Chapter 4

On liftings of powers of irreducible polynomials

4.1 Motivation of the problem and statements of results

Let \( v \) be a henselian Krull valuation of arbitrary rank of a field \( K \) with valuation ring \( R_v \), maximal ideal \( M_v \), residue field \( \bar{K} = R_v/M_v \) and \( \bar{v} \) be its unique prolongation to a fixed algebraic closure \( \bar{K} \) of \( K \) having value group \( G_v \). Using the canonical homomorphism from \( R_v \) onto \( \bar{K} \), as usual one can lift any monic irreducible polynomial having coefficients in \( K \) to yield a monic irreducible polynomial with coefficients in \( R_v \). The description of the residue field of any residually transcendental prolongation of \( v \) to \( K(x) \) given in Theorem 1.1.A led Popescu and Zaharescu [Po-Za] to generalize the notion of usual lifting. In this attempt, they introduced the concept of lifting of a polynomial belonging to \( K(\alpha)[Y] \) (\( Y \) an indeterminate) with respect to a \((K, v)\)-minimal pair \((\alpha, \delta)\) as given in Definition 1.4.

In 1997, Khanduja and Saha proved that a monic polynomial belonging to \( K[x] \), which is a lifting of a monic irreducible polynomial \( T(Y) \neq Y \) belonging to \( K(\alpha)[Y] \) with respect to a \((K, v)\)-minimal pair \((\alpha, \delta)\), is irreducible over \( K \) (see [Kh-Sa1, Theorem 2.2]). As in the case of the usual lifting, a lifting of a power of an irreducible
polynomial may not be irreducible. In this chapter, we give sufficient conditions so that a given polynomial which is a lifting of a power of an irreducible polynomial belonging to \( K(\alpha)[Y] \) with respect to a \((K, v)\)-minimal pair \((\alpha, \delta)\) is irreducible over \( K \). As an application, we extend Eisenstein-Dumas Irreducibility Criterion and Generalized Schönemann Irreducibility Criterion (see Theorem 4.1.3 and Theorem 3.1.A respectively). There are instances where the above criteria fail to establish irreducibility but Theorem 4.1.1 to be proved in this chapter does (see Examples 4.4.2, 4.4.3).

**Theorem 4.1.1.** Let \((K, v)\) be a henselian valued field of arbitrary rank with value group \( G_v \) and \((\tilde{K}, \tilde{v}), (\alpha, \delta), w_{\alpha, \delta}, f(x), m, \lambda, e\) and \( h(x) \) be as in Theorem 1.1.A. Let \( T(Y) \) not divisible by \( Y \) be a monic polynomial of degree \( t \) with coefficients in \( \tilde{K}(\alpha) \) which is the \( n \)th power of a monic irreducible polynomial \( U(Y) \) belonging to \( \tilde{K}[\alpha][Y] \) for some \( n \geq 1 \). Let \( F(x) \) be a lifting of \( T(Y) \) with respect to \((\alpha, \delta)\) having \( f(x)\)-expansion \( \sum_{i \geq 0} A_i(x)f(x)^i \). Let \( \psi(x, Y) = \sum_{i=0}^d b_i(x)Y^i \) belonging to \( \mathcal{R}_v[x, Y] \) be a monic polynomial such that \( \deg b_i(x) < m, \sum b_i(\alpha)Y^i = U(Y) \). Let \( G(x) \in K[X] \) be a polynomial of degree less than \( \deg F(x) \) defined by \( G(x) = F(x) - h(x)^i \left( \psi(x, \frac{f(x)^e}{h(x)}) \right)^n \) with \( f(x)\)-expansion \( \sum_{0 \leq i < e+t} G_i(x)f(x)^i \) satisfying the following properties:

(i) If \( j \) is the smallest index such that \( w_{\alpha, \delta}(G(x)) = \tilde{v}(G_j(\alpha)) + j\lambda \), then the \( w_{\alpha, \delta}\)-residue of \( \frac{G(x)}{G_j(x)f(x)^j} \) as a polynomial in \( Z = \left( \frac{f(x)^e}{h(x)} \right) \) with coefficients from \( \tilde{K}(\alpha) \) is coprime to the polynomial \( T(Z) \).

