On prolongations of valuations via Newton polygons and liftings of polynomials

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Abstract

Let $v$ be a real valuation of a field $K$ with valuation ring $R_v$. Let $K((t))$ be a finite separable extension of $K$ with $t$ integral over $R_v$ and $F(t)$ be the minimal polynomial of $t$ over $K$. Using Newton polygons and residually transcendental prolongations of $v$ to a simple transcendental extension $K(x)$ of $K$ together with liftings with respect to such prolongations, we describe a method to determine all prolongations of $v$ to $K((t))$ along with their residual degrees and ramification indices over $v$. The problem is classical but our approach uses new ideas. The paper gives an analogue of Ore’s Theorem when the base field is an arbitrary rank-1 valued field and extends the main result of [S.D. Cohen, A. Movahhedi, A. Salinier, Factorization over local fields and the irreducibility of generalized difference polynomials, Mathematika 47 (2000) 173-196].

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Introduction

Let $K = \mathbb{Q}(t)$ be an algebraic number field with $t$ in the ring $A_K$ of algebraic integers of $K$ and $F(x)$ be the minimal polynomial of $t$ over the field $\mathbb{Q}$. The determination of the prime ideal decomposition in $A_K$ of any rational prime $p$ is one of the most important problems of Algebraic Number Theory and is related to the factorization of the polynomial $F(x)$, obtained on replacing each coefficient of $F(x)$ by its residue modulo $p$. In 1878, Dedekind proved that if $p$ does not divide $[A_K : \mathbb{Z}[t]]$ and

$$F(x) = \phi_1(x)^{e_1} \cdots \phi_r(x)^{e_r},$$

where $\phi_i(x)$ are distinct irreducible polynomials over $\mathbb{Z}/p\mathbb{Z}$ with $\phi_i(x)$ monic, then $pA_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are distinct prime ideals of $A_K$ with $pA_K$, having residual degree equal to $\deg(\phi_i(x))$ (see [5, Theorem 4.8.13]). Dedekind also gave a criterion to verify when the hypothesis $p$ does not divide $[A_K : \mathbb{Z}[t]]$ is satisfied and proved that this is so if and only if for each $i$, the condition either $e_i = 1$ or $\phi_i(x) : M(x)$ holds, where $M(x) = \frac{\phi_i(x) - \phi_j(x)}{x}$ (cf. [5, Theorem 6.1.4], [11]). In 1894, Hensel developed a powerful approach by showing that the prime ideals of $A_K$ lying over $p$ are in one-to-one correspondence with the monic irreducible factors of $F(x)$ over the ring $\mathbb{Z}_p$ of $p$-adic integers and that the ramification index together with the residual degree of a prime ideal of $A_K$ lying over $p$ are same as those of a simple extension of $\mathbb{Z}_p$ obtained by adjoining a root of the corresponding irreducible factor of $F(x)$ belonging to $\mathbb{Z}_p[x]$. Keeping in view Hensel’s result, in 1928, Ore [16] generalized Dedekind’s Theorem stated above using $\phi$-Newton polynomials, where $\phi(x) \in \mathbb{Z}[x]$ are monic polynomials which are irreducible modulo $p$. By virtue of Hensel’s Lemma, (1) leads to a factorization $F(x) = F_1(x) \cdots F_r(x)$ over the ring $\mathbb{Z}_p$ of $p$-adic integers with $F_i(x) = \phi_i(x)^{e_i}$. If $p$ divides $[A_K : \mathbb{Z}[t]]$, then these factors $F_i(x)$ need not be irreducible over $\mathbb{Z}_p$ and the problem Ore attempted to solve is to determine further decomposition of $F_i(x)$ into
a product of irreducible factors over $\mathbb{Q}_p$. For this purpose, he considered the $0,\overline{v}$-Newton polygon of $F_i(x)$ for each $i$, having $k_i$ sides with positive slope which leads to a factorization of $F_i(x)$ into $k_i$ factors, say $F_i(x) = F_1(x) \cdots F_k(x)$ in $\mathbb{Z}_p[x]$. Moreover to each side $S_j$ of the $0,\overline{v}$-Newton polygon of $F_j(x)$, he associated a polynomial $(F_j)_{S_j}(Y)$ over the finite field $\mathbb{F}_q = \mathbb{F}_{p^k}$, $k = p^{\nu_0 K_0}$, in an indeterminate $Y$. The factorization of the associated polynomial $(F_j)_{S_j}(Y)$ over $\mathbb{F}_q$ provides a further factorization of the factor of $F_j(x)$ corresponding to the side $S$. Finally Ore showed that if for some $i$, all these polynomials $(F_i)_{S_j}(Y)$ corresponding to various sides $S_j$, $1 \leq j \leq k_i$, of the $0,\overline{v}$-Newton polygon of $F_i(x)$ have no multiple factor, say $(F_i)_{S_j}(Y)$ splits into $n_q$ distinct irreducible factors over $\mathbb{F}_q$, then all the $\sum_{i=1}^{k_i} n_q$ factors of $F_i(x)$ obtained in this way are irreducible over $\mathbb{Q}_p$. Further the slopes of the sides of the $0,\overline{v}$-Newton polygon of $F_j(x)$ and the degrees of the irreducible factors of $(F_j)_{S_j}(Y)$ over $\mathbb{F}_q$ for $S$ ranging over all the sides of such a polygon lead to the explicit determination of the residual degrees and the ramification indices of all those prime ideals of $A_0$ lying over $\mathfrak{p}$ which correspond to the irreducible factors of $F_i(x)$.

In this paper, our aim is to extend the scope of Ore’s Theorem when the base field is an arbitrary field $K$ with a real valuation $v$ which is not necessarily discrete. The main motivation behind this work is the paper of Cohen et al. which also extends Ore’s approach of finding irreducible factors of polynomials over complete discrete valued fields (see [6, Theorem 1.5]). The problem is similar but our method of tackling it is different and more lucid. It involves Newton polygons as well as prolongations of $v$ to a simple transcendental extension $K(x)$ of $K$ whose residue fields are transcendental over the residue field of $v$. It is well known that the residue field of such a prolongation is a simple transcendental extension $L(Z)$ of a finite extension $L$ of the residue field of $v$ (see [1,15]). We also use liftings of polynomials belonging to $L(Z)$ with respect to $\nu$ which yield suitable polynomials over $K$. Our method of proof of the main theorem has the possibility of being extended to hetensal valued fields of arbitrary rank. The analogues of all the preliminary results used for the proof of this theorem except probably Theorem 2.C and Corollary 2.D are valid in the general case. In the last section, we quickly deduce Ore’s Theorem for Dedekind domains as well as a generalization of Dedekind’s Theorem and give some examples to illustrate the process of obtaining prolongations of $v$ to finite extensions.

