} \)

(iii) \( \Sigma_{i,j}^{(i)} = \rho_{m,c} \)

(iv) \( \Sigma_{i,j}^{(j)} = \rho_{m,c} \)

(v) \( \Sigma_{i,j}^{(i)} = -\rho_{m,c} \)

(vi) \( \Sigma_{i,j}^{(j)} = -\rho_{m,c} \)

Now, since

\[ \psi_{ij}^{(i)} = E[\psi_{ij}^{(i)}(X)^2], \]

where

\[ \psi_{ij}^{(i)}(x) = E[\phi(x, X_{i2}, \ldots, X_{ic}, X_{j1}, \ldots, X_{jc})] - E[\phi(x, X_{i1}, \ldots, X_{ic}, X_{j1}, \ldots, X_{jc})], \]

\[ = P^{\text{th}} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) < P^{\text{th}} \text{ o.s. of } (X_{j1}, \ldots, X_{jc}) \]

\[ + P^{(c-m+1)\text{th}} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) < (c-m+1)\text{th} \text{ o.s. of } (X_{j1}, \ldots, X_{jc}) \]

\[ = P_1 - 1. \quad \ldots (4.5.5) \]

Where

\[ P_1 = P^{\text{th}} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) < P^{\text{th}} \text{ o.s. of } (X_{j1}, \ldots, X_{jc}) \]

\[ + P^{(c-m+1)\text{th}} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) \]

\[ (c-m+1)\text{th} \text{ o.s. of } (X_{j1}, \ldots, X_{jc}) \]

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\[
\begin{align*}
&= \int_{-\infty}^{\infty} p[m^{\text{th}} \text{ o.s. of } (x,x_{i_2}, \ldots, x_{i_c}) < y]
\times dp[m^{\text{th}} \text{ o.s. of } (x_1, \ldots, x_j) \leq y] + \int_{-\infty}^{\infty} p[(c-m+1)^{\text{th}} \text{ o.s. of } (x,x_{i_2}, \ldots, x_{i_c}) < y]dp[(c-m+1)^{\text{th}} \text{ o.s. of } (x_1, \ldots, x_j) \leq y] \\
&= \left\{ \begin{array}{l}
\sum_{v=m}^{c-1} \binom{c-1}{v} (H(y))^v (1-H(y))^{c-1-v} \\
+ \sum_{v=m-1}^{\infty} \binom{c-1}{v} (H(y))^v (1-H(y))^{c-1-v}
\end{array} \right\}
\times m_{m}^{(c-m)} (G(y))^{(m-1)} (1-G(y))^{(c-m)} dG(y)
\]

\[
+ \left\{ \begin{array}{l}
\sum_{s=c-m+1}^{c-1} \binom{c-1}{s} (H(y))^s (1-H(y))^{c-1-s} \\
+ \sum_{s=c-m}^{\infty} \binom{c-1}{s} (H(y))^s (1-H(y))^{c-1-s}
\end{array} \right\}
\times m_{m}^{(c-m)} (G(y))^{(m-1)} (1-G(y))^{(c-m)} dG(y)
\]

\[
= m_{m}^{(c-m)} \left[ \left\{ \sum_{v=m}^{\infty} \binom{c-1}{v} (H(y))^v (1-H(y))^{c-1-v} \\
+ \sum_{s=m-1}^{\infty} \binom{c-1}{s} (H(y))^s (1-H(y))^{c-1-s}\right\}
\times (G(y))^{(m-1)} (1-G(y))^{(c-m)} dG(y)
\]

\[
+ \left\{ \sum_{s=c-m+1}^{\infty} \binom{c-1}{s} (H(y))^s (1-H(y))^{c-1-s}\right\}
\times (1-G(y))^{(m-1)} (1-G(y))^{(c-m)} dG(y)
\]

\]

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\[
\begin{align*}
+ \int_{\infty}^0 \left( \frac{c-1}{c-m} \right) \left( H(y) \right)^{(c-m)} \left( 1-H(y) \right)^{(m-1)} dH(y) \\
\times \left( G(y) \right)^{(c-m)} \left( 1-G(y) \right)^{(m-1)} dG(y) \\
\end{align*}
\]

Under \( \mu_1 = \ldots = \mu_k \), we have

\[
= m \left( \sum_{v=m}^{c-1} \left( \frac{1}{v} \right) \int_{-\infty}^{\infty} \left( H(y) \right)^{(v+m-1)} \left( 1-H(y) \right)^{(2c-v-1)} dH(y) \\
+ \int_{-\infty}^{\infty} \left( \frac{1}{c-m} \right) \left( H(y) \right)^{2(m-1)} \left( 1-H(y) \right)^{2(c-m)} dH(y) \right) \\
+ \left\{ \sum_{s=c-m+1}^{c-1} \left( \frac{1}{s} \right) \int_{-\infty}^{\infty} \left( H(y) \right)^{(c+s)} \left( 1-H(y) \right)^{(c-m-s-2)} dH(y) \right\} \\
+ \left\{ \int_{-\infty}^{\infty} \left( \frac{1}{c-m} \right) \left( H(y) \right)^{2(c-m)} \left( 1-H(y) \right)^{2(m-1)} dH(y) \right\} \\
= \sum_{v=m}^{c-1} \left( \frac{1}{v} \right) \beta[v+m,2c-m-v] + \sum_{s=c-m+1}^{c-1} \left( \frac{1}{s} \right) \beta[c-m+s+1,c+m-s-1] \\
+ \int_{H(x)}^{1} z^{(m-2)} \left( 1-z \right)^{2(c-m)} dz \\
+ \int_{H(x)}^{1} z^{2(c-m)} \left( 1-z \right)^{2(m-1)} dz \\
= \sum_{v=m}^{c-1} \left( \frac{1}{v} \right) \beta[v+m,2c-m-v] + \sum_{s=c-m+1}^{c-1} \left( \frac{1}{s} \right) \beta[c-m+s+1,c+m-s-1] \\
\times \beta[c-m+s+1,c+m-s-1] + \sum_{u=0}^{c-1} \left( \frac{1}{u} \right)^{2(c-m)} \\
\times \left[ 1-H(x) \right)^{(2m-u+1)} / (2m+u-1) + \sum_{z=0}^{c-1} (-1)^{c-m} \frac{2(m-1)}{z} \right] \\
\times \left[ 1-H(x) \right)^{(2c-2m+z+1)} / (2c-2m+z+1) \right). \\
\]
Therefore from (4.5.5), we have

\[
\psi_{ij}(x) = m(m) \sum_{v=m}^{c-1} \binom{v}{m+v, 2c-m-v} + \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta \left[ c-m+s+1, c+m-s-1 \right] + \frac{2(c-m)}{m} \sum_{u=0}^{2(m-1)} \binom{c-1}{m} \frac{(-1)^{u}}{u!} \times \prod_{v=m}^{c-1} \binom{c-1}{u \in \mathbb{Z}} \sum_{z=0}^{2(m-1)} \frac{(-1)^{z}}{z!} \times \prod_{z'=0}^{2(m-1)} \frac{(-1)^{z'}}{z'!} \left[ 1-(H(x))^{(2c-2m+z+1)}/(2c-2m+z+1) \right] - 1 \ldots (4.5.6)
\]

Now, since

\[
\psi_{ij}^{(1)} = E \left[ \psi_{ij}(x)^2 \right].
\]

Therefore

\[
\psi_{ij}^{(1)} = m(m) \sum_{v=m}^{c-1} \binom{v}{m+v, 2c-m-v} + \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta \left[ c-m+s+1, c+m-s-1 \right] + \frac{2(c-m)}{m} \sum_{u=0}^{2(m-1)} \binom{c-1}{m} \frac{(-1)^{u}}{u!} \times \prod_{v=m}^{c-1} \binom{c-1}{u \in \mathbb{Z}} \sum_{z=0}^{2(m-1)} \frac{(-1)^{z}}{z!} \times \prod_{z'=0}^{2(m-1)} \frac{(-1)^{z'}}{z'!} \left[ 1-(H(x))^{(2c-2m+z+1)}/(2c-2m+z+1) \right] - 1 \ldots (4.5.6)
\]
\[
+ \frac{(H(x))(4(4c-4m+z+z')+2)}{[(2c-2m+z+1)(2c-2m+z'+1)]}
\]

\[
+ 2 \binom{c-1}{m-1} \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[v, m+v, 2c-m-v]
\]

\[
2(c-m) \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} 1 \left[ 1 - (H(x))^{2m+u-1} \right] / (2m+u-1)
\]

\[
+ 2 \binom{c-1}{m-1} \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[s, c-m+s+1, c+m-s-1]
\]

\[
x \left[ 1 - (H(x))^{2m+u-1} \right] - (H(x))^{(2c-2m+z+1)}
\]