(ii) The order of \( \frac{\tilde{v}(G_j(\alpha)) + j\lambda}{n} \) modulo the group \( G(K(\alpha)) + Z\lambda \) is \( n \).

Then \( F(x) \) is irreducible over \( K \). If \( \theta \) is any root of \( F(x) \), then \( G(K(\alpha)) \) is a subgroup of index \( e_n \) in \( G(K(\theta)) \) and \( \overline{K(\theta)} \) is an extension of \( \overline{K(\alpha)} \) of degree equal to degree of \( U(Y) \).
It may be pointed out that in the particular case when \( n = 1 \), condition (ii) of the above theorem is obviously satisfied and condition (i) holds in view of the fact that the degree of the \( w_{\alpha, \delta} \)-residue of \( \frac{G(x)}{G_j(x)f(x)^j} \) as a polynomial in \( \left( \frac{f(x)^j}{h(x)} \right) \) is strictly less than the degree of \( T(Y) = U(Y) \). So Theorem 4.1.1 implies the already known result that lifting of an irreducible polynomial is irreducible. The following theorem will be deduced from Theorem 4.1.1. It yields Corollary 4.1.3 which extends Eisenstein-Dumas Irreducibility Criterion.

**Theorem 4.1.2.** Let \( v \) be a discrete valuation of a field \( K \) with value group \( \mathbb{Z} \) and residue field \( \overline{K} \). Let \( F(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_0 \) be a polynomial over \( K \) and \( t \) be the gcd of \( v(a_0) \) and \( s \) and \( e \) be the number \( \frac{s}{t} \). Let \( h \) be an element of \( K \) with \( v(h) = \frac{v(a_0)}{t} \). Assume that the following conditions are satisfied:

(a) \( \frac{v(a_i)}{s-i} \geq \frac{v(a_0)}{s} \) for \( 0 \leq i \leq s - 1 \).

(b) The polynomial \( T(Y) = Y^t + \cdots + c_0 \) with \( c_{t-i} = \frac{a_{s-i}}{h^i} \), \( 0 \leq i \leq t - 1 \), is the \( n \)th power of a monic irreducible polynomial \( U(Y) \) over \( \overline{K} \) for some integer \( n \geq 1 \).

(c) There exists a monic polynomial \( \psi(Y) \in R_v[Y] \) with \( \overline{\psi}(Y) = U(Y) \) such that \( G(x) = F(x) - h\left( \psi\left( \frac{x}{h} \right) \right)^n = \sum_{i=0}^{s-1} g_i x^i \) satisfies the property that

\[
\min_{i} \left\{ v(g_i) + j - v(a_0) \right\} = v(g_j) + j \quad \text{is attained at only one index } j \text{ and } n \text{ is coprime to } ev(g_j) + j \quad \text{and } s.
\]

Then \( F(x) \) is irreducible over \( K \).

The following corollary is an immediate consequence of the theorem above because according to the hypothesis of Corollary 4.1.3 (with notations as in the above theorem), \( t = 1, e = s \) and for any \( g_i, g_j \in K \), not both zero, \( v(g_i) + j - v(a_0) \neq v(g_j) + j \frac{v(a_0)}{s} \), if \( 0 \leq i, j \leq s - 1, i \neq j \).
Corollary 4.1.3 (Eisenstein-Dumas Irreducibility Criterion). Let $(K, v)$ be as in Theorem 4.1.2. Let $F(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_0$ be a polynomial with coefficients in $K$ such that $v(a_0)$ and $s$ are coprime. If \[ \frac{v(a_i)}{s-i} \geq \frac{v(a_0)}{s} \] for $0 \leq i \leq s-1$, then $F(x)$ is irreducible over $K$.

Extending the argument used for proving Theorem 4.1.2, we also deduce Theorem 4.1.4 from Theorem 4.1.1. As pointed out in Remark 4.4.1, the following theorem extends Generalized Schönenmann Irreducibility Criterion (stated as Theorem 3.1.A). Moreover Example 4.4.2 shows that this theorem is indeed stronger than Generalized Schönenmann Irreducibility Criterion.