1. Definitions and statement of results

Let $(K, v)$ be a henselian valued field of arbitrary rank with valuation ring $R_v$, maximal ideal $M_v$, residue field $\overline{K} = R_v/M_v$, and $\nu$ be the unique prolongation of $v$ to the algebraic closure $\overline{K}$ of $K$ with value group $G_v$. For an element $\alpha$ belonging to the valuation ring $R_v$ of $\overline{K}$, $\nu(\alpha)$ will denote its $\nu$-residue, i.e., the image of $\nu$ under the canonical homomorphism from $R_v$ onto its residue field. For a polynomial $f(x)$ belonging to $R_v[x]$, $f(x)$ will stand for the polynomial over the residue field of $v$ obtained on replacing each coefficient of $f(x)$ by its $\nu$-residue. When $L$ is a subfield of $K$, $L(G)$ will denote respectively the residue field and the value group of the valuation of $L$ obtained on restricting $\nu$. We shall denote by $v^{\alpha}$ the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate $x$. It extends $v$ and is defined on $K[x]$ by

$$v^{\alpha}(\sum_{i} a_i x^i) = \min\{v(a_i)\}, \quad a_i \in K.$$  (2)

The residue field of $v^{\alpha}$ is a simple transcendental extension of the residue field of $v$ (see [7, Corollary 2.2.2]). In general, a prolongation $w$ of $v$ to $K(x)$ is called residually transcendental if the residue field of $w$ is a transcendental extension of the residue field of $v$. It is known that residually transcendental prolongations are given by means of minimal pairs (see [2]) defined in the following way:

A pair $(w, \delta)$ in $K \times G_v$ will be called a minimal pair (more precisely a $(K, v)$-minimal pair) if whenever $\beta$ belongs to $K$ with $[K(\beta) : K] \leq [K(\alpha) : K]$, then $\nu(\alpha - \beta) \leq \delta$. For example, if $f(x)$ belongs to $K[x]$ and $x$ is a monic polynomial of degree $m \geq 1$ with $f(x)$ irreducible over the residue field of $v$ and $\alpha$ is a root of $f(x)$, then $(\alpha, \delta)$ is a $(K, v)$-minimal pair for each positive $\delta$ in $G_v$, because whenever $\beta$ belongs to $K$ with degree $[K(\beta) : K] < m$, then $\nu(\alpha - \beta) \leq 0$, otherwise $\nu(\alpha - \beta) > 0$, which in view of the Fundamental Inequality ([7, Theorem 3.3.4]) would imply that $[K(\beta) : K] \geq [K(\alpha) : K] = m$ leading to a contradiction.

Let $(\alpha, \delta)$ be a $(K, v)$-minimal pair. The valuation $\overline{w}_{\nu, \delta}$ of $K(x)$ defined by

$$\overline{w}_{\nu, \delta}(\sum_{i} c_i(x - \alpha)^i) = \min\{\nu(c_i) + i\delta\}, \quad c_i \in \overline{K}$$  (3)

will be referred to as the valuation with respect to the minimal pair $(\alpha, \delta)$. The restriction of $\overline{w}_{\nu, \delta}$ to $K(x)$ will be denoted by $w_{\nu, \delta}$. It is proved in [1, Theorem 2.1] that a prolongation $w$ of $v$ to $K(x)$ is residually transcendental if and only if $w = w_{\nu, \delta}$ for some $(K, v)$-minimal pair $(\alpha, \delta)$. The valuation $w_{\nu, \delta}$ and its residue field are described by the theorem stated below, the proof of which is omitted (see [1, Theorem 2.1], [2, Theorem 2.2]).

**Theorem 1.A.** Let $(K, v)$, $(K, v)$ be as in the opening lines of this section and $(\alpha, \delta)$ be a $(K, v)$-minimal pair. Let $f(x)$ be the minimal polynomial of $\alpha$ over $K$ of degree $m$ with $w_{\nu, \delta}(f(x)) = \delta$. Then the following hold:

(a) For any polynomial $g(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum g_i(x)f(x)^i$, $\deg g_i(x) < m$, one has $w_{\nu, \delta}(g(x)) = \min\{\nu(g_i(\alpha)) + i\delta\}$. 


(b) If h(x) belonging to $K[x]$ is a polynomial of degree less than $m$, then the $w_{\nu,s}$-residue of $h(x)/h(0)$ equals 1.

(c) Let $\nu$ be the smallest positive integer such that $\phi \in G(K(\alpha))$ and $h(x)$ belonging to $K[x]$ be a polynomial of degree less than $m$ with $v(\alpha(x)) = \nu$. Then the $w_{\nu,s}$-residue of \( \frac{\text{f}(x)}{\text{h}(x)} \) of $K(\alpha)$ is transcendental over $K(\alpha)$ and the residue field of $w_{\nu,s}$ is $K(\alpha)$.

Using the canonical homomorphism from the valuation ring $R_0$ of $\nu$ onto its residue field $\tilde{K}$, as usual one can lift any monic irreducible polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with coefficients in $K$ to yield a monic irreducible polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $K$. In 1995, Popescu and Zaharescu [17] generalized the notion of usual lifting by introducing the concept of lifting of a polynomial belonging to $K(\nu)[Y]$ (Y an indeterminate) with respect to a $(K, \nu)$-minimal pair $(\alpha, \lambda)$ as follows.

**Definition 1.8.** For a $(K, \nu)$-minimal pair $(\alpha, \lambda)$, let $f(x)$, $m$, $\nu$ and $h(x)$ be as in Theorem 1.6A. A monic polynomial $F(x)$ belonging to $K[x]$ is said to be a lifting of a monic polynomial $T(Y)$ belonging to $K(\nu)[Y]$ having degree $\nu \geq 1$ with respect to $(\alpha, \lambda)$ (more precisely with respect to $(\alpha, \lambda)$ and $h(x)$) if the following three conditions are satisfied:

(i) \( \deg F(x) = \nu m \),

(ii) \( w_{\nu,s}(F(x)) = w_{\nu,s}(h(x)') = \nu \lambda \),

(iii) \( w_{\nu,s} - \text{residue of } F(x) \text{ modulo } h(x) \).