\[
+ (H(x))^{(2c+u+z)}/[(2m+u-1)(2c-2m+z+1)]
\]

\[
+ 2 \binom{c-1}{c-m} \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[v, m+v, 2c-m-v]
\]

\[
\times \sum_{z=0}^{2(m-1)} (-1)^z \binom{2(m-1)}{z} \left[ 1 - (H(x))^{2c-2m+z+1} \right] / (2c-2m+z+1)
\]

\[
+ 2 \left[ \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[v, m+v, 2c-m-v] \right] \left[ \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \right]
\]

\[
x \beta[c-m+s+1, c+m-s-1]
\]

\[
+ 2 \binom{c-1}{c-m} \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1, c+m-s-1]
\]

\[
\times \sum_{z=0}^{2(m-1)} (-1)^z \binom{2(m-1)}{z} \left[ 1 - (H(x))^{2c-2m+z+1} \right] / (2c-2m+z+1)
\]} - 1.
Therefore

\[
\begin{align*}
\{i\}^{(i)}_{i,j;i,j} &= \left(\begin{array}{c}
c_m \end{array}\right)^2 \left(\begin{array}{c}
c_m-1 \end{array}\right)^2 \sum_{u=0}^{2(c-m)} \sum_{u'=0}^{2(c-m)} \frac{(-1)^u + u'}{u}  \\
& \times \frac{1}{(2m+u-1)} \left[ \frac{1}{(2m+u')} - \frac{1}{(2m+u)(4m-1+u+u')} \right] \\
& + \left[ \sum_{v=m}^{c-1} \left(\begin{array}{c}
c_v \end{array}\right)^2 \beta[m+v,2c-m-v] \right]^2 \\
& + \left[ \sum_{s=c-m+1}^{c-1} \left(\begin{array}{c}
c_s \end{array}\right)^2 \beta[c+s-m+1,c+m-s-1] \right]^2 \\
& + \left[ \sum_{z=0}^{2m-2} \sum_{z'=0}^{2m-2} \frac{(-1)^{z+z'} (2m-2)!}{z! z'} \right] \frac{1}{(2c+z-2m+1)} \\
& \times \left[ \frac{1}{(2c+z'-2m+2)} - \frac{1}{(2c+z-2m+2)(4c+z+z'-4m+3)} \right] \\
& + 2 \left[ \sum_{v=m}^{c-1} \left(\begin{array}{c}
c_v \end{array}\right)^2 \beta[m+v,2c-m-v] \right] \frac{2(c-m)}{2m+u} \\
& + 2 \left[ \sum_{s=c-m+1}^{c-1} \left(\begin{array}{c}
c_s \end{array}\right)^2 \beta[c+s-m+1,c+m-s-1] \right] \\
& \times \frac{2(c-m)}{2m+u} \\
& + 2 \sum_{u=0}^{2(c-m)} \frac{(-1)^u (2(c-m))}{u} \frac{1}{(2m+u)} \\
& \times \left[ \frac{1}{(2m+u)} - \frac{1}{(2c+u+z+1)(2c+z-2m+2)} \right]
\end{align*}
\]
\[ + 2 \binom{c-1}{c-m} \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[v, m + v, 2c - m - v] \]
\[ \times \left[ \sum_{z=0}^{2m-2} (-1)^z \binom{2m-2}{z} \frac{1}{(2c+z-2m+2)} \right] \]
\[ + 2 \binom{c-1}{c-m} \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c + s - m + 1, c + m - s - 1] \]
\[ \times \left[ \sum_{z=0}^{2m-2} (-1)^z \binom{2m-2}{z} \frac{1}{(2c+z-2m+2)} \right] - 1 \quad \ldots(4.5.7) \]
\[ = \rho_{m,c} \text{(say)}. \]

Hence (i) is proved.

Now
\[ \{^{(j)} \}_{i,j;i,j} = E\left[ \psi_{ij}^{(j)} (X)^2 \right], \]
where
\[ \psi_{ij}^{(j)} (X) = E[\phi(X_{i1}, \ldots, X_{ic}; X_{j1}, \ldots, X_{jc})] \]
\[ - E[\phi_{ij} (X_{i1}, \ldots, X_{ic}; X_{j1}, \ldots, X_{jc})], \]
\[ = P[ \text{m}^{\text{th}} \text{o.s. of } (X_{i1}, X_{i2}, \ldots, X_{ic}) < \text{m}^{\text{th}} \text{o.s. of } (X, X_{j2}, \ldots, X_{jc})] \]
\[ + P[ (c-m+1)^{\text{th}} \text{o.s. of } (X_{i1}, X_{i2}, \ldots, X_{ic}) < (c-m+1)^{\text{th}} \text{o.s. of } (X, X_{j2}, \ldots, X_{jc})] - 1 \]
\[ = P_2 - 1. \quad \ldots(4.5.8) \]
Where

\[ P_2 = P[ \text{m.o.s. of } (X_{11}, \ldots, X_{1c}) < \text{m.o.s. of } (x, X_{j2}, \ldots, X_{jc}) ] \]

\[ + \ P[ (c-m+1) \text{th o.s. of } (X_{11}, X_{12}, \ldots, X_{1c}) < (c-m+1) \text{th o.s. of } (x, X_{j2}, \ldots, X_{jc}) ] \]

\[ = \int_{-\infty}^{\infty} P[m \text{th o.s. of } (x, X_{j2}, \ldots, X_{jc}) > y] \]

\[ + \int_{-\infty}^{\infty} P[(c-m+1) \text{th o.s. of } (x, X_{j2}, \ldots, X_{jc}) > y] \]

\[ \times P[(c-m+1) \text{th o.s. of } (X_{11}, \ldots, X_{1c}) > y] \]

\[ = \left\{ \sum_{v=m}^{c-1} C_{c}^{v} (\frac{H(y)}{1-H(y)})^{v} (1-\frac{\bar{H}(y)}{1-\frac{\bar{H}(y)}})^{c-1-v} \right\} \]

\[ \times m_{m}^{c-1} G(y)^{(m-1)} (1-G(y))^{(c-m)} dG(y) \]

\[ + \left\{ \sum_{s=c-m+1}^{c-1} C_{c}^{s} \bar{H}(y)^{s} (1-\frac{\bar{H}(y)}{1-\frac{\bar{H}(y)}})^{(c-1-s)} \right\} \]

\[ \times m_{m}^{c} G(y)^{(c-m)} (1-G(y))^{(m-1)} dG(y) \]

\[ = m_{m}^{c} \left\{ \sum_{v=m}^{c-1} C_{c}^{v} (\frac{H(y)}{1-H(y)})^{v} (1-\frac{\bar{H}(y)}{1-\frac{\bar{H}(y)}})^{c-1-v} \right\} \]
\[
+ \int_{x}^{\infty} \left( \frac{c-1}{m-1} \right) (H(y))^{m-1} (1-H(y))^{(c-m)} \\
\times (G(y))^{m-1} (1-G(y))^{(c-m)} \, dG(y)
\]

\[
+ \left\{ \int_{-\infty}^{\infty} \frac{c-1}{s} \sum_{s=c-m+1}^\infty (H(y))^{(c-s)} (1-H(y))^{(c-s)} \, dH(y) \right\}
\]

\[
+ \left\{ \int_{x}^{\infty} \left( \frac{c-1}{c-m} \right) (H(y))^{(c-m)} (1-H(y))^{(m-1)} \right\}
\times (G(y))^{(c-m)} (1-G(y))^{(m-1)} \, dG(y)
\].

Under \( \mu_1 \ldots \mu_k \), we have

\[
= m(C_m) \left\{ \left\{ \sum_{v=m}^{c-1} \binom{c-1}{v} \right\} \int_{-\infty}^{\infty} (H(y))^{(v+m-1)} (1-H(y))^{(2c-v-m-1)} \, dH(y) \right\}
\]

\[
+ \left\{ \sum_{s=c-m+1}^\infty (H(y))^{(c-m+s)} (1-H(y))^{(c+m-s-2)} \, dH(y) \right\}
\]

\[
= m(C_m) \left\{ \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v,2c-m-v] + \sum_{s=c-m+1}^\infty \binom{c-1}{s} \right\}
\times \beta[c-m+s+1,c+m-s-1] + \left\{ \sum_{m-1}^{c-1} \frac{1}{H(x)} \right\} \int_{0}^{1} z^{(m-2)} (1-z)^{2(c-m)} \, dz
\]

\[
+ \left( \frac{c-1}{c-m} \right) \int_{0}^{1} z^{(c-m)} (1-z)^{2(m-1)} \, dz
\]

\[
= m(C_m) \left\{ \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v,2c-m-v] + \sum_{s=c-m+1}^\infty \binom{c-1}{s} \right\}
\].
\[
\beta[c-m+s+1,c+m-s-1] + \sum_{u=0}^{c-1} (-1)^u \binom{2(c-m)}{u} \\
\times \left[ 1 - (\bar{H}(x))^{(2m+u-1)}/(2m+u-1) \right] + \binom{2(m-1)}{c-m} \sum_{l=0}^{c-1} (-1)^l \binom{2(m-1)}{l} \\
\times \left[ 1 - (\bar{H}(x))^{(2c-2m+l+1)}/(2c-2m+l+1) \right].
\]