Theorem 4.1.4. Let $v$ be a valuation of arbitrary rank of a field $K$ with value group $G_v$, valuation ring $R_v$, residue field $\overline{K}$ and $v^\sigma$ be the Gaussian prolongation of $v$ to $K(x)$ given by equation (1.2). Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial with $\overline{f}(x)$ irreducible over $\overline{K}$ and $\alpha$ be a root of $f(x)$. Let $F(x) \in R_v[x]$ be a polynomial with $f(x)$-expansion $f(x)^s + A_{s-1}(x)f(x)^{s-1} + \cdots + A_0(x)$, $A_0(x) \neq 0$ and $e$ be the smallest positive integer such that $e \frac{v^\sigma(A_0(x))}{s} = e\lambda$ (say) belongs to $G_v$ with $e\lambda = v(h)$ for some $h$ in $K$. Assume that the following conditions are satisfied:

(a) $\frac{v^\sigma(A_i(x))}{s-i} > \frac{v^\sigma(A_0(x))}{s} > 0$ for $0 \leq i \leq s-1$.

(b) If $s/e$ is denoted by $t$, the polynomial $T(Y) = Y^t + C_{t-1}(\alpha)Y^{t-1} + \cdots + C_0(\alpha)$ with $C_i(x) = \frac{A_0(x)}{h^{t-i}}$, $0 \leq i \leq t-1$, is the $n^{th}$ power of a monic irreducible polynomial $U(Y)$ belonging to $K(\alpha)[Y]$ for some $n \geq 1$.

(c) There exists a monic polynomial $\psi(x, Y) = \sum_{i=0}^{d} b_i(x)Y^i$ belonging to $R_v[x, Y]$ with $\deg b_i(x) < m$, $\sum_{i} b_i(\alpha)Y^i = U(Y)$ and $G(x) = F(x) - h\left(\psi\left(x, \frac{f(x)^r}{h}\right)\right)^n$ be a polynomial in $K[x]$ having degree less than $\deg F(x)$ with $f(x)$-expansion $\sum_{i=0}^{s-1} G_i(x)f(x)^i$ satisfying the property that $\min\{v^\sigma(G_i(x)) + i\lambda\} = v^\sigma(G_j(x)) + j\lambda$ for exactly one
index \( j \) and the order of \( \frac{n^r(G_j(x)) + j\lambda}{n} \) modulo the group \( G_v + \mathbb{Z}\lambda \) is \( n \).

Then \( F(x) \) is irreducible over \( K \).

### 4.2 Preliminary Results

We prove a proposition to be used in the proofs of Theorems 4.1.1, 5.1.5.

**Proposition 4.2.1.** Let \((K, v), (\tilde{K}, \tilde{v}), (\alpha, \delta), w_{\alpha, \delta}, f(x), \lambda, e \) and \( h(x) \) be as in Theorem 1.1.A. Let \( F(x) \) belonging to \( K[x] \) be a monic polynomial which is a lifting of a monic non constant polynomial \( T(Y) \) not divisible by \( Y \) belonging to \( \tilde{K}[\alpha][Y] \) with respect to \( (\alpha, \delta) \) and \( h(x) \). Let \( \theta \) be a root of \( F(x) \) with \( \tilde{v}(\theta - \alpha) = \delta \). Let \( G(x) \) belonging to \( K[x] \) be any polynomial having \( f(x) \)-expansion \( \sum G_i(x)f(x)^i \) with \( j \) as the smallest index for which \( w_{\alpha, \delta}(G(x)) = \tilde{v}(G_j(\alpha)) + j\lambda \). If the \( w_{\alpha, \delta} \)-residue of \( \frac{G(x)}{G_j(x)f(x)^j} \) as a polynomial in \( \tilde{K} \) is coprime to the polynomial \( T(h) \), then \( w_{\alpha, \delta}(G(x)) = \tilde{v}(G(\theta)) \).