Keeping in mind that the valuation $w_{\nu,s}$ is uniquely determined by $f(x)$ and $\lambda$, in view of Theorem 1.6A(a), sometimes we avoid referring to the minimal pair $(\alpha, \lambda)$ and say that the above lifting is with respect to $f(x)$, $\lambda$. It may be pointed out that the usual lifting is indeed a lifting with respect to the minimal pair $(0, 0)$.

**Definition 1.6.** Let $\nu$ be a real field of a field $K$. Let $\phi(x)$ belonging to $R_0[x]$ be a monic polynomial with $\phi(x)$ irreducible over $K$ and $\nu'$ be the Gaussian prolongation defined by (2). Let $F(x) = \nu'[x]$ be a polynomial with $\phi(x)$-expansion $\sum_{i=0}^{\infty} a_i(x)\phi(x)^i$. $a_i(x)\phi(x)^i \neq 0$. To each term $a_i(x)\phi(x)^i$. We associate the point $(n - i, \nu'[a_i(x)])$ ignoring however the point $(n - \nu, \nu'[a_i(x)])$ in case $a_i(x)$ is 0 and form the set

\[ P = \{|j, \nu'[A_{\nu,s}(\alpha_j(x))]| 0 \leq j \leq n, A_{\nu,s}(\alpha_j(x)) \neq 0 \}. \]

The $\nu$-Newton polygon of $F(x)$ (with respect to the underlying valuation $\nu$) is the polygonal path formed by the lower edges along the convex hull of the points of $P$. Note that the slopes of the edges are increasing when calculated from left to right. The $\nu$-Newton polygon of $F(x)$ minus its horizontal part (if any) is called its principal part.

With the above notations, in this paper we prove:

**Theorem 1.1.** Let $(K, \nu)$ be a complete rank 1 valued field with valuation ring $R_\nu$, value group $G_\nu$, residue field $\tilde{K} = R_\nu/M_\nu$, and let $(K, \nu)$ be as in Theorem 1.6A. Let $\phi(x) 

(ii) \text{If } 0 \text{ is a root of } F(x), \text{ then } \nu(\phi(0)) = \lambda. \text{ The index } [G(K(\nu)) : G_\nu] \text{ is divisible by } c_\nu, \text{ where } c_\nu \text{ is the smallest positive integer such that } \phi(c_\nu) \in G_\nu \text{ and } [K(\nu) : K] \text{ is divisible by } m.

(iii) $F(x)$ is a lifting of $T(y) \in K(\nu)[Y]$ not divisible by $Y$ of degree $\nu$, $c_\nu$ with respect to $\phi(x)$, $\lambda$.

(iv) If $U_1(Y)^{\nu_1} \cdots U_n(Y)^{\nu_n}$ is the factorization of $T(y)$ into powers of distinct monic irreducible polynomials over $K(\nu)$, then $F(x)$ factors as $F_1(x) \cdots F_n(x)$ over $K$, each $F_j(x)$ is a lifting of $U_j(Y)^{\nu_j}$ with respect to $\phi(x)$, $\lambda$, and has degree $a_j\nu_j$.

If some $a_j = 1$, then $F_j(x)$ is irreducible over $K$ and for any root $b_j$ of $F_j(x)$, the index $[G(K(b_j)) : G_\nu] = c_\nu$ and the degree $[K(\nu) : K] = m$ is divisible by $U_n(Y)$ in this case.

It may be pointed out that the above theorem together with the classical Hensel’s lemma and Corollary 2.5 immediately yields Theorem 1.5 of [6]. Further in view of Theorem 2A stated in the second section, Theorem 1.1 extends Ore’s theorem ([6, Theorem 1.8]) to arbitrary real valued fields. More specifically, we quickly derive by using Generalized Dedekind Criterion [11] an analogue of Ore’s theorem for Dedekind domains similar to the one in the classical case as well as a refinement of Dedekind’s theorem regarding factorization of prime ideals (see Corollaries 1.3 and 1.4).

**Theorem 1.2.** Let $K$ be a Dedekind domain with quotient field $K$ and $K' = K(\nu)$ be a finite separable extension of $K$ with $\nu$ in the integral closure $K'$ of $K$ in $K'$. Let $F(x)$ be the minimal polynomial of $\nu$ over $K$ and $\nu$ be a non-zero prime ideal of $K$. Suppose that $F(x) = \phi(x)^{\nu_1} \cdots \phi(x)^{\nu_n}$ mod $\nu$, where $\phi(x) \in K[x]$ are monic irreducible which are distinct and irreducible modulo $\nu$. Let $m$ denote the degree of $\phi(x)$ and let $a_i$ be one of its roots, then $p^n = a_{i_1} \cdots a_{i_n}$ where $a_{i_1}, \ldots, a_{i_n}$ are coprime ideals of $K'$ and $N_{K'[K]}(a_i) = p^{n_i}$. Let $\nu$ denote the valuation of $K$ with valuation ring $R_\nu$ and value group $\nu$. Suppose that the $\nu_i$-Newton polygon...
of \( F(x) \) with respect to \( v \) has \( k_j \) sides \( \lambda_1, \ldots, \lambda_{k_j} \) of positive slopes \( \lambda_1, \ldots, \lambda_{k_j} \). Let \( e \) denote the smallest positive integer for which \( e\lambda_i \in \Xi \) and \( l \) the length of the horizontal projection of \( S_j \). Then, over the completion \((K, \overline{v})\) of \((K, v)\), \( F(x) \) factors as \( \prod_{i=1}^n \prod_{j=1}^{k_j} F_j(x) \), where \( F_j(x) \in R_i[x] \) is a monic polynomial of degree \( m_i \), which is a lifting of a polynomial \( T_j(Y) \in K(\alpha_i)[Y] \) having degree \( l_i/e \) with respect to \( \phi_{\alpha_i} \). If \( U_{ij}(Y)^{n_{ij}} \cdots U_{1j}(Y)^{n_{1j}} \) is the factorization of \( T_j(Y) \), into a product of powers of distinct monic irreducible polynomials over \( K(\alpha_i) \), then \( a_i' = \prod_{j=1}^{k_j} \left( \prod_{i=1}^n p_i^{n_{ij}} \right)^{t_{ij}} \), where \( v \) is the unique prolongation of \( v \) to the algebraic closure of \( K \) for \( 1 \leq j \leq k_i \), \( 1 \leq l \leq n_i \) and \( N_{K/F}(v_i') = p^{v(n_i\deg(v_i'))} \). If some \( a_i' = 1 \), then \( v_i' \) is a prime ideal of \( R' \).

