ANNEXURE

CONDITIONS ACROSS A STANDING NORMAL SHOCK WAVE

Relationship between Laboratory Fixed and Shock Fixed Coordinates

Mathematically it is easier to analyze the shock wave in shock fixed coordinates rather than laboratory fixed one. The normal shock wave is shown in Fig.8.1.

![Standing Normal Shock Wave](image)

Fig 8.1: Flow through a Standing Normal Shock Wave

The conservation of mass, momentum and energy for the flow through the shockwave are given by

\[ \rho_1 V_1 = \rho_2 V_2 \quad (8.1) \]

\[ p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \quad (8.2) \]

\[ h_1 + \frac{1}{2} V_1^2 = h_2 + \frac{1}{2} V_2^2 \quad (8.3) \]

Assuming perfect gas with a ratio of specific heat \( \gamma \) as constant, then the enthalpy can be expressed as

\[ h = C_p T = \frac{\gamma}{\gamma - 1} RT, \quad \text{but} \quad R = \frac{P}{\rho T}, \quad \text{substituting for} \ R \]
Substituting for \( h \) from equation (8.4) into equation (8.3) gives:

\[
\frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} \frac{V_1^2}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} \frac{V_2^2}{\gamma - 1}
\]

(8.5)

Eliminating the velocities \( V_1 \) and \( V_2 \) from Equations (8.1), (8.2), and (8.5) yields

\[
\frac{\rho_2}{\rho_1} = \frac{\gamma - 1}{\gamma - 1} \frac{p_2}{p_1} + 1
\]

Further simplified as:

\[
\frac{p_2}{p_1} = \frac{1 - \frac{\gamma - 1}{\gamma + 1} \frac{p_1}{\rho_1}}{\frac{\rho_1}{\rho_2} - \frac{\gamma - 1}{\gamma + 1}}
\]

(8.7)

Expressions 8.6 and 8.7 are known as the Rankine-Hugoniot equations. By combining the equations 8.1 and 8.2 and solving for the pressure ratio

\[
\frac{p_2}{p_1} = 1 + \frac{\rho_1}{\rho_2} \frac{V_1^2}{(1 - \frac{\rho_1}{\rho_2})}
\]

(8.8)

The Mach number of the flow in region 1 was defined by:

\[
M_1 = \frac{V_1}{\sqrt{\frac{\gamma}{\rho} \frac{p}{\rho}}}
\]

and

\[
a_1 = \sqrt{\gamma R T} = \sqrt{\frac{\gamma p}{\rho}}
\]
Substituting the above relation in equation 8.8 and simplifying, yields

\[ \frac{p_2}{p_1} = 1 + \gamma M_1^2 (1 - \frac{\rho_1}{\rho_2}) \]  

(8.10)

Combining the equations 8.6 and 8.10 the pressure ratio equation becomes

\[ \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1) \] 

(8.11)

Simplifying the equation 8.10 and 8.11 the density ratios is obtained;

\[ \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \]  

(8.12)

From equations 8.11 and 8.12 and using the ideal gas equation of state to obtain the temperature ratios as

\[ \frac{T_2}{T_1} = 1 + \frac{2(\gamma - 1) \gamma M_1^2 + 1}{(\gamma + 1)^2 M_1^2} (M_1^2 - 1) \]  

(8.13)

Equations 8.11, 8.12 and 8.13 are the relations in terms of flow properties relating the upstream and downstream of normal shock wave.
Expression for laboratory fixed coordinates

Laboratory Fixed Coordinate

\[ U_2 \rightarrow U_1 \rightarrow U_3 \]

Shock wave (velocity = 0)

Shock Fixed Coordinate

\[ V_2 = U_s - U_2 \]

\[ V_1 = U_s - U_1 \]

\[ V_1 = U_s \]

Fig. 8.2 Shock Wave in Different Coordinate Systems

Fig. 8.2 shows the laboratory fixed and shock wave fixed coordinates. When a normal shock wave propagates into a gas at rest with a velocity \( U_s \) then in shock fixed coordinates we have \( V_1 = U_s \) and Equation (8.9) becomes:

\[ M_1 = \frac{V_1}{a_1} = \frac{U_s}{a_1} = M_s \]  \hspace{1cm} (8.14)

The conditions across a propagating shock in laboratory fixed coordinates shown in equations 8.11, 8.12 and 8.13 and replacing \( M_1 \) by \( M_s \). In order to determine the conditions behind a moving normal shockwave, the conservation of mass should be expressed in terms of coordinates fixed in the laboratory. If \( U \) denote a velocity in the laboratory system and \( V \) denote a velocity in the shock system, \( V_1 \) is given by

\[ V_1 = U_s - U_1 \]  \hspace{1cm} (8.15)
\[ V_2 = U_s - U_2 \]  
(8.16)

Where \( U_s \) denotes the shock velocity.