Therefore from (4.5.8), we have
\[
\psi^{(j)}_{ij}(x) = m(m) \sum_{v=m}^{c-1} (c-1) \beta[m+v,2c-m-v] \\
+ \sum_{s=c-m+1}^{c-1} (c-1) \beta[c-m+s+1,c+m-s-1] + \binom{2(c-m)}{c-m} \sum_{u=0}^{c-1} (-1)^u \binom{2(c-m)}{u} \\
\times \left[ 1 - (\bar{H}(x))^{(2m+u-1)}/(2m+u-1) \right] + \binom{2(m-1)}{c-m} \sum_{l=0}^{c-1} (-1)^l \binom{2(m-1)}{l} \\
\times \left[ 1 - (\bar{H}(x))^{(2c-2m+l+1)}/(2c-2m+l+1) \right] - 1. \ldots(4.5.9)
\]

Therefore
\[
\phi_{ij}(x) = \left[ m(m) \right]^2 E \sum_{v=m}^{c-1} (c-1) \binom{2(c-m)}{c-m} \sum_{u=0}^{c-1} (-1)^u \binom{2(c-m)}{u} \\
\times \left[ 1 - (\bar{H}(x))^{2m+u-1}/(2m+u-1) \right] + \left[ \sum_{s=c-m+1}^{c-1} (c-1) \beta[c-m+s+1,c+m-s-1] \right]^2 \\
\times \left[ 1 - (\bar{H}(x))^{2c-2m+l+1}/(2c-2m+l+1) \right] \\
\times \left[ \sum_{v=m}^{c-1} (c-1) \beta[m+v,2c-m-v] \right]^2
\]

\[
+ \sum_{s=c-m+1}^{c-1} (c-1) \beta[c-m+s+1,c+m-s-1]^2 \\
\times \left[ 1 - (\bar{H}(x))^{2c-2m+l+1}/(2c-2m+l+1) \right] \\
\times \left[ 1 - (\bar{H}(x))^{2m+u-1}/(2m+u-1) \right]
\]
\[
\times [1 - (\overline{H(x)}) (2c-2m+1+1)] - (\overline{H(x)}) (2c-2m+1'+1)
\]
\[+
(\overline{H(x)}) (4c-4m+1'+1+2)] / [(2c-2m+1+1) (2c-2m+1'+1)]
\]
\[+ 2(c-m) \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v]
\]
\[\times \sum_{u=0}^{2(c-m)} (-1)^{u+1} 2(c-m) [1 - (\overline{H(x)}) (2m+u-1)] / (2m+u-1)
\]
\[+ 2(c-m) \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \sum_{u=0}^{2(c-m)} (-1)^{u+1} 2(c-m) [1 - (\overline{H(x)}) (2m+u-1)] / (2m+u-1)
\]
\[+ 2(c-m) \sum_{m-1}^{c-1} \binom{c-1}{c-m} \sum_{u=0}^{2(c-m)} (-1)^{u+1} 2(c-m) [1 - (\overline{H(x)}) (2m+u-1)] / (2m+u-1)
\]
\[\times [1 - (\overline{H(x)}) (2c-2m+1+1)] - (\overline{H(x)}) (2c-2m+1+1)
\]
\[+ (\overline{H(x)}) (2c+u+1)] / [(2m+u-1) (2c-2m+1'+1)]
\]
\[+ 2(c-m) \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v]
\]
\[\times \left[ \sum_{l=0}^{2(m-1)} (-1)^{l+1} \binom{2(m-1)}{l} [1 - (\overline{H(x)}) (2c-2m+1+1)] / (2c-2m+1+1) \right]
\]
\[+ 2 \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v] \sum_{s=c-m+1}^{c-1} \binom{c-1}{s}
\]
\[\times \beta[c-m+s+1, c+m-s-1] \times 2(c-m) \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1, c+m-s-1]
\]
\[\times \left[ \sum_{l=0}^{2(m-1)} (-1)^{l+1} \binom{2(m-1)}{l} [1 - (\overline{H(x)}) (2c-2m+1+1)] / (2c-2m+1+1) \right]
\]
\[\} - 1.
\]
Therefore
\[ f(i,j) = \rho_{m,c}. \]  

Hence (4.5.10) establishes (ii).

Now

\[ f(i) \]

\[ i, j, i, e = E[\psi_{ij}(X)^e_{ie}(X)]. \]

From (4.5.6), it follows that

\[ \psi_{ij}(X) = \psi_{ie}(X) = m(A) \left\{ \begin{array}{l}
\sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v] \\
+ \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1, c+m-s-1] \\
+ \binom{c-1}{m-1} \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} [1-(H(x))(2m+u-1)] / (2m+u-1) \\
+ \binom{c-1}{c-m} \sum_{l=0}^{2(m-1)} (-1)^l \binom{2(m-1)}{l} [1-(H(x))(2c-2m+l+1)] / (2c-2m+l+1) \end{array} \right\} - 1. \]

Therefore

\[ f(i) \]

\[ i, j, i, e = E[\psi_{ij}(X)^e_{ie}(X)]. \]

\[ = \rho_{m,c}. \]  

Furthermore

\[ f(j) \]

\[ i, j, h, j = E[\psi_{ij}(X)^h_{hj}(X)]. \]

From (4.5.9), we have

\[ \psi_{ij}(X) = \psi_{hj}(X) = m(A) \left\{ \begin{array}{l}
\sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v, 2c-m-v] \\
+ \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1, c+m-s-1] \\
+ \binom{c-1}{m-1} \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} [1-(H(x))(2m+u-1)] / (2m+u-1) \\
+ \binom{c-1}{c-m} \sum_{l=0}^{2(m-1)} (-1)^l \binom{2(m-1)}{l} [1-(H(x))(2c-2m+l+1)] / (2c-2m+l+1) \end{array} \right\} - 1. \]
Therefore

\[ \psi_{ij}^{(i)}(X) = E[\psi_{ij}^{(i)}(X)] = \rho_{m,c} \quad \ldots (4.5.12) \]

Equations (4.5.11) and (4.5.12) establish (iii) and (iv) respectively.

Now

\[ \psi_{ij}^{(i)}(X) = E[\psi_{ij}^{(i)}(X)] \]

From (4.5.6), we have

\[
\psi_{ij}^{(i)}(X) = m^c \left\{ \sum_{v=m}^{c-1} \binom{c-1}{v} \beta_{[m+v,2c-m-v]} \right.
\]

\[ + \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta_{[c-m+s+1,c+m-s-1]} \]

\[ + \binom{c-1}{m-1} \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} [1-(H(x))^{(2m+u-1)}](2m+u-1) \]

\[ + \binom{c-1}{m-1} \sum_{l=0}^{2(m-1)} (-1)^l \binom{2(m-1)}{l} [1-(H(x))^{(2c-2m+1+1)}](2c-2m+1+1) \}

\[
\ldots (4.5.13)
\]

and

\[
\psi_{hi}^{(i)}(X) = E[\phi(x_{h1}, \ldots, x_{hc}; x_{i1}, \ldots, x_{ic})] - E[\phi(x_{h1}, \ldots, x_{hc}; x_{i1}, \ldots, x_{ic})]
\]

\[ = P[m \text{ th o.s. of } (x_{h1}, \ldots, x_{hc}) < m \text{ th o.s. of } (x_{i1}, \ldots, x_{ic})] \]

\[ + P[ (c-m+1) \text{ th o.s. of } (x_{h1}, \ldots, x_{hc}) < (c-m+1) \text{ th o.s. of } (x_{i1}, \ldots, x_{ic})] - 1 \]

\[ = P_3 - 1. \quad \ldots (4.5.14) \]
Where

\[ P_3 = P[ m^{th} \text{ o.s. of } (X_{h1}, \ldots, X_{hc}) < m^{th} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) ] \]

\[ + P[ (c-m+1)^{th} \text{ o.s. of } (X_{h1}, \ldots, X_{hc}) < (c-m+1)^{th} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) ] \]

\[ = \int_{-\infty}^{\infty} P[ m^{th} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) > y ] \]

\[ dp[ m^{th} \text{ o.s. of } (X_{h1}, \ldots, X_{hc}) < y ] \]

\[ + \int_{-\infty}^{\infty} P[(c-m+1)^{th} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) > y ] dp[ (c-m+1)^{th} \text{ o.s. of } (x, X_{i2}, \ldots, X_{ic}) < y ] \]