Corollary 1.3. Let \( R, K, K', R'. F(x) = \phi_0(x)^{n_0} \cdots \phi_i(x)^{n_i} \) modulo \( v \) and \( \overline{v} \) be as in the above theorem. Let \( F(x) = \sum A_j(x)\phi_j(x)^{j} \) be the \( \phi(x) \)-expansion of \( F(x) \). Assume that

\[
\frac{v^j(A_j(x))}{j} \geq \frac{v^j(A_0(x))}{j}, \quad 1 \leq j \leq v_i - 1, \quad \gcd(v^j(A_0(x)), v_i) = 1, \quad 1 \leq i \leq r.
\]

Then \( pR' = v^{j_i} \cdots v^{j_r} \cdot a_1 \cdots a_r \) are distinct prime ideals of \( R' \) and the residual degree of \( \phi_j(x) \) is \( \deg \phi_j(x) \) for each \( i \).

As an application of the above corollary, we obtain the following well known result (see [9, Chapter I, Theorem 7.4]).

Corollary 1.4. Let \( R, K, K', R'. F(x) = \phi_0(x)^{n_0} \cdots \phi_i(x)^{n_i} \) modulo \( p \) and \( \overline{v} \) be as in Theorem 1.2 and \( K_0 \) denote the integral closure of \( R_0 \) in \( K \). If \( K_0 \neq K \), then \( pR' = v^{j_i} \cdots v^{j_r} \cdot a_1 \cdots a_r \) are distinct prime ideals of \( K' \) and the residual degree of \( \phi_j(x) \) is \( \deg \phi_j(x) \).

In the particular case when \( R = \mathbb{Z}, K = \mathbb{Q}, f = p \mathbb{Z} \), it can be easily verified that the condition \( "K_0 \neq K" \) of the above corollary is equivalent to saying that \( p \) does not divide \( [A_{\mathbb{C}} : \mathbb{Z}[i]] \), where \( A_{\mathbb{C}} \) denotes the ring of algebraic integers of \( \mathbb{K} = \mathbb{C}(i) \).

2. Preliminary results

We shall use the following well known theorem stated below (cf. [14, Chap. II, 8.2, 8.5]).

Theorem 2.1. Let \( \overline{v} \) be a valuation of \( \mathbb{Q} \) and \( \overline{v} = v \overline{\mathbb{Q}} \) be as in the above theorem. Let \( \overline{v} \) be a valuation of \( \mathbb{Q} \) and \( \overline{v} = v \overline{\mathbb{Q}} \) be as in Theorem 1.2 and \( K_0 \) denote the integral closure of \( R_0 \) in \( K \). If \( K_0 \neq K \), then \( pR' = v^{j_i} \cdots v^{j_r} \cdot a_1 \cdots a_r \) are distinct prime ideals of \( K' \) and the residual degree of \( \phi_j(x) \) is \( \deg \phi_j(x) \).

Then there are exactly \( r \) prolongations of \( \phi(x) \) to \( K(\overline{v}) \). If \( \overline{v} \) is a root of \( F(\overline{v}) \) and \( \tilde{v} \) is the unique prolongation of \( \overline{v} \) to the algebraic closure of \( K \), then the valuations \( u_1, u_2, \ldots, u_r \) of \( \overline{v}(K) \) defined by

\[
u_i \left( \sum a_i x^i \right) = e \left( \sum a_i x^i \right), \quad a_i \in K
\]

are the prolongations of \( v \) to \( \overline{v}(K) \).

The following lemmas establish the close analogy between the phenomenon of lifting and the concept of Newton polygon.

Lemma 2.1. Let \( (K, v) \) be a henselian rank \( 1 \) valued field. Let \( \phi(x) \in R_i[x] \) be a monic polynomial of degree \( m \) with \( \phi(0) \) irreducible over \( K \) and \( \alpha \in K \) be a root of \( \phi(x) \). Let \( \lambda \) be a positive element of \( G \), and \( \nu_{\lambda, \alpha} \) be as defined by (3) for the minimal pair \((\alpha, \lambda)\). Let \( e \) be the smallest positive integer such that \( e\nu_{\lambda, \alpha}(\phi(x)) = \phi(\alpha) \) (say) belongs to \( G \). If \( F(\overline{v}) \) is a lifting of \( \phi(\alpha) \) with respect to \( (\alpha, \lambda) \) of a monic polynomial \( T(Y) \) belonging to \( K(\alpha)[Y] \), having degree \( t \) and not divisible by \( Y \), then the \( \phi \)-Newton polygon of \( F(x) \) consists of a single side which has slope \( \lambda \) and the length of the horizontal projection is \( \epsilon t \).

Proof. We first show that for any polynomial \( A(x) = \sum a_i x^i \) belonging to \( K(x) \) having degree less than \( m \) one has

\[\nu(\overline{A(\alpha)}) = v(\phi(\alpha)) + e \nu(\phi(\alpha)) \tag{5}\]

Clearly (5) needs to be verified when \( m > 1 \). Keeping in view that \( \phi(x) \) is irreducible over \( K \) of degree \( m > 1 \). It follows that \( \nu(\alpha) = 0 \). Suppose to the contrary that (5) does not hold, then the triangle inequality would imply that \( \nu(\alpha) > \min_{\epsilon \in \mathbb{C}} \epsilon v(\alpha) = v(\alpha) \) (say), which gives \( \sum_{i=0}^{m-1} \frac{1}{\nu_i} \overline{\nu_i} \geq 0 \). This is impossible as \( \overline{v} \) is a root of \( \phi(x) \). Hence (5) holds.

Let \( F(x) = \phi(x)^e + A_{e-1}(x)\phi(x)^{e-1} + \cdots + A_0(x) \) be the \( \phi(x) \)-expansion of \( F(x) \). Since \( F(\overline{v}) \) is a lifting of \( T(Y) \) of degree \( t \) not divisible by \( Y \), in view of Definition 1.8, we have \( s = \epsilon t \) and

\[\nu_{\lambda, \alpha}(F(x)) = \nu_{\lambda, \alpha}(\phi(x)^e) = \epsilon t = \epsilon \nu(\phi(\alpha))\]

1 As pointed out in the proof of Corollary 1.4 in Section 4, the hypothesis (v) of the corollary is stronger than condition (4) of Corollary 1.3.
Using Theorem 1.A(a) and (5), it follows that
\[ s \lambda = w_{\phi} A(F(x)) = \min \{ v(A_i(x)) + i \lambda \} = v_i(A_0(x)); \]
consequently
\[ v_i(A_i(x)) \geq v_i(A_0(x)) \]
for \( 1 \leq i \leq s - 1 \), which shows that the \( \phi \)-Newton polygon of \( F(x) \) has a single side whose slope is \( \lambda \) and the length of its horizontal projection is \( s \).