Substitution of Equations (8.15) and (8.16) into (8.1) gives:

\[ \rho_1(U_s - U_1) = \rho_2(U_s - U_2) \]  
(8.17)

But for the above case \( U_1 = 0 \), the equation 8.17 for velocity \( U_2 \) can be written as:

\[ U_2 = U_s (1-\rho_1/\rho_2) \]  
(8.18)

\[ = a_1 M_s(1-\rho_1/\rho_2). \]

Substitution of 8.12 and 8.14 in equation 8.18 gives:

\[ U_2 = \frac{2a_1}{\gamma + 1} \left( M_s - \frac{1}{M_s} \right) \]  
(8.19)

The equation 8.19 gives the velocity \( U_2 \) imparted to the gas if the speed of sound \( a_1 \) of the gas and the shock propagation Mach number (\( M_s \)) are known. Simplifying the equation 8.19 by dividing either side of the equation from the velocity of sound in region 2 to obtain an expression

\[ M_2 = \frac{2}{\gamma + 1} \left[ M_s - \frac{1}{M_s} \right] a_1 a_2 \]  
(8.20)

Where \( M_2 = U_2/a_2 \). Using the relations the \( \gamma_1 = \gamma_2 \) and \( a^2 = \gamma RT \), Equation (8.20) is expressed as:

\[ M_2 = \frac{2}{\gamma + 1} \left[ M_s - \frac{1}{M_s} \right] \left( \frac{T_1}{T_2} \right)^{\frac{\gamma}{2}} \]  
(8.21)

Substituting Equation 8.13 in 8.21 to eliminates the temperature ratio and obtains the relation for \( M_2 \):
\[ M_2 = \frac{2(M_s^2 - 1)}{\left[2\gamma M_s^2 - (\gamma - 1)\right]^{1/2} \left[(\gamma - 1)M_s^2 + 2\right]^{1/2}} \]

(8.22)

The x-t Diagram and the Relationship between \( M_s \) and the valve Pressure ratio

The basic parameter of the shock tube is the valve pressure ratio \( p_d/p_1 \) ratio of the driver to the driven pressure when the valve opens, (see Figure 8.4). The temperatures of the fluid in both the sections are at \( T_1 \) and \( T_4 \) and gas constants are \( R_1 \) and \( R_4 \). When the valve is allowed to eject the rubber ball a normal shock wave is formed. The shock propagates into the low pressure section shown in fig. 8.4. The normal shock wave moves at a speed \( U_s \) and the generated shockwave front of the expansion wave at speed \( a_4 \).

![Fig. 8.3 x - t Diagram](Ref: Shock Tube by Grazia Lamanna University of Stuttgart)
Region 1 (Figure 8.4) denotes the undisturbed gas in the driven section ahead of the normal shock wave and region 4 denotes the undisturbed gas in the driver section ahead of the expansion wave. Region 2 denotes the condition of the fluid traversed by the normal shock and region 3 represents the fluid through which the expansion wave has passed through. The boundary between region 2 and 3 is called the contact surface and it separates the fluid. Along the contact surface the pressures and velocities are equal:

\[ U_3 = U_2; \quad p_3 = p_2 \]  

(8.23)

The temperatures \( T_2 \) and \( T_3 \) and the densities \( \rho_2 \) and \( \rho_3 \) may, however, be different. The expansion of the shock wave occurs isentropically which is conserved given by

\[ \frac{2a}{\gamma - 1} + U = \text{Constant} \]

Hence across the region 4 to region 3

\[ \frac{2a_4}{\gamma_4 - 1} + U_4 = \frac{2a_3}{\gamma_3 - 1} + U_3 \]  