**Proof.** Since \( \theta \) be a root of \( F(x) \) with \( \tilde{v}(\theta - \alpha) = \delta \), by assertion (iii) of Theorem 2.2.6, we have \( \tilde{v}(f(\theta)) = w_{\alpha, \delta}(f(x)) = \lambda \). With \( G(x) = \sum G_i(x)f(x)^i \), in view of Lemma 2.2.B(ii), \( \tilde{v}(G_j(\theta)) = \tilde{v}(G_j(\alpha)) \); consequently

\[
\tilde{v}(G(\theta)) = \tilde{v}\left( \sum G_i(\theta)f(\theta)^i \right) \geq \min_i \{\tilde{v}(G_i(\theta)) + i\lambda\} = \min_i \{\tilde{v}(G_i(\alpha)) + i\lambda\}.
\]

Therefore it follows from Theorem 1.1.A(\( a \)) and the hypothesis of \( G(x) \) that

\[
\tilde{v}(G(\theta)) \geq \min_i \{\tilde{v}(G_i(\theta)) + i\lambda\} = w_{\alpha, \delta}(G(x)) = \tilde{v}(G_j(\theta)) + j\lambda \quad (4.1)
\]

and for each \( i \)

\[
\tilde{v}(G_i(\theta)f(\theta)^i) = \tilde{v}(G_i(\alpha)) + i\lambda \geq w_{\alpha, \delta}(G(x)) = \tilde{v}(G_j(\theta)f(\theta)^j). \quad (4.2)
\]

Keeping in mind the choice of \( j, e \), observe that in (4.2) equality can hold only for those \( i \) when \( i - j \) is non negative and divisible by \( e \). Suppose to the contrary

55
that equality does not hold in (4.1), then $\nu(G(\theta)) > w_{\alpha,\beta}(G(x)) = \nu(G_j(\theta)f(\theta)^f)$. Therefore $\nu \left( \sum G_i(\theta) f(\theta)^{i-j} \right) > 0$, which in view of (4.2) and the above observation implies that

$$\nu \left( 1 + \frac{G_j(\theta) f(\theta)^e}{G_j(\theta)} + \frac{G_{j+2e}(\theta) f(\theta)^{2e}}{G_j(\theta)} + \cdots \right) > 0. \quad (4.3)$$

On denoting $\left( \frac{f(\theta)^e}{h(\theta)} \right)$ by $\xi$ and $\frac{G_{j+e}(\theta) h(\theta)}{G_j(\theta)}$ by $C_j(\theta)$, we can rewrite (4.3) as

$$1 + C_1(\theta) \xi + C_2(\theta) \xi^2 + \cdots = 0.$$

In view of Lemma 2.2.B(ii), $C_i(\theta) = C_i(\alpha)$ and $\xi = \left( \frac{f(\theta)^e}{h(\theta)} \right) = \left( \frac{f(\theta)^e}{h(\alpha)} \right)$. Therefore $\xi$ satisfies the polynomial $W(Y) = 1 + C_1(\alpha)Y + C_2(\alpha)Y^2 + \cdots$. Using Theorem 1.1.A(b), it can be easily checked that the $w_{\alpha,\beta}$-residue of $\frac{G(x)}{G_j(x)f(x)^f}$ is the polynomial $W(Z)$ with $Z = \left( \frac{f(x)^e}{h(x)} \right)$. Since $\xi$ is a root of $T(Y)$ by assertion (iv) of Theorem 2.2.6, we conclude that $T(Y)$ and $W(Y)$ are not coprime. This contradicts the hypothesis and hence the proposition is proved.

**Definition.** For a finite extension $(K',\nu')$ of a henselian valued field $(K,\nu)$, we define its defect to be $\frac{[K' : K]}{ef}$, where $e,f$ are the index of ramification and the residual degree of $(K',\nu')/(K,\nu)$ respectively (cf. [Kuh2]). We shall denote it by $d((K',\nu')/(K,\nu))$ or by $d(K'/K)$ when the underlying valuations are clear. The extension $(K',\nu')/(K,\nu)$ is said to be defectless if its defect is one.

The following result proved in [Kh-Sa3] will be used in the sequel.