**Lemma 2.2.** Let \( (\alpha, \delta) \) be as in Lemma 2.1. Suppose that the \( \phi \)-Newton polygon of \( F(x) \) belonging to \( R[x] \) having \( (\phi(x))-\)expansion \( \phi(x)^s + A_{s-1}(x)\phi(x)^{s-1} + \cdots + A_0(x), A_0(x) \neq 0 \) has only one side and the side has slope \( \lambda > 0 \). Let \( e \) be the smallest positive integer such that \( e \lambda = e(\alpha) \in G_r \), for some \( h \in K \) and \( t \) denote the numbers \( \delta/e \). Then \( F(x) \) is a lifting of \( T(Y) = Y^t + \left( \frac{\alpha_{s-t-1}(h)}{h} \right) Y^{t-1} + \cdots + \left( \frac{\alpha_0(h)}{h} \right) \) with respect to \( \phi(x), \lambda \).

**Proof.** Write \( \phi(x) = \sum_{i=1}^m c_i(x - \alpha'), c_i \in K(\alpha) \). Define \( \delta \in G_r \) by
\[ \min \{ i \delta(c_i) + i \lambda \} = \lambda \]
and \( \delta = \max \{ (\lambda - i \delta(c_i))/i \} \).

Then \( \delta \) is positive so that \((\alpha, \delta)\) is a \((K, \cdot)-\)minimal pair. Keeping in view the hypothesis about the \( \phi \)-Newton polygon of \( F(x) \), we have
\[ v_i(A_i(x)) \geq v_i(A_0(x)) / s \]
for \( 1 \leq i \leq s - 1 \), i.e.,
\[ v_i(A_i(x)) + i \lambda \geq v_i(A_0(x)) = s \lambda \]
which shows that \( w_{\phi} A_i(x) = s \lambda \). Since \( e \) is the smallest positive integer for which \( e \lambda \in G_r \), it follows that when \( i \) is not divisible by \( e \), then \( v_i(A_i(x)) + i \lambda > v_i(A_0(x)) = v_i(\alpha(\delta)) \). Therefore one can take \( u_e = \text{residue of } (\frac{\alpha_{m-i}(h)}{h}) \text{ in view of Theorem 1.A(b) and denoting } (\frac{\alpha_{m-i}(h)}{h}) \text{ by } Z, \) we see that
\[ Z^t + \left( \frac{\alpha_{s-t-1}(h)}{h} \right) Z^{t-1} + \cdots + \left( \frac{\alpha_0(h)}{h} \right) = T(Z) \]
and hence the lemma.

**Lemma 2.3.** Let \( \phi(x), \alpha \) be as in Lemma 2.1 and \( F(x) \) belonging to \( R[x] \) be a polynomial having \( (\phi(x))-\)expansion \( A_0(x)\phi(x)^s + A_{s-1}(x)\phi(x)^{s-1} + \cdots + A_0(x), A_0(x) \neq 0 \). Suppose that \( S \) is a side of the \( \phi \)-Newton polygon of \( F(x) \) having slope \( \lambda > 0 \). Then the minimum of the set \( M = \{ v_i(A_i(x)) + i \lambda : 0 \leq s \leq i \leq s \} \) is attained at only those indices \( i \) for which \((s - i, v(A_0(x))) = (s - j, v(A_0(x))) \) lies on \( S \).

**Proof.** Let \( j \) and \( k \) be the smallest and the largest indices at which the minimum of \( M \) is attained. Since \( v_i(A_i(x)) + i \lambda = v_i(A_0(x)) + k \lambda \) it follows that the line joining the points \((s - k, v_i(A_0(x))) \) and \((s - j, v_i(A_i(x))) \) has slope \( \lambda \). On substituting, it can be checked that a point \((s - i, v_i(A_i(x))) \) lies on \( S \) if and only if \( v_i(A_0(x)) = v_i(A_0(x)) + k \lambda = v_i(A_0(x)) + j \lambda \), which holds if and only if \( v_i(A_i(x)) + i \lambda = v_i(A_0(x)) + j \lambda \) as desired.

**Notation.** Let \((\alpha, \delta), f(x), w_{\phi}, \) and \( \lambda \) be as in Theorem 1.A. Let \( F(x) \) belonging to \( K[x] \) be a non-zero polynomial with \( (\phi(x))-\)expansion \( \sum_j A_j(x)\phi(x)^j \). We shall denote by \( I_{\delta}(F(x)), S_{\lambda}(F(x)) \) respectively the minimum and the maximum integers belonging to the set \([i \mid v_i(F(x)) = (\delta, \lambda) + i \lambda]\).

With the above notation, the following already known result will be used in the proof of Lemma 2.4 (cf. [12, Lemma 2.1]). Its proof is omitted.

**Theorem 2.8.** For any non-zero polynomials \( F(x), G(x) \) belonging to \( K[x] \), one has
\[ I_{\delta}(F(x)G(x)) = I_{\delta}(F(x)) + I_{\delta}(G(x)) \]
and \( S_{\lambda}(F(x)G(x)) = S_{\lambda}(F(x)) + S_{\lambda}(G(x)) \).

**Lemma 2.4.** Let \( (K, \cdot), \phi(x) \) be as in Lemma 2.1. Let \( F(x), G(x) \in R[x] \) be monic polynomials not divisible by \( \phi(x) \). Then the principal part of the \( \phi \)-Newton polygon of \( F(x)G(x) \) is obtained by constructing a polygonal path beginning with a point on the positive side of \( x \)-axis and using translates of edges in the \( \phi \)-Newton polygons of \( F(x), G(x) \) in the increasing order of slopes.