Hence, from (4.5.14)

\[ \psi_{hi}(x) = m(C) \left( \sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v,2c-m-v] \right. \]

\[ + \left. \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1,c+s-1] + \binom{c-1}{m-1} \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} \right. \]

\[ \times \left[ [1-(H(x))(2(m+u-1))/(2m+u-1)] + \binom{c-1}{c-m} \sum_{l=0}^{2(m-1)} (-1)^l \binom{2(m-1)}{l} \right. \]

\[ \times \left. [1-(\bar{H}(x))(2(m-2l+1))/(2m-2l+1)] \right] - 1. \]  \( \ldots (4.5.15) \)

Now

\[ \{i\}^{(i)}_{l,j;h,i} = E[\psi_{i,j}^{(i)}(X) \psi_{hi}^{(i)}(X)]. \]

using (4.5.13) and (4.5.15), it can be seen that

\[ \{i\}^{(i)}_{l,j;h,i} = - \rho_{m,c}. \]  \( \ldots (4.5.16) \)

Consider
\[ \psi_{ij}(X) = \mathbb{E}[\psi_{ij}(X)\psi_{je}(X)], \]

from (4.5.9), we have

\[
\psi_{ij}(X) = \binom{c}{m} \sum_{v=m}^{c-1} \binom{c-1}{v} \beta_{[m+v,2c-m-v]} + \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta_{[c-m+s+1,c+m-s-1]} + \binom{c-1}{m-1} \sum_{u=0}^{2(c-m)} (-1)^u \binom{2(c-m)}{u} \\
\times \left[ 1 - (\bar{H}(x))^{(2m+u-1)}/(2m+u-1) + \binom{c-1}{c-m} \sum_{l=0}^{2(m-1)} (-1)^l \binom{2(m-1)}{l} \right] \\
\times \left[ 1 - (\bar{H}(x))^{(2c-2m+u+1)}/(2c-2m+u+1) \right] - 1 \quad \text{(4.5.17)}
\]

and

\[
\psi_{je}(X) = \mathbb{E}[(X_{j2},\ldots,X_{jc};e_1,\ldots,e_c)] - \mathbb{E}[(X_{j1},\ldots,X_{jc};e_1,\ldots,e_c)],
\]

\[
P_4 = \mathbb{P} \left[ m^{th} \text{ o.s. of } (X_{j2},\ldots,X_{jc}) < m^{th} \text{ o.s. of } (e_1,\ldots,e_c) \right] \\
+ \mathbb{P} \left[ (c-m+1)^{th} \text{ o.s. of } (X_{j2},\ldots,X_{jc}) < (c-m+1)^{th} \text{ o.s. of } (e_1,\ldots,e_c) \right] - 1
\]

\[
= \int_{-\infty}^{\infty} \mathbb{P} \left[ m^{th} \text{ o.s. of } (X_{j2},\ldots,X_{jc}) < y \right] \\
\times dp \left[ m^{th} \text{ o.s. of } (e_1,\ldots,e_c) = y \right]
\]

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Finally, from (4.5.18), we have

\[
\psi_{je}^{(j)}(X) = -m\sum_{v=m}^{c-1} \binom{c-1}{v} \beta[m+v,2c-m-v] + \sum_{s=c-m+1}^{c-1} \binom{c-1}{s} \beta[c-m+s+1,c+s-1] + \sum_{u=0}^{2(c-m)} (-1)^u \binom{c-m}{u} x \frac{1-[H(x)](2m+u)}{2m+u} + \sum_{v=m}^{c-m} x \frac{1-[H(x)](2m-2v+2)}{2m-2v+2} \] ...

(4.5.19)

Using (4.5.17) and (4.5.19), it can be seen that

\[
\mathbb{E}[\psi_{ij}^{(j)}(X)|\psi_{je}^{(j)}(X)] = -\rho_{m,c} \] ...

(4.5.20)

Equations (4.5.16) and (4.5.20) establish (v) and (vi) respectively.

Hence the theorem 4.5.1 is proved.

**Theorem 4.5.2:** The Pitman ARE of procedures \(R'_m,c\) relative to \(R_a\) is given by

\[
\text{ARE}(R'_m,c;R_a) = \frac{\int_{0}^{1}(J u) [J u]^{-j} du}{c^2 \rho_{m,c}} \left[ \sum_{m}^{c} \binom{c}{m} \right]^2 \]

\[
\times \left[ \frac{-\int_{-\infty}^{\infty}(F(x)) [2m-2] (1-F(x)) f^2(x) dx}{0} \right] \frac{1}{J(u) J(u,f) du} \]

\[
+ (c-m+1) \binom{c}{c+1} \frac{-\int_{-\infty}^{\infty}(F(x)) [2m] (1-F(x)) ^2 f^2(x) dx}{0} \right] \frac{1}{J(u) J(u,f) du} \]

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Proof: From the arguments used in theorem 4.4.1,

$$\inf_{\mu \in \Omega} P_{\Omega} [CS|R'_{m,c}] = P_0 [CS|R'_{m,c}].$$

Also from theorem 4.5.1, we see that under the configuration

$$\mu_1 = \ldots = \mu_k,$$

the random vector $$(N/c^2 \rho_{m,c}^{1/2})$$ is asymptotically normal with mean vector 0 and variance-covariance matrix as

$$(N/c^2 \rho_{m,c}^{1/2}) E(T_{m,c}^{1/2}) = \left[ \begin{array}{ccc}
1/p_i + 1/p_j & \text{if } i=h, j=e \\
1/p_i & \text{if } i=h, j=e \\
1/p_j & \text{if } i\neq h, j=e \\
-1/p_i & \text{if } i=e, j=h \\
-1/p_j & \text{if } i\neq e, j=h \\
0 & \text{otherwise.}
\end{array} \right]$$

Thus

$$\liminf_{\mu \in \Omega} P_{\Omega} [CS|R'_{m,c}] = \lim P_0 [CS|R'_{m,c}]$$

$$= \lim P_0 \left[ \gamma_{m,c}^{1/2} = b \left[ c^2 \rho_{m,c} (n_j^{-1} + n_{j+1}^{-1}) \right]^{1/2} \right. \text{ for all } i, i \neq j].$$

$$= \lim P_0 \left[ \gamma_{m,c}^{1/2} = b \left[ c^2 \rho_{m,c} (n_j^{-1} + n_{j+1}^{-1}) \right]^{1/2} \right.$$

Also from lemma 2.5.3 it follows that under $\mu_1 = \ldots = \mu_k$ and $\min(n_1', \ldots, n_k') \rightarrow \infty$, in such a way that $(n_i'/n') \rightarrow p_i, 0 < p_i < 1$, for $i=1, \ldots, k$, the random vector

$$(N'/\sigma_{N'}^2)^{1/2} S = (N'/\sigma_{N'}^2)^{1/2} (S_1^{(2)}, \ldots, S_1^{(k)}, \ldots, S_k^{(1)}, \ldots, S_k^{(k-1)})^t$$