**Theorem 4.2.A.** Let $(K,\nu)$ be a henselian valued field. If $\theta, \alpha \in \bar{K}$ are such that $\nu(\theta - \alpha) > \nu(\alpha - \beta)$ for every $\beta$ belonging to $\bar{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$, then $d(K(\alpha)/K)$ divides $d(K(\theta)/K)$.  

56
4.3 Proof of Theorem 4.1.1

In view of Theorem 2.2.6, there exists a root θ of $F(x)$ such that $\tilde{v}(\theta - \alpha) = \delta$. Let $l$ denote the degree of $U(x)$. To establish the irreducibility of $F(x)$ over $K$, we shall prove that $[K(\theta) : K] = \deg F(x)$. In view of Lemma 2.2.B(ii), $\overline{K(\alpha)} \subseteq \overline{K(\theta)}$, $G(K(\alpha)) \subseteq G(K(\theta))$ and $\overline{\xi} = \left(\frac{f(\theta)^{\prime}}{h(\alpha)}\right) = \left(\frac{f(\theta)^{\prime}}{h(\theta)}\right)$ belongs to $\overline{K(\theta)}$. By Theorem 2.2.6(iv), $\overline{\xi}$ is a root of $T(Y)$ and hence of $U(Y)$; consequently

$$[K(\theta) : K(\alpha)] \geq l. \quad (4.4)$$

So we can write $[K(\theta) : K] = [G(K(\theta)) : G(\alpha)] \overline{K(\theta)}d(K(\theta)/K)$ as a product

$$[G(K(\theta)) : G(K(\alpha))]\overline{K(\theta)}/[K(\alpha)]G(K(\alpha)) : G(\alpha)] \overline{K(\theta)}d(K(\theta)/K).$$

Our claim is that

$$[G(K(\theta)) : G(K(\alpha))] \geq en \quad (4.5)$$

and

$$d(K(\alpha)/K) \text{ divides } d(K(\theta)/K). \quad (4.6)$$

In view of (4.4) and the above expression for $[K(\theta) : K]$, the claim implies that

$$[K(\theta) : K] \geq elmn. \quad (4.7)$$

Keeping in mind that $[K(\theta) : K] \leq \deg F(x) = elmn$, it will immediately follow that equality holds in (4.7), (4.5) and (4.4).

We first prove (4.6). In view of Theorem 4.2.A, it is enough to verify that for each $\beta \in \overline{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$, the inequality $\tilde{v}(\alpha - \beta) < \tilde{v}(\theta - \alpha)$ holds. Since $(\alpha, \delta)$ is a $(K, v)$-minimal pair, $\tilde{v}(\alpha - \beta) < \delta$ whenever $[K(\beta) : K] < [K(\alpha) : K]$. As $\tilde{v}(\theta - \alpha) = \delta$, we have $\tilde{v}(\alpha - \beta) < \tilde{v}(\theta - \alpha)$ and hence (4.6) holds.

Keeping in mind the hypothesis that the order of $\frac{\tilde{v}(G_2(\alpha))}{n} + j\lambda$ modulo the group $G(K(\alpha)) + Z\lambda$ is $n$ and the fact that $[G(K(\alpha)) + Z\lambda : G(K(\alpha))] = e$,
inequality (4.5) and hence the theorem is proved once we show that
\[ \frac{\hat{v}(G_j(a)) + j\lambda}{n} \in G(K(\theta)). \] (4.8)

By virtue of hypothesis (i) of the theorem and Proposition 4.2.1, we have \( \hat{v}(G_j(a)) + j\lambda = w_{\alpha,\delta}(G(x)) = \hat{v}(G(\theta)). \) It now follows that
\[ \hat{v}(G_j(a)) + j\lambda = \hat{v}\left(F(\theta) - h(\theta)^e\left(\psi\left(\theta, \frac{f(\theta)^e}{h(\theta)}\right)\right)^n\right) = \hat{v}(h(\theta)^e) + n\hat{v}\left(\psi\left(\theta, \frac{f(\theta)^e}{h(\theta)}\right)\right). \]

Thus \( \hat{v}\left(\psi\left(\theta, \frac{f(\theta)^e}{h(\theta)}\right)\right) = \frac{\hat{v}(G_j(a)) + j\lambda - \epsilon t\lambda}{n} \in G(K(\theta)). \) Since \( t = ln \) and \( \epsilon \lambda \) belongs to \( G(K(\alpha)) \subseteq G(K(\theta)), \) we see that \( \frac{\hat{v}(G_j(a)) + j\lambda}{n} \in G(K(\theta)) \) which proves (4.8) as desired.