**Proof.** Let \( F(x) = \sum_{i=0}^s A_i(x)\phi(x)^i, G(x) = \sum_{i=0}^s B_i(x)\phi(x)^i \) be the \( (\phi(x))-\)expansions of \( F(x) \) and \( G(x) \) with \( A_i(x), B_i(x) \neq 0 \). Let \( \lambda > 0 \) be the slope of a side \( S \) of the \( \phi \)-Newton polygon of \( F(x) \) with initial point \( s \) and end point \( s - j, v_i(A_0(x)) \). Let \( \alpha \) be a root of \( \phi(x) \) and \( \delta > 0 \) be defined by \( \delta \). Then \( w_{\phi}(F(x)) = \min \{ i v_i(A_i(x)) + i \lambda \} \) in view of Theorem 1.A(a) and (5); consequently by Lemma 2.3, \( I_{\delta}(F(x)) = j + S_{\lambda}(F(x)) = k \). We first show that the \( \phi \)-Newton polygon of \( F(x)G(x) \) has a side of slope \( \lambda \) and also find the length of the horizontal projection of this side. Two cases arise:

Case I. \( \lambda \) is the slope of any side of the \( \phi \)-Newton polygon of \( G(x) \).

In this case, in view of Lemma 2.3, \( I_{\delta}(G(x)) = S_{\lambda}(G(x)) = r \text{ (say)} \). By Theorem 2.8, \( I_{\delta}(F(x)G(x)) = I_{\delta}(F(x)) + I_{\delta}(G(x)) = j + r \) and \( S_{\lambda}(F(x)G(x)) = k + r \). Therefore the \( \phi \)-Newton polygon of \( F(x)G(x) \) has a side with slope \( \lambda \) having the length of the horizontal projection equal to that of \( S \).

Case II. \( \lambda \) is the slope of some side of the \( \phi \)-Newton polygon of \( G(x) \).
Let the side $S'$ of the $\phi$-Newton polygon of $G(x)$ with slope $\lambda$ have initial point $(t - \tilde{z}_1, v^*(B_i(x)))$ and end point $(t - j_1, v^*(B_j(x)))$. Therefore by Lemma 2.3, $\ell_{j_1}(G(x)) = f_j$, $\ell_{j_1}(F(x)G(x)) = f_j + j_1$, $\ell_{j_1}(F(x)G(x)) = k + k_i$. So the $\phi$-Newton polygon of $F(x)G(x)$ has a side of slope $\lambda$ whose length of the horizontal projection is equal to the sum of the lengths of the horizontal projections of $S$ and $S'$.

The proof of the lemma is complete once we show that if a real number $\lambda > 0$ is the slope of a side $S'$ of the $\phi$-Newton polygon of $F(x)G(x)$, then either the $\phi$-Newton polygon of $F(x)$ or of $G(x)$ has a side with slope $\lambda$. If $\lambda$ denotes the length of the horizontal projection of $S'$ and $\delta$ is as defined by (6), then by Lemma 2.3, $\ell_{j_1}(F(x)G(x)) - \ell_{j_1}(F(x)G(x)) = l > 0$. So in view of Theorem 2.B, either $\ell_{j_1}(F(x)) - \ell_{j_1}(F(x)) > 0$ or $\ell_{j_1}(G(x)) - \ell_{j_1}(G(x)) > 0$ which proves that either $F(x)$ or $G(x)$ has a side of slope $\lambda > 0$.

**Corollary 2.5.** Let $\phi(x)$, $F(x)$ and $G(x)$ be as in the above lemma. Assume in addition that $\phi(x) >> G(x)$. Then the sides of positive slopes of the $\phi$-Newton polygon of $F(x)G(x)$ are translates of the sides of positive slopes of the $\phi$-Newton polygon of $F(x)$.

**Proof.** Keeping in mind that the $\phi$-Newton polygon of $G(x)$ is a horizontal line segment, the corollary follows immediately from the above lemma.

The first four assertions of the following proposition are proved in [3, Proposition 2.3]. We omit their proofs and prove only the last assertion which improves assertions (ii) and (iii).

**Proposition 2.6.** Let $(K, v)$ be a henselian valued field of arbitrary rank and $(\alpha, \delta)$ be a $(K, v)$-minimal pair. Let $f(x)$ be the minimal polynomial of $\alpha$ over $K$ of degree $m$ and $\lambda, \epsilon, \delta > 0$. Let $g(x) \in K[x]$ be a lifting of a monic polynomial $T(Y)$ not divisible by $v$ of degree $t$ belonging to $K(\alpha)[Y]$ with respect to $(\alpha, \delta)$. Then the following hold:

(i) $v(\theta - \alpha) < \delta$ for each root $\theta$ of $g(x)$.

(ii) There exists a root $\theta_i$ of $g(x)$ such that $v(\theta_i - \alpha) = \delta$.

(iii) If $t$ is as in (ii), then $v(f(\theta_i)) = \lambda$.

(iv) If $q$ is as in (ii), then $v(f(\theta_i))$ is a root of $T(Y)$.

(v) Given any root $\theta$ of $g(x)$, there exists a $K$-conjugate $\theta'$ of $\theta$ such that $v(\theta' - \alpha) = \delta$ and $v(f(\theta')) = v(f(\theta)) = \lambda$.

**Proof.** Since $g(x)$ is a lifting of a monic polynomial $T(Y)$ not divisible by $v$, it is clear that $f(x) \mid g(x)$. Suppose to the contrary that the first part of assertion (v) is false, then in view of assertion (ii), there exists a root $\theta$ of $g(x)$ such that $v(\theta - \alpha) < \delta$ for every $K$-conjugate $\theta'$. So if $g(x) = \prod_i(x - \theta_i)$ denotes the minimal polynomial of $\theta$ over $K$, then on writing $g(x) = \prod_i(1 + x - \theta_i)$ and denoting the $\varepsilon_{\theta, \delta}$-residue of an element $f \in K(\alpha)$ by $\varepsilon_i$, we see that

$$\left(\frac{g(x)}{G_i(\alpha)}\right) = 1. \tag{7}$$

Let $r$ be a positive integer such that $r \varepsilon_{\theta_i}(\alpha) \in G_i$. Say $r \varepsilon_{\theta_i}(\alpha) = -v(\alpha)$. Write

$$\left(\frac{g(x)}{h(x)}\right) = \left(\frac{g(x)/G_i(x)}{ah(x)/G_i(x)}\right)^r \left(\frac{G_i(x)}{G_i(\alpha)}\right)^r \left(\frac{G_i(x)}{G_i(\alpha)}\right)^r.$$

On taking the $\varepsilon_{\theta, \delta}$-residue and using (7), we see that

$$\left(\frac{g(x)}{h(x)}\right) = \left(\frac{g(x)/G_i(x)}{ah(x)/G_i(x)}\right)^r \left(\frac{G_i(x)}{G_i(\alpha)}\right)^r.$$  

Keeping in mind that the degree of the polynomial $g(x)/G_i(x) < \deg g(x) = \text{etm}$, the above equation implies that $\left(\frac{g(x)/G_i(x)}{h(x)}\right)^r$ is a polynomial in $\frac{f(x)}{h(x)}$ over $K(\alpha)$ of degree strictly less than $tr$, which contradicts the hypothesis that $g(x)$ is a lifting of a polynomial of degree $t$. This proves the first part of assertion (v) and hence $v(f(\theta)) = v(f(\theta')) = \lambda$ in view of assertion (iii).