is asymptotically normal with mean vector 0 and...
variance-covariance matrix as given in (4.5.21). Here
\[ N' = \sum_{i=1}^{k} n'_i, \text{and} \quad \sigma^2_{N'} = \frac{1}{(N' - 1)} \sum_{\beta=1}^{N'} (a_{\beta N'}(\beta) - \bar{a}_{N'}(\beta))^2 \to \int_0^1 (J(u) - \bar{J})^2 \, du, \]
where \( J(u) \) is the score function and \( \bar{J} = \int_0^1 J(u) \, du \). Also from (4.5.1), we have
\[
\lim \inf_{\mu \in D} P_{\mu} [CS|R_a] \leq \lim_{a \to 0} P_{\mu} [CS|a] \leq \lim_{a \to 0} P_{\mu} [S_{ij}^{(i)} \geq C_{ij}^{(i)}(n', \mu)] \text{ for all } i, i \neq j. \quad \ldots (4.5.23)
\]
Using lemma 2.5.4 it follows that for the sequence of local alternatives \( \mu_N = (\mu_1, \ldots, \mu_k)^T \) for \( \mu \) satisfying the conditions
\[
\text{that as } \min n'_i \to \infty, \quad \max (\mu_i - \bar{\mu}_N)^2 \to \infty, \quad I(f) \sum_{i=1}^{N'} (\mu_i - \bar{\mu}_N)^2 \to \int_0^1 (J(u) - \bar{J})^2 \, du,
\]
\( \rightarrow d^0, \quad 0 < d^0 < \infty, \) (here \( \bar{\mu}_N = 1/N \sum n'_i \mu_i \)), the random vector \( (N'/2/\sigma_{N'})_i \) is asymptotically normally distributed with means
\[
(N'/2/\sigma_{N'})_i = (N'/2/\sigma_{N'})\Delta_{ij} \int_0^1 (J(u)J(u,f)) \, du
\]
and the covariances same as given in (4.5.21)
\[
(\Delta_{ij}^2(N) = \mu_{N'} - \mu_i). \]
Using lemma 2.5.5 the random vector \( (N/c^2\rho_{m,c})^{1/2} \) is asymptotically normal with the same covariance as in (4.5.21) and using result of lemma 4.5.1 with means as \( (N/c^2\rho_{m,c})^{1/2} \text{E}[T_{ij}^{m,c}] \) and
\[
\begin{align*}
&= (N/c^2\rho_{m,c})^{1/2} \left[ m_0^{m_0} \left\{ \sum_{t=m}^{C} \int_{-\infty}^{\infty} (H(x))^t (1-H(x))^{C-t} \right. \\
&\quad \times \left. (H(x-\Delta_{ij}(N)))^{(m-1)} (1-H(x-\Delta_{ij}(N)))^{(c-m)} dH(x-\Delta_{ij}(N)) \right. \\
&\quad + \sum_{t=m+1}^{C} \int_{-\infty}^{\infty} (H(x))^t (1-H(x))^{C-t} (H(x-\Delta_{ij}(N)))^{(c-m)} dH(x-\Delta_{ij}(N)) \right] \\
&= 164
\end{align*}
\]
\[
\begin{align*}
\times (1 - H(x - \Delta_{ij}(N)))^{(m-1)}dH(x - \Delta_{ij}(N)) \rightleftharpoons 1 \\
= \left(N/c^2p_{m,c}\right)^{1/2}\left(\sum_{t=m}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right.
\left. + \sum_{t=c-m+1}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right) - 1 \\
- \left(N/c^2p_{m,c}\right)^{1/2}\left(\sum_{t=m}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right.
\left. + \sum_{t=c-m+1}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right) - 1 \\
\text{(using Taylor Series expansion of } H(x + \Delta_{ij}(N)) \text{ and ignoring higher order terms)} \\
- \left(N/c^2p_{m,c}\right)^{1/2}\left(\sum_{t=m}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right.
\left. + \sum_{t=c-m+1}^{C} \int_{-\infty}^{\infty} (H(x) + \Delta_{ij}(N))^t (1 - H(x) - \Delta_{ij}(N))^{(c-t)} \times (H(x))^{(m-1)}(1 - H(x))^{(c-m)}dH(x) \right) - 1 \\
\text{(using binomial expansion)}.
\end{align*}
\]
Finally we obtain

\[
\begin{aligned}
\left(\frac{N^2}{c^2\rho_{m,c}}\right)^{1/2} &\ E[T_{ij}^m, c] = \left(\frac{N^2}{c^2\rho_{m,c}}\right)^{1/2}\ \Delta_{ij}(N)\ m^C_m \left\{ m^C_m \right\} \\
&\cdot \int_{-\infty}^{\infty} (H(x))^{(2m-2)} (1-H(x))^2 (c-m) h^2(x) dx + \left( c-m+1 \right) (c-m+1)
\end{aligned}
\]

\[
\times \int_{-\infty}^{\infty} (H(x))^{2(c-m)} (1-H(x))^2 (m-1) f^2(x) dx \right\}.
\]

Thus it follows from (4.5.21), (4.5.22) and (4.5.23) that

\[
\lim \inf \frac{P[CS|R']}{P[R']} = \lim \inf \frac{P[CS|R]}{P[R]}.
\]

The expected subset sizes are

\[
E_{\mu}[S|R', m, c] = E_{\mu} \left[ \sum_{a \in A} Z_{R', m, c}(X, a) \right],
\]

and

\[
E_{\mu}[S|R] = E_{\mu} \left[ \sum_{a \in A} Z_{R, m, c}(X, a) \right].
\]

These two expressions are approximately equal if at least in the limit, \( Z_{R', m, c}(X, a) \approx Z_{R, m, c}(X, a) \). This is possible when the asymptotic distributions of \( (N^2/c^2\rho_{m,c})^{1/2}T \) and \( (N'/c^2\rho_{m,c})^{1/2}S \) are equal. From this it follows that

\[
\left(\frac{N'/c^2\rho_{m,c}}{N} \right)^{1/2} E[S_{ij}^{(i)}] = \left(\frac{N^2}{c^2\rho_{m,c}}\right)^{1/2} E[T_{ij}^m, c]
\]

i.e.

\[
\left(\frac{N'/c^2\rho_{m,c}}{N} \right)^{1/2} \int_{-\infty}^{\infty} J(u) J(u, f) du = \left(\frac{N^2}{c^2\rho_{m,c}}\right)^{1/2} \ m^C_m \left\{ m^C_m \right\} \\
\times \int_{-\infty}^{\infty} (F(x))^{(2m-2)} (1-F(x))^2 (c-m) f^2(x) dx + \left( c-m+1 \right) (c-m+1)
\]

\[
\times \int_{-\infty}^{\infty} (F(x))^{2(c-m)} (1-F(x))^2 (m-1) f^2(x) dx \right\}.
\]
or \( \frac{N'}{N} = \frac{\sigma^2_{N'}}{c^2 \rho_{m,c}} \left[ \frac{m(C)}{m(m)} \right]^2 \)

\[
= \left[ m(C) \right] \frac{-\int_{-\infty}^{\infty} (F(x))^2 (m-1) (1-F(x))^2 \frac{f^2(x)}{m} \, dx}{\int_{0}^{1} J(u)J(u,f) \, du} \]

\[
+ (c-m+1) \left( \frac{c}{c-m+1} \right) \frac{-\int_{-\infty}^{\infty} (F(x))^2 (m-1) (1-F(x))^2 \frac{f^2(x)}{m} \, dx}{\int_{0}^{1} J(u)J(u,f) \, du} \]

\[
\times \left[ \frac{-\int_{-\infty}^{\infty} (F(x))^2 (m-1) (1-F(x))^2 \frac{f^2(x)}{m} \, dx}{\int_{0}^{1} J(u)J(u,f) \, du} \right]^2 \]

Hence the Pitman ARE of \( R\) relative to \( R_a \) is

\[
\text{ARE}(R_{m,c}, R_a) = \frac{\int_{0}^{1} (J(u)-J)^2 \, du \left[ \frac{m(C)}{m(m)} \right]^2}{c^2 \rho_{m,c}} \]

\[
- \int_{-\infty}^{\infty} (F(x))^2 (m-1) (1-F(x))^2 \frac{f^2(x)}{m} \, dx \]

\[
+ (c-m+1) \left( \frac{c}{c-m+1} \right) \frac{-\int_{-\infty}^{\infty} (F(x))^2 (m-1) (1-F(x))^2 \frac{f^2(x)}{m} \, dx}{\int_{0}^{1} J(u)J(u,f) \, du} \]

This completes the proof of the theorem.

Let the selection procedure \( R_a \) be based on Wilcoxon scores for which \( J(u) = u \) \((0<u<1)\). The ARE in this case is given by the expression

\[ 167 \]
\[ \text{ARE}(R_{m,c}, R_{d}) = \frac{1}{12c^2 \rho_{m,c}} \left[ m_n^C \right]^2 \]

\[ x \left[ m_n^C \int_{-\infty}^{\infty} (F(x))^{2(m-1)} (1-F(x))^{2(c-m)} f^2(x) dx - \int_{-\infty}^{\infty} f^2(x) dx \right] \]

\[ + (c-m+1) \left[ c \int_{-\infty}^{\infty} (F(x))^{2(c-m)} (1-F(x))^{2(m-1)} f^2(x) dx - \int_{-\infty}^{\infty} f^2(x) dx \right] \]

\[ \implies \text{... (4.5.24)} \]

Now we shall find out the ARBs (4.5.24) for the following distributions:

(i) Uniform distribution:

\[ f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \]

Here \( \int_{-\infty}^{\infty} f^2(x) dx = 1 \)

and

\[ \int_{-\infty}^{\infty} (F(x))^{2m-2} (1-F(x))^{2c-2m} f^2(x) dx \]

\[ = \int_{0}^{1} x^{2m-2} (1-x)^{2c-2m} dx \]

\[ = \beta(2m-1, 2c-2m+1). \]

Similarly

\[ \int_{-\infty}^{\infty} (F(x))^{2c-2m} (1-F(x))^{2m-2} f^2(x) dx \]

\[ = \beta(2c-2m+1, 2m-1). \]
Therefore ARE in expression (4.5.24), when the underlying distribution is uniform, reduces to

\[ \text{ARE}(R', R, \text{Uniform}) = \frac{1}{12c^2 \varrho_{m,c}} \left[ \frac{m(c)}{m(m)} \right]^2 \beta(2m-1, 2c-2m+1) + (c-m+1) \left( \frac{c}{c-m+1} \right)^2 \beta(2c-2m+1, 2m-1)^2. \]

The ARE values for uniform distribution for different values of \( m \) and \( c \) are given in table 4.5.1.