4.4 Proof of Theorems 4.1.2, 4.1.4

Proof of Theorem 4.1.2. Since the value group and the residue field remain the same when \((K, v)\) is replaced by its henselization, we may prove the theorem assuming that \((K, v)\) is henselian. Set \( \delta = v(a_0)/s. \) Let \( w_{0,\delta} \) be the valuation of \( K(x) \) corresponding to the \((K, v)\)-minimal pair \((0, \delta)\). In view of hypothesis (a) of the theorem, it can be easily seen that \( w_{0,\delta}(F(x)) = \min\{v(a_i) + i\delta\} = v(a_0) = s\delta. \) Note that \( e \) is the smallest positive integer such that \( e\delta \in \mathbb{Z}. \) So \( v(a_i) + i\delta > s\delta \) if \( i \) is not divisible by \( e; \) consequently on taking \( w_{0,\delta}\)-residue, we have
\[ \left(\frac{F(x)}{h_t}\right) = \left(\frac{x^e}{h}\right)^t + \left(\frac{a_{2e}}{h}\right)\left(\frac{x^e}{h}\right)^{t-1} + \cdots + \left(\frac{a_0}{h^t}\right) = T\left(\frac{x^e}{h}\right). \]

Thus \( F(x) \) is a lifting of \( T(Y) \) not divisible by \( Y \) with respect to \((0, \delta). \) With \( \psi \) as in hypothesis (c), the polynomial \( G(x) = F(x) - h^e\left(\psi\left(\frac{x^e}{h}\right)\right)^n = \sum g_j x^j \) satisfies the property that \( w_{0,\delta}(G(x)) = v(g_j) + j\delta \) for only one index \( j; \) consequently the \( w_{0,\delta}\)-residue of \( \frac{G(x)}{g_j x^j} \) is \( 1 \) and hence condition (i) of Theorem 4.1.1 is satisfied.

Keeping in view the hypothesis that \( n \) is coprime to \( j \frac{v(a_0)}{t} + ev(g_j), \) it can be easily
checked that the order of $\frac{jv(a_0)/s + v(g_j)}{n}$ for $n \in \mathbb{Z} + z(aQ) = \mathbb{Z}$. Therefore condition (ii) of Theorem 4.1.1 is also satisfied which proves that $F(x)$ is irreducible over $K$.

**Proof of Theorem 4.1.4.** Since a valued field and its henselization have the same value groups and residue fields, we may assume without loss of generality that $(K, v)$ is henselian. Set $\lambda = \frac{v^s(A_0(x))}{s}$ and let $\delta \in G_v$ be as in equation (3.10) so that $(\alpha, \delta)$ is a $(K, v)$-minimal pair with $w_{\alpha, \delta}(f(x)) = \lambda$. Keeping in mind Theorem 1.1.A(a), equation (2.2) and hypothesis (a) of the theorem, it can be easily checked that

$$w_{\alpha, \delta}(F(x)) = \min_i \{v(A_i(\alpha)) + i\lambda\} = \min_i \{v^s(A_i(x)) + i\lambda\} = v^s(A_0(x)) = s\lambda.$$ 

Using hypothesis (b) of the theorem and arguing as in the proof of Theorem 4.1.2, it can be seen that $F(x)$ is a lifting of $T(Y) = U(Y)^n$ with respect to the minimal pair $(\alpha, \delta)$. With $G(x)$ as in hypothesis (c) of the theorem, the $w_{\alpha, \delta}$-residue of $F(x)$ is $\bar{1}$, and hence condition (i) of Theorem 4.1.1 is satisfied; further condition (ii) of this theorem is also satisfied in view of (c). So $F(x)$ is irreducible over $K$ by Theorem 4.1.1.