**Corollary 2.7.** Let the hypothesis be as in Proposition 2.6. Assume in addition that $f(x) \in K[x]$ with $f(x)$ irreducible over $K$ and that $T(Y)$ is a power of an irreducible polynomial of degree $d$ over $K(\alpha) = K(\alpha)$, then for each root $\theta$ of $g(x)$, the degree $|K(\alpha) : K(\theta)|$ is divisible by $dm$.

**Proof.** Let $\theta$ be a root of $g(x)$ and $\theta'$ be a $K$-conjugate of $\theta$ with $v(\theta' - \alpha) = \delta$. Then by assertion (iv) of Proposition 2.6, $\left(\frac{G_i(x)}{G_i(\alpha)}\right)^r$ is a root of $T(Y)$. Clearly the corollary follows immediately once we prove that $\left(\frac{h(x)}{h(\alpha)}\right)^r = \delta$. For this write $h(x) = \prod_i(x - \theta_i)$. Then $h(x)/h(\alpha) = \prod_i(1 + x - \theta_i)$ and in view of assertion (v),

$$\varepsilon_{\theta, \delta} \left(\frac{h(x)}{h(\alpha)}\right)^r > 0.$$  

As desired.
The result stated below proved in [13] will be used in the proof of Theorem 1.1.

**Theorem 2.C.** Let \( (k, \nu) \) be a complete rank-1 valued field with value group \( G_{k} \) and \((\alpha, \delta)\), \(u_{\nu, k}, f(x), m, \lambda \) and \(e\) be as in Theorem 1.A. Assume that \( e \delta \) belongs to \( G_{k}\), with \( e \delta = \nu(h) \) for some \( h \in K\). Let \( Z \) denote the \( u_{\nu, k}\)-residue of \( f(x)^{\nu}/h \) and \( F(x) \) belonging to \( K[x] \) be such that \( u_{\nu, k}(F(x)) = 0 \). If the \( u_{\nu, k}\)-residue of \( F(x) \) is the product of two coprime polynomials \( T(Z), U(Z) \) belonging to \( K[\alpha][Z] \) with \( T(Z) \) monic of degree \( t \geq 1 \), then there exist \( G(x), H(x) \in K[x] \) such that \( F(x) = G(x)H(x) \), deg\( G(x) = em \) and the \( u_{\nu, k}\)-residue of \( G(x) \), \( H(x) \) are \( T(Z), U(Z) \) respectively. Further if \( T(Z) \neq Z \) is irreducible over \( K(\alpha) \), then \( G(x) \) is irreducible over \( K(\alpha) \).

We prove the foregoing corollary needed in the sequel.

**Corollary 2.D.** Let \((k, \nu)\), \((\alpha, \delta)\), \(f(x), e \) and \(h \) be as in Theorem 2.C. Let \( T(Y) \in K(\alpha)[Y] \) be a polynomial which is a product of two coprime non-constant monic polynomials \( U_{1}(Y), U_{2}(Y) \in K(\alpha)[Y] \). If \( F(x) \) belonging to \( K[x] \) is a monic polynomial which is a lifting of \( T(Y) \) with respect to \((\alpha, \delta)\), then \( F(x) \) factors over \( K \) as \( F_{1}(x)F_{2}(x) \), where \( F_{1}(x) \) is a lifting of \( U_{1}(Y) \) with respect to \((\alpha, \delta)\). Let \( t = 1, 2 \).

**Proof.** Let \( \nu_{1}, \nu_{2} \) denote the degrees of \( T(Y) \), \( U_{1}(Y), U_{2}(Y) \) respectively. Let \( Z \) denote the \( u_{\nu, k} \)-residue of \( f(x)^{\nu}/h \). Applying Theorem 2.C to \( F(x)/h^{i} \), we see that \( F(x)/h^{i} = G_{1}(x)G_{2}(x) \) where \( G_{1}(x), G_{2}(x) \in K[x] \) are polynomials of degree \( e\nu_{1}m, e\nu_{2}m \) respectively with \( G_{1}(x) = U_{1}(Z) \), \( G_{2}(x) = U_{2}(Z) \). Let \( \alpha_{1} \) denote the leading coefficient of \( G_{1}(x) \). Since \( U_{1}(Z) \) is monic, it can be easily seen that \( \alpha_{1}h^{(\nu_{1}m)} = 1 \). The corollary follows immediately if we take \( F_{1}(x) = \alpha_{1}^{-1}G_{1}(x) \) and \( F_{2}(x) = \alpha_{1}h^{\nu_{1}m}G_{2}(x) \).

**3. Proof of Theorem 1.1**

We prove assertions (i), (ii), (iii) of the theorem by induction on \( k \) = the number of sides of the \( \phi \)-Newton polygon of \( F(x) \). For \( k = 1 \), let \( \lambda_{1} \) denote the slope of the single side of the \( \phi \)-Newton polygon of \( F(x) \), which must be positive as \( F(x) = \phi(x)^{i_{1}} + \sum_{i_{2}=1}^{m} \phi(x)^{i_{2}} \phi(x)^{i_{1}} \) with \( \nu(i_{2}) \alpha_{1}(x) \) for every \( i_{1} \). Let \( \lambda_{1} \) denote the smallest positive integer such that \( \nu\lambda_{1} = \nu(h) \) for some \( h \in K \) and \( t_{1} \) the number \( s/\nu_{1} \). By Lemma 2.2, \( F(x) \) is a lifting of the polynomial \( T(Y) = Y^{t_{1}} + \left( \frac{A_{t_{1}}h^{(\nu_{1}m)}}{h^{t_{i}}} \right) Y^{t_{2}} + \cdots + \left( \frac{A_{t_{1}}h^{(\nu_{1}m)}}{h^{t_{i}}} \right) \) with respect to \( \phi(\lambda) \), \( \lambda_{1} \). Therefore by Proposition 2.6(v), for each root \( \theta \) of \( F(x) \), \( \nu(\theta(\lambda)) = \lambda_{1} \). Thus assertions (i), (ii), (iii) of the theorem are proved for \( k = 1 \).