(ii) Double Exponential:

\[ f(x) = \begin{cases} (1/2) e^{-|x|}, & -\infty < x < \infty \end{cases} \]

\[ F(x) = \begin{cases} (1/2) e^x, & x < 0 \\ 1 - (1/2) e^{-x}, & x \geq 0. \end{cases} \]

Here

\[ \int_{-\infty}^{\infty} f^2(x) \, dx = 1/4 \int_{-\infty}^{\infty} e^{-2|x|} \, dx = 1/4 \]

and

\[ \int_{-\infty}^{\infty} (F(x))^{2m-2} (1-F(x))^{2c-2m} f^2(x) \, dx = \int_{-\infty}^{0} ((1/2) e^x)^{2m-2} (1-(1/2) e^x)^{2c-2m} ((1/2)e^{-|x|})^2 \, dx + \int_{0}^{\infty} (1-(1/2) e^{-x})^{2m-2} ((1-(1/2) e^{-x})^{2c-2m} ((1/2)e^{-|x|})^2 \, dx \]
\[
\int_0^\infty \left( \frac{1}{e} \right)^{2m-2} (1 - \left( \frac{1}{e} \right))^{2c-2m} \left( \frac{1}{e} \right)^{2c-2m} \left( \frac{1}{e} \right)^{2c-2m} \int_0^\infty \left( \frac{1}{e} \right)^{2c-2m} (1 - \left( \frac{1}{e} \right))^{2m-2} \left( \frac{1}{e} \right)^{2m-2} \left( \frac{1}{e} \right) dx
\]
\[
= \int_{1/2}^{2c-2m-1} t^{2m-1} 2c-2m dt + \int_{0}^{2c-2m+1} t^{2c-2m+1} (1-t)^{2m-2} dt
\]
(by putting \( \frac{1}{e} = t \))
\[
= \sum_{j=0}^{2c-2m} \left( -1 \right)^j \left( \frac{2c-2m}{j} \right) \left( \frac{1}{2} \right)^{2m+j} (1/2m+j)
\]
\[
+ \sum_{v=0}^{2m-2} \left( -1 \right)^v \left( \frac{2m-2}{v} \right) \left( \frac{1}{2} \right)^{2m+2+v} (1/2c-2m+2+v)
\]
Similarly
\[
\int_{-\infty}^{\infty} \left( P(x) \right)^{2c-2m} (1 - P(x))^{2m-2} \left( f^2(x) \right) dx
\]
\[
= \sum_{j=0}^{2m-2} \left( -1 \right)^j \left( \frac{2m-2}{j} \right) \left( \frac{1}{2} \right)^{2c+2m+2+j} (1/2c-2m+2+j)
\]
\[
+ \sum_{v=0}^{2c-2m} \left( -1 \right)^v \left( \frac{2c-2m}{v} \right) \left( \frac{1}{2} \right)^{2m+v} (1/2m+v)
\]
Therefore ARE in expression (4.5.24), when the underlying distribution is double exponential, reduces to
\[
\text{ARE}_{\text{Double Exponential}} = \frac{1}{12c^2 \rho_{m,c}} \left[ \frac{m_C^2}{m_C} \right] \left\{ \sum_{j=0}^{2c-2m} \left( -1 \right)^j \left( \frac{2c-2m}{j} \right) \left( \frac{1}{2} \right)^{2m+j} (1/2m+j) + \sum_{v=0}^{2m-2} \left( -1 \right)^v \left( \frac{2m-2}{v} \right) \left( \frac{1}{2} \right)^{2c-2m+2+v} (1/2c-2m+2+v) \right\}
\]
The ARE values for double exponential distribution for different values of m and c are given in table 4.5.1.

(iii) Normal:
\[ f(x) = \phi(x) = (2\pi)^{-1/2} e^{-x^2/2}, \quad -\infty < x < \infty \]
\[ F(x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-z^2/2} dz, \quad -\infty < x < \infty. \]

Here
\[ \int_{-\infty}^{\infty} f^2(x) dx \]
\[ = \int_{-\infty}^{\infty} \phi^2(x) dx \]
\[ = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-x^2} dx \]
\[ = 1/2 (\pi)^{-1/2} \]

and
\[ \int_{-\infty}^{\infty} (F(x))^{2m-2} (1-F(x))^{2c-2m} f^2(x) dx \]
\[ = \int_{-\infty}^{\infty} (\Phi(x))^{2m-2} (1-\Phi(x))^{2c-2m} \phi^2(x) dx \]
\[ = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (\Phi(x))^{2m-2} (1-\Phi(x))^{2c-2m} \phi(x 2^{1/2}) dx \]

(because \( \phi^2(x) = (2\pi)^{-1/2} \phi(x 2^{1/2}) \))
These integrals have been tabulated by Gupta (1963b) (reading in those tables $N=2m-2+j$, $H=0$ and $p/(1-p)=1/2 \Rightarrow p=1/3$).

Similarly

$$
\int_{-\infty}^{\infty} (P(x))^{2c-2m}(1-F(x))^{2m-2} f^2(x) \, dx
$$

The integrals have been tabulated by Gupta (1963b) (reading in those tables $N=2m-2+j$, $N=2c-2m+v$, $H=0$ and $p/(1-p)=1/2 \Rightarrow p=1/3$). Therefore ARE in expression (4.5.24), when the underlying distribution is normal, reduces to

$$
ARE(R', R |\text{Normal}) = \frac{1}{12c^2\rho^{m,c}} \left[ \frac{m^c}{m!} \right]^2
$$

$$
\times \left[ \frac{m^c}{m!} \sum_{j=0}^{2c-2m} (-1)^j \frac{(2c-2m)}{j} \int_{-\infty}^{\infty} \phi(x/2^{1/2}) (2m-2+j) \phi(x) \, dx \right]
$$

$$
+ (c-m+1) \left( \frac{c}{c-m+1} \right) \sum_{v=0}^{2m-2} (-1)^v \frac{(2m-2)}{v} \int_{-\infty}^{\infty} (\Phi(x/2^{1/2})) (2c-2m+v) \phi(x) \, dx
$$

The ARE values for normal distribution for different values of $m$ and $c$ are given in table 4.5.1.
(iv) Logistic:

\[ f(x) = \frac{e^x}{(1+e^x)^2}, \quad -\infty < x < \infty \]

\[ F(x) = \frac{e^x}{1+e^x}, \quad -\infty < x < \infty. \]

Here

\[ \int_{-\infty}^{\infty} f^2(x) \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{2e^{2x}}{(1+e^x)^4} \, dx \]

\[ = \int_{1}^{\infty} \frac{(1-t)}{t^4} \, dt \quad \text{(put } 1+e^x = t) \]

\[ = 1/6 \]

and

\[ \int_{-\infty}^{\infty} F(x) \cdot 2^{m-2}(1-F(x)) \cdot 2^{c-2m} \cdot f^2(x) \, dx \]

\[ = \int_{-\infty}^{\infty} \frac{(e^{x})^{2m-2} (1-e^{x})^{2c-2m} e^{2x}}{(1+e^x)^4} \, dx \]

\[ = \int_{0}^{\frac{\pi}{4}} \frac{t^{2m-1}}{(1+t)^{2c+2}} \, dt \quad \text{(put } t = \tan^2 \theta) \]

\[ = \frac{\pi/2}{(2m-1)! \cdot (2c-2m+1)!/(2c+1)!}. \]

Similarly

\[ \int_{-\infty}^{\infty} (F(x)) \cdot 2^{c-2m}(1-F(x)) \cdot 2^{c-2m} \cdot f^2(x) \, dx \]

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Therefore ARE in expression (4.5.24), when the underlying distribution is logistic, reduces to

\[
\text{ARE}(R'_m, R_a \mid \text{Logistic}) = \frac{1}{12c^2 \rho_{m,c}} \left[ m^C_m \right]^2 \left[ 6 \times m^C_m \times (2m-1)! \times (2c-2m+1)/(2c+1)! \right.
\]
\[
+ 6 \times (c-m+1) m^C_{c-m+1} \times (2m-1)! \times (2c-2m+1)/(2c+1)! \right]^{2}.
\]

The ARE values for logistic distribution for different values of \(m\) and \(c\) are given in table 4.5.1.