**Remark 4.4.1.** The hypothesis of the above theorem is weaker than that of Generalized Schönemann Irreducibility Criterion because according to the hypothesis of this criterion (with notation as in Theorem 4.1.4), $e = s, t = 1$ and for any non-zero polynomials $A(x), B(x) \in K[x]$ of degree less than $m$, one has

$$v^s(A(x)) + i\lambda \neq v^s(B(x)) + j\lambda$$

when $0 \leq i, j \leq s - 1, i \neq j$. The following example shows that there are instances when the criterion is not applicable whereas Theorem 4.1.4 works.

**Example 4.4.2.** We prove that the polynomial $F(x) = x^8 + 4x^6 + 12x^4 + 25x^2 + 25$ is irreducible over the field $\mathbb{Q}_3$ of 3-adic numbers. Note that $F(x) \equiv (x^2 + 1)^4 \pmod{3}$. The $(x^2 + 1)$-expansion of $F(x)$ is $(x^2 + 1)^4 + 6(x^2 + 1)^2 + 9(x^2 + 1) + 9$. Clearly
condition (a) of Theorem 4.1.4 is satisfied. With notations as in this theorem, it can be easily seen that \( s = 4, e = 2, t = 2, \lambda = 1/2 \) and on taking \( h = 3 \) one can check that \( T(Y) = (Y + 1)^2 \) which shows that condition (b) of Theorem 4.1.4 holds.

Consider \( \psi(Y) = Y + 1 \). In this situation

\[
G(x) = F(x) - 3^2 \left( \psi \left( \frac{(x^2 + 1)^2}{3} \right) \right)^2 = 9(x^2 + 1).
\]

So condition (c) of Theorem 4.1.4 is also satisfied. Therefore \( F(x) \) is irreducible over \( \mathbb{Q}_3 \).

The following example illustrates that Theorem 4.1.1 may work when Eisenstein-Dumas Irreducibility Criterion stated as Corollary 4.1.3 is not applicable.

**Example 4.4.3.** Consider the polynomial \( F(x) = x^3 + 9cx^2 + 27cx - 27b^3 \) over \( Z \), where \( b \equiv 2 \pmod{9} \) and \( c \equiv 1 \pmod{9} \). We shall show that \( F(x) \) is irreducible over the field \( K = \mathbb{Q}_3 \) of 3-adic numbers. Let \( w_{0,1} \) be valuation of \( K(x) \) with respect to the minimal pair \((0,1)\). Clearly \( w_{0,1}(F(x)) = 3 \). Take \( h = 3b \), then the \( w_{0,1} \)-residue of \( F(x)/h^3 \) is given by

\[
\left( \frac{F(x)}{27b^3} \right) = \left( \frac{x}{3b} \right)^3 - 1 = Z^3 - 1 = (Z - 1)^3 = T(Z) \text{ (say)},
\]

where \( Z \) is the \( w_{0,1} \)-residue of \( \frac{x}{3b} \). On taking \( \psi(Z) = Z - 1 \), a simple calculation shows that \( G(x) = F(x) - (3b)^3 \left( \psi \left( \frac{x}{3b} \right) \right)^3 = 9(c + b)x^2 + 27(c - b^2)x \). Keeping in mind the choice of \( b, c \), it can be easily seen that \( w_{0,1}(G(x)) = w_{0,1}(27(c - b^2)x) = 5 \) and the \( w_{0,1} \)-residue of \( \frac{G(x)}{27(c - b^2)x} = \frac{b(c + b)}{c - b^2} \left( \frac{x}{3b} \right) \) is \( Z + 1 \). So condition (i) of Theorem 4.1.1 is satisfied. As \( \frac{w_{0,1}(G(x))}{3} = \frac{5}{3} \) has order 3 modulo the group \( Z \), it is clear that condition (ii) of the Theorem 4.1.1 is also satisfied. So \( F(x) \) is irreducible over \( K \).