Suppose that \( k \geq 2 \) and let \( \lambda_{k} = \lambda \) (say) denote the slope of the side \( S_{k} \) of the \( \phi \)-Newton polygon of \( F(x) \) having the largest slope. Let \( e \) denote the smallest positive integer such that \( e\nu_{1} = \nu(h) \in G_{k} \), for some \( h \in K \). Let \( \delta > 0 \) be defined by (6), then \((\alpha, \delta)\) is a \((K, \nu)\)-minimal pair. Using Theorem 1.A(a) and (5), we see that \( u_{\nu, k}(F(x)) = \min_{i \leq 1} \{ \nu(\lambda_{i}A(x)) + \nu(i) \} \).

By virtue of Lemma 2.3, the above minimum is attained at only those indices \( i \) for which the point \((s - i, \nu(\lambda_{i}A(x))) \) lies on the side \( S_{k} \). Let \( l \) denote the length of the horizontal projection of \( S_{k} \). Since \((s - i, \nu(\lambda_{i}A(x))) \) is the terminal point of \( S_{i} \), its initial point must be \((s - l, \nu(\lambda_{i}A(x))) \). Therefore

\[
\nu_{i_{1}}, \lambda_{k}(F(x)) = i_{1}(\lambda_{i}A(x)) + \lambda_{k} = i_{1}(\lambda_{i}A(x)) = \iota(\lambda)(\text{say}) \tag{8}
\]

and

\[
i_{1}(\lambda_{i}A(x)) : Z : u_{\nu, k}(F(x)) \quad \text{if} \quad e \mid i. \tag{9}
\]

Clearly (8) implies that \( D \in G_{e} \), and hence \( e[l, \lambda] = l \nu_{1} \). Keeping in mind (8), (9) and denoting the \( u_{\nu, k} \)-residue of \( \phi(x)^{i_{1}}/h \) by \( Z \), we see that

\[
\left( \frac{T(\lambda_{k})}{\nu_{1}} \right) = \left( \frac{A_{t_{1}}h^{t_{i}}}{h^{t_{i}}} \right) \nu_{1} = \left( \frac{A_{t_{1}}h^{t_{i}}}{h^{t_{i}}} \right) Z + \cdots + \left( \frac{A_{t_{1}}h^{t_{i}}}{h^{t_{i}}} \right) Z^{(2)} \nu_{1}.
\]

Using Theorem 1.A(b), it follows that

\[
\left( \frac{T(\lambda_{k})}{\nu_{1}} \right) = \left( \frac{A_{t_{1}}h^{t_{i}}}{h^{t_{i}}} \right) D(Z), \quad \text{where} \quad D(Z) \in K(\alpha)[Z] \}
\]

is a monic polynomial of degree \( d \). Applying Theorem 2.C, we see that \( F(x)/h^{i} \) has a factor \( G(x) \) with \( \phi(x) \)-expansion \( \sum_{i_{2}=1}^{m} G_{i}(x)\phi(x)^{i_{2}} \) belonging to \( K[x] \) of degree \( \nu_{1}m \). Having \( \nu_{1}m \)-valuation equal to \( \nu_{1}m \) and \( u_{\nu, k} \)-residue \( D(Z) \). Then the leading coefficient \( g_{0} \) of \( G(x) \) must have valuation \( -ed \nu_{1} = -el \) and \( \nu_{1}(G_{1}(x)) + \nu_{1}(\nu_{1}m) \geq 0 \). Therefore \( \nu_{1}(G_{1}(x)) + \nu_{1}(\nu_{1}m) \geq 0 \). The polynomial \( \nu_{1}(G_{1}(x)) + \nu_{1}(\nu_{1}m) \geq 0 \). So \( g_{0}(G_{1}(x)) = F_{1}(x) \) (say) belongs to \( K[x] \). It is easily seen that \( F_{1}(x) \) is a lifting of \( D(Z) \) with respect to the minimal pair \((\alpha, \delta)\). If \( \lambda_{k} \) is a root of \( F_{1}(x) \), then by Proposition 2.6(i), \( \nu(\theta(\lambda_{i})) = \lambda_{k} \). Now by Lemma 2.1, the \( \phi \)-Newton polygon of \( F_{1}(x) \) consists of a single side whose slope is \( \lambda_{k} \) and the length of its horizontal projection equal to \( l \). Therefore it is a translate of \( S_{k} \). By Lemma 2.4, the polynomial \( F(x)/F_{1}(x) \) has \( k - 1 \) sides with slopes \( \lambda_{1} < \cdots < \lambda_{k-1} \) which are translates of \( S_{1}, \ldots, S_{k-1} \) respectively. Therefore by induction hypothesis applied to \( F(x)/F_{1}(x) \), assertions (i)–(iii) of the theorem follow. Assertion (iv) is obtained by successive application of Corollary 2.D to polynomials \( F_{i}(x) \) and then applying Corollary 2.7.
4. Proof of Theorem 1.2, Corollaries 1.3 and 1.4

Using Hensel’s Lemma, Theorem 1.2 follows quickly from Theorem 1.1. Consider the polynomial $F(x)$, which is a Schönhem polynomial with respect to $x^2 + 1$ and the $p$-adic valuation. Hence it has only one side of positive slope which is $\frac{1}{2}$. We can easily verify that the $p$-Newton polygon of $F(x)$ is a Schönhem polygon with respect to $x^2 + 1$ and the $p$-adic valuation. Therefore, it is irreducible over $K$. Let $v_1$ be a root of $F(x)$, and let $A_{K'}$ denote the ring of algebraic integers of $K' = K(v_1)$. Let $\mathfrak{p}$ be a prime ideal of $K'$. We want to determine how $\mathfrak{p}$ splits in $A_{K'}$. It can be easily seen that the factorization of $F(x)$ into powers of irreducible polynomials over the residue field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is given by $x^2 + 1 \equiv x^2 \pmod{p}$.

The $0$-Newton polygon of $F(x)$ with respect to $x^2 + 1$ is divisible by $2$. It can be easily seen that the factorization of $F(x)$ into powers of irreducible polynomials over the residue field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is given by $x^2 + 1 \equiv x^2 \pmod{p}$.