(v) Exponential:

\[
f(x) = e^{-x}, \quad x > 0
\]
\[
F(x) = 1 - e^{-x}, \quad x > 0.
\]

Here

\[
\int_{-\infty}^{\infty} f^2(x) \, dx = \int_{0}^{\infty} e^{-2x} \, dx = \frac{1}{2}
\]

and

\[
\int_{-\infty}^{\infty} (F(x))^2 (1-F(x))^2 f^2(x) \, dx = \int_{0}^{\infty} (1 - e^{-x})^2 (1-(1 - e^{-x}))^2 e^{-2x} \, dx
\]

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\[ \int_0^1 t^{2m-1}(1-t)^{2c-2m} \, dt = \beta(2m, 2c-2m+1). \]

Similarly

\[ \int_{-\infty}^{\infty} (F(x))^{2c-2m}(1-F(x))^{2m-2}f^2(x) \, dx = \beta(2c-2m+1, 2m). \]

Therefore ARE in expression (4.5.24), when the underlying distribution is exponential, reduces to

\[
\text{ARE}(R', R_\text{a} | \text{Exponential}) = \frac{1}{12c^2 \rho_{m,c}} \left[ \binom{C}{m} \right]^2 \left[ 2 \times \binom{C}{m} \beta(2m, 2c-2m+1) \right. \\
\left. + 2 \times (c-m+1) \left( \binom{C}{c-m+1} \beta(2c-2m+1, 2m) \right) \right]^2.
\]

The ARE values for exponential distribution for different values of \( m \) and \( c \) are given in table 4.5.1.

(vi) Cauchy:

\[ f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty \]

\[ F(x) = \frac{1}{2} + \frac{\tan^{-1} x}{\pi}, \quad -\infty < x < \infty. \]

Here

\[ \int_{-\infty}^{\infty} f^2(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{\pi^2 (1+x^2)^2} \, dx \]
\[ = 2 \pi^{-2} \int_{0}^{\pi/2} \cos^2 \theta \, d\theta \]

and

\[
\int_{-\infty}^{\infty} (F(x))^{2m-2} (1-F(x))^{2c-2m} f^2(x) \, dx \\
= (\pi)^{-2} \sum_{j=0}^{\infty} (-1)^j (2c-2m) f^{\Pi/2} (-\frac{1}{2} + \frac{\theta}{\pi}) \cos^2 \theta \, d\theta. \]

Similarly

\[
\int_{-\infty}^{\infty} (F(x))^{2c-2m} (1-F(x))^{2m-2} f(x) \, dx \\
= (\pi)^{-2} \sum_{v=0}^{\infty} (-1)^v (2m-2) f^{\Pi/2} (-\frac{1}{2} + \frac{\theta}{\pi}) \cos^2 \theta \, d\theta. \]

Therefore ARE in expression (3.5.24), when the underlying distribution is Cauchy, reduces to

\[
\text{ARE} (R'_m, c', R_a | \text{Cauchy}) \\
= \frac{1}{12c^2 \rho_{m,c}} \left[ m_{m}\left(\frac{c}{m}\right) \right]^2 \times \left[ \sum_{j=0}^{2c-2m} (-1)^j (2c-2m) f^{\Pi/2} (-\frac{1}{2} + \frac{\theta}{\pi}) \cos^2 \theta \, d\theta \\
+ (c-m+1) \left( m_{m}\left(\frac{c}{m+1}\right) \right)^2 \sum_{v=0}^{2m-2} (-1)^v (2m-2) f^{\Pi/2} (-\frac{1}{2} + \frac{\theta}{\pi}) \cos^2 \theta \, d\theta \right] . \]
The ARE values for Cauchy distribution for different values of $m$ and $c$ are given in table 4.5.1.

**Table 4.5.1**

ARE ($R'_{m,c}$, $R_a$) with $R_a$ based on Wilcoxon scores

<table>
<thead>
<tr>
<th>(m,c)</th>
<th>Uniform</th>
<th>Normal</th>
<th>Logistic</th>
<th>Double Exp.</th>
<th>Cauchy</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>(1,2)</td>
<td>1.228</td>
<td>1.022</td>
<td>0.995</td>
<td>0.941</td>
<td>0.884</td>
<td>1.228</td>
</tr>
<tr>
<td>(2,2)</td>
<td>1.228</td>
<td>1.022</td>
<td>0.995</td>
<td>0.941</td>
<td>0.884</td>
<td>1.228</td>
</tr>
<tr>
<td>(1,3)</td>
<td>1.841</td>
<td>1.032</td>
<td>0.939</td>
<td>0.768</td>
<td>0.599</td>
<td>1.841</td>
</tr>
<tr>
<td>(2,3)</td>
<td>0.588</td>
<td>0.914</td>
<td>0.972</td>
<td>1.111</td>
<td>1.255</td>
<td>0.588</td>
</tr>
<tr>
<td>(3,3)</td>
<td>1.841</td>
<td>1.032</td>
<td>0.939</td>
<td>0.768</td>
<td>0.599</td>
<td>1.841</td>
</tr>
<tr>
<td>(1,4)</td>
<td>2.501</td>
<td>0.997</td>
<td>0.851</td>
<td>1.092</td>
<td>1.215</td>
<td>0.630</td>
</tr>
<tr>
<td>(2,4)</td>
<td>0.630</td>
<td>0.936</td>
<td>0.984</td>
<td>1.092</td>
<td>1.215</td>
<td>0.630</td>
</tr>
<tr>
<td>(3,4)</td>
<td>0.630</td>
<td>0.936</td>
<td>0.984</td>
<td>1.092</td>
<td>1.215</td>
<td>0.630</td>
</tr>
<tr>
<td>(4,4)</td>
<td>2.501</td>
<td>0.997</td>
<td>0.851</td>
<td>1.092</td>
<td>1.215</td>
<td>0.630</td>
</tr>
<tr>
<td>(1,5)</td>
<td>3.167</td>
<td>0.950</td>
<td>0.763</td>
<td>0.505</td>
<td>0.254</td>
<td>3.167</td>
</tr>
<tr>
<td>(2,5)</td>
<td>0.745</td>
<td>0.983</td>
<td>0.977</td>
<td>0.979</td>
<td>1.026</td>
<td>0.745</td>
</tr>
<tr>
<td>(3,5)</td>
<td>0.511</td>
<td>0.905</td>
<td>0.950</td>
<td>1.160</td>
<td>1.336</td>
<td>0.511</td>
</tr>
<tr>
<td>(4,5)</td>
<td>0.745</td>
<td>0.983</td>
<td>0.977</td>
<td>0.979</td>
<td>1.026</td>
<td>0.745</td>
</tr>
<tr>
<td>(5,5)</td>
<td>3.167</td>
<td>0.950</td>
<td>0.763</td>
<td>0.505</td>
<td>0.254</td>
<td>3.167</td>
</tr>
</tbody>
</table>

From the above table of ARE’s we observe the following:

(i) For $m=1$ and $c=1$, the ARE is unity for all the distributions, since the selection procedures $R_a$ specialized to Wilcoxon scores is a member of the class of selection procedures $R'_{m,c}$, when $m=1$ and $c=1$.

(ii) When the underlying distributions are uniform and exponential, AREs are same for all values of $c$ and $m$. This is because the mathematical expressions of AREs for these two
distributions coincide.

(iii) For extreme values of $m$ and as $c$ increases AREs for light tail distributions increases and reverse is the case for moderate and heavy tail distributions. However for normal distribution no specific behavior emerges.

(iv) At the middle value of $m$, when $c$ increases and is restricted to odd value only, the AREs decreases for light tail distributions and increases for moderate and heavy tail distributions. Moreover, when $c$ is odd and $m$ takes the middle value, AREs of procedures $R'_{m,c}$ and $R_{m,c}$ coincide. This is because under such situations the U-statistics used for both the procedures become same.

(v) From the tables of AREs given in chapter III and the above table of AREs we see that selection procedures proposed here always perform better than the selection procedures proposed in chapter III irrespective of type of underlying distribution.

**Remark 4.5.1:** The efficiency of the procedures $R'_{m,c}$ with respect to other procedures can be computed by the product rule of AREs (that is if $R_1$, $R_2$ and $R_3$ are three procedures for the same problem, then $\text{ARE}(R_1,R_3) = \text{ARE}(R_1,R_2) \times \text{ARE}(R_2,R_3)$).

4.6 Approximate Use of Selection Procedures $R'_{m,c}$

Let $Y^{m,c}_{ij} = \left[c^2 \rho_{m,c} (n_1^{-1} + n_j^{-1})\right]^{1/2} T^{m,c}_{ij} i,j, i,j=1,\ldots,k$. It follows from theorem 4.5.1 that the asymptotic distribution of the random vector $Y_j = (Y^{m,c}_{1j}, \ldots, Y^{m,c}_{j-1,j}, Y^{m,c}_{j+1,j}, \ldots, Y^{m,c}_{kj})^t$, under
\( \mu_1 = \ldots = \mu_k \) is multivariate standard normal with correlation matrix given by

\[ \text{corr}(Y_{ij}, Y_{hj}) = \rho_{ij}^{(j)} = \frac{1}{n_j} \left[ (n_i^{-1} + n_j^{-1}) (n_h^{-1} + n_j^{-1}) \right]^{-1/2} \]

Thus

\[ P_0 \left[ \sum_{i,j} \frac{X_i X_j}{(1/n_i + 1/n_j)} \right] = \frac{1}{n_j} \left[ (1/n_i + 1/n_j) \right]^{-1/2} \]

for all \( i, i \neq j \).

Now

\[ \min_i \left[ \frac{n_i^{-1} + n_j^{-1}}{n_j^{-1} + n_i^{-1}} \right]^{-1/2} = 1 \]

and

\[ \min_i \rho_{ij}^{(j)} = \left[ \frac{1}{n_i^{-1} + n_j^{-1}} \right]^{-1/2} = \rho \text{ (say)} \]

Using Slepian inequality (see Slepian (1962)), in the limiting case we get

\[ P_0 \left[ \sum_{i,j} \frac{X_i X_j}{(1/n_i + 1/n_j)} \right] = \frac{1}{n_j} \left[ (1/n_i + 1/n_j) \right]^{-1/2} \]

for all \( i, i \neq j \).

where \( B_{ij} = (B_{m,c}^{m,c}, \ldots, B_{m,c}^{m,c}) \) is multivariate normal of equally correlated N(0,1) variables (the value of this correlation = \( \rho \)).

We use the tables of Gupta et al (1973) to compute \( b \), so that

\[ P_0 \left[ \sum_{i,j} \frac{X_i X_j}{(1/n_i + 1/n_j)} \right] \geq \rho^* \]

Let us consider example 2.6.1 of chapter II to illustrate the proposed selection procedures. The following table gives the details of populations selected in the subset, for various choices of \( m \) and \( c \).
### Table 4.6.1

Populations selected using $R'_{m,c}$ to data on four tour guides

<table>
<thead>
<tr>
<th>(m,c)</th>
<th>$\min_{i\neq j} \frac{T_i^c}{T_j^c}$</th>
<th>$b(c^2\rho_{m,c}X^2/n)^{1/2}$</th>
<th>Populations in the selected subset</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$j=1$</td>
<td>$j=2$</td>
<td>$j=3$</td>
</tr>
<tr>
<td>(1,1)</td>
<td>-0.5555</td>
<td>-0.1666</td>
<td>0.5555</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-0.4666</td>
<td>0.0833</td>
<td>0.9111</td>
</tr>
<tr>
<td>(2,2)</td>
<td>-0.6444</td>
<td>-0.4222</td>
<td>0.1333</td>
</tr>
<tr>
<td>(1,3)</td>
<td>-0.2600</td>
<td>0.2100</td>
<td>0.9750</td>
</tr>
<tr>
<td>(2,3)</td>
<td>-0.9200</td>
<td>-0.4800</td>
<td>0.9200</td>
</tr>
<tr>
<td>(3,3)</td>
<td>-0.5000</td>
<td>-0.3500</td>
<td>-0.150</td>
</tr>
<tr>
<td>(1,4)</td>
<td>-0.8444</td>
<td>0.2177</td>
<td>1.0000</td>
</tr>
<tr>
<td>(2,4)</td>
<td>-0.8400</td>
<td>-0.3600</td>
<td>1.0000</td>
</tr>
<tr>
<td>(3,4)</td>
<td>-1.0000</td>
<td>-0.6000</td>
<td>0.6000</td>
</tr>
<tr>
<td>(4,4)</td>
<td>-0.3333</td>
<td>-0.2444</td>
<td>-0.222</td>
</tr>
<tr>
<td>(1,5)</td>
<td>0.0000</td>
<td>0.1388</td>
<td>1.0000</td>
</tr>
<tr>
<td>(2,5)</td>
<td>-0.5555</td>
<td>-0.1111</td>
<td>1.0000</td>
</tr>
<tr>
<td>(3,5)</td>
<td>-1.0000</td>
<td>-1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(4,5)</td>
<td>-1.0000</td>
<td>-0.3333</td>
<td>0.0000</td>
</tr>
<tr>
<td>(5,5)</td>
<td>-0.1666</td>
<td>-0.1388</td>
<td>-0.138</td>
</tr>
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#### 4.7 Modification of Procedures $R'_{m,c}$

The procedures $R'_{m,c}$ can be modified to select a subset of $k$
populations better than the unknown control population. Let
\( F_0(x) = F(x - \mu) \) be the cumulative distribution function (cdf) of
the control population \( \Pi_0 \) and \( F_i(x) = F(x - \mu_i) \) be the cdf of the
population \( \Pi_i, i=1,\ldots,k \). Population \( \Pi_j \) is considered to be
better than \( \Pi_0 \) if \( \mu_j \geq \mu_0 \). Let \( n_j \) be the number of observations
taken from the population \( \Pi_j, j=0,1,\ldots,k \) and let
\( n^* = (n_{0}^*, n_{1}^*, \ldots, n_{k}^*)^t \). Again denote \( n_{[1]} \leq \ldots \leq n_{[k]} \) as the ordered
values of \( n_{1}, \ldots, n_{k} \). The proposed selection procedures, based on
the statistic \( T_{0j}^{m,c}, T_{0j}^{m,c} = t_{0j}^{m,c} - 1 \), is
\[
R'_{m,c} : \text{Include } \Pi_j \text{ in the subset if and only if } \quad \frac{T_{0j}^{m,c}}{b' [c^2 \rho_{m,c} (n_{0}^{-1} + n_{[1]}^{-1})]}^{1/2},
\]
where \( R'_{m,c} \) denotes the selection procedures in case of the
control population and the constant \( b' \) is chosen such that
\[
P_0 \left[ T_{0j}^{m,c} \geq b' [c^2 \rho_{m,c} (n_{0}^{-1} + n_{[1]}^{-1})]^{1/2} \right] = \mathbb{P} \quad (2^{-k} \geq \mathbb{P} < 1) \text{ is the pre-assigned probability.}
\]
The constant \( b' \) can be determined by adopting the method used in
section 4.5.

Remark 4.7.1: The proposed procedures have one major advantage in
comparison to Kumar et al (1992) that these can also be used even
with unequal sample sizes from different populations.

Remark 4.7.2: The problem of selecting a subset containing the
population with the smallest location parameter is a simple
analog of the proposed procedures.
4.8 Simulation Study

In section 4.6, the implementation of the proposed procedures $R_{m,c}'$ with the help of existing tables, for the case of equal sample sizes have been demonstrated using asymptotic normality of U-statistics. In practice an experimenter should know the common sample sizes sufficient for the use of asymptotic results. Therefore, simulation technique has been used to have an idea about the common sample sizes required as explained below.

Two sets for each of the parametric families namely normal, exponential and Cauchy have been considered. Three populations with particular parametric configurations are taken from each family. Random samples of common size $n$ ($n=5(2)15$) are generated through computer from each of the three populations in a set and the size of the expected subset using selection procedure $R_{m,c}'$ for different values of $n$, $m$ and $c$ is noted by taking $P^* = 0.90$. This process is repeated 300 times in batches of 100 repetitions. The estimated expected subset sizes for the different values of $n$, $m$ and $c$, by taking the average of the subset sizes selected in 300 repetitions under each set of the parametric families for the procedure $R_{m,c}'$ are given in the following tables.
Table 4.8.1
Form of populations: $\Pi_1$ is $N(2.0,1.0)$, $\Pi_2$ is $N(4.5,1.0)$ and $\Pi_3$ is $N(7.5,1.0)$.

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Table 4.8.2
Form of populations: $\Pi_1$, $\Pi_2$ and $\Pi_3$ are Cauchy with means 2.0, 4.4 and 9.5 respectively.

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### Table 4.8.3

Form of populations: \( \Pi_1, \Pi_2, \) and \( \Pi_3 \) are Exponential with means 1.0, 5.5 and 8.5 respectively.

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### Table 4.8.4

Form of populations: \( \Pi_1 \) is \( N(1.0, 1.0) \), \( \Pi_2 \) is \( N(3.5, 1.0) \) and \( \Pi_3 \) is \( N(8.5, 1.0) \).

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### Table 4.8.5

Form of populations: $\Pi_1$, $\Pi_2$ and $\Pi_3$ are Cauchy with means $3.0$, $6.4$ and $10.5$ respectively.

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### Table 4.8.6

Form of populations: $\Pi_1$, $\Pi_2$ and $\Pi_3$ are Exponential with means $3.0$, $8.5$ and $10.5$ respectively.

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**Remark 4.8.1:** Simulation was carried out in batches of $100$ repetitions and very little difference, if any, was observed among the respective values of the estimated expected subset sizes in both the batches. Therefore, it was decided to stop...
simulation after 300 repetitions.

**Remark 4.8.2:** Due to non-availability of very fast computers, simulation could not be carried out for all possible values of $n$, $m$ and $c$.

**Remark 4.8.3:** From the above tables we see that the estimated expected subset sizes go on taking stable values for a common sample size beyond 9. Thus, a common sample size of 9 or more is sufficient enough to use the asymptotic results.