(8.24)

Further \( U_4 = 0 \), as the unexpanded driver gas in region 4 will be at rest. For an ideal gas \( \gamma_4 = \gamma_3 \) substituting these quantities in with Equation 8.23 and 8.24 we get;

\[ \frac{2}{\gamma_4 - 1} a_4 = \frac{2}{\gamma_3 - 1} a_3 + U_2 \]  

(8.25)

The speed of sound in region 3 and 4 is given by:

\[ a_3 = \sqrt{\gamma_3 R_3 T_3} \quad \text{and} \quad a_4 = \sqrt{\gamma_4 R_4 T_4} \]  

(8.26)

Thus the temperature ratio across the expansion wave is;

\[ \frac{T_4}{T_3} = \left( \frac{a_4}{a_3} \right)^2 \]  

(8.27)

Similarly from the isentropic relation \( p\rho^\gamma = \text{constant} \), the equation of state \( P/\rho = RT \), and equation 8.27 is written as:
\[ \frac{P_4}{P_3} = \left( \frac{a_4}{a_3} \right)^{2\gamma_4/(\gamma_4 - 1)} = \frac{P_4}{P_2} \quad (8.28) \]

From equations (8.25) and (8.28) and simplifying yields

\[ \frac{P_4}{P_2} = \left[ \frac{a_4}{a_4 - \frac{\gamma_4 - 1}{2} U_2} \right]^{2\gamma_4/(\gamma_4 - 1)} \quad (8.29) \]

From the equation 8.19 for the propagation of a normal shock wave into a gas initially at rest, the velocity of the gas behind the shock wave \( U_2 \) is obtained as:

\[ U_2 = \frac{2a_1}{\gamma_1 + 1} \left[ M_s - \frac{1}{M_s} \right] \quad (8.30) \]

Combining Equation (8.29) with (8.30) we obtain an important relationship for the dependence of the shock speed \( M_s \) on the initial valve pressure ratio. Note that \( p_4/p_1 = (p_4/p_2) (p_2/p_1) \), where \( p_2/p_1 \) comes from Equation (8.11) and (8.14), so that

\[ \frac{P_4}{P_1} = \frac{2\gamma_4 M_s^2}{\gamma_1 + 1} \left[ 1 - \frac{\gamma_4 - 1}{\gamma_1 + 1} \frac{a_4}{a_4 - \frac{1}{2} M_s} \left( M_s - \frac{1}{M_s} \right) \right]^{2\gamma_4/(\gamma_4 - 1)} \quad (8.31) \]

The above equation relates the shock wave strength on Mach number \( M_s \) of the normal shock to the initial pressure ratio across the diaphragm. For a given \( p_4 \) and \( p_1 \), the normal shock strength obtained in a given gas depends on the specific heat ratio and the sound speed in the high-pressure gas. The strongest possible shock is obtained when \( p_4/p_1 \to \infty \). The shock Mach number remains finite as the pressure ratio goes to infinity - the right-hand side of the equation goes to infinity as the term inside the square brackets goes to zero, which occurs at finite Mach number. The strongest shocks are obtained by using a driver gas with a high speed of sound and a low specific heat ratio. For these reasons a low-density gas such as hydrogen or helium is most frequently used as a driver gas. By increasing the temperature, and hence the speed of sound of the driver gas, the shock strength
may be increased. For this purpose some shock tubes incorporate facilities to heat the driver gas. Shock tubes that can create very strong shocks are used for studying hypersonic flows such as reentry flow.

Analysis of Reflected Normal Shock Waves

The equations of reflected shock waves do not readily give solutions for the physical parameters behind the reflected shock in terms of the initial conditions in the tube. It is convenient, and illustrative to derive the pressure, density, and temperature for the reflected shock in terms of the conditions from the incident one. The equations derived with these properties may then be related to the initial conditions by the way of Mach number or speed of the incident shock.

Incident

\begin{align*}
V_{11} &\quad (2) \\
\quad &\quad (1) \\
&\quad \quad \quad U_s \\
&\quad \quad \quad S_1 \\
&\quad \quad \quad (a_1) \\
V_{11} = U_s U_2 &\quad (2) \\
\quad &\quad (1) \\
&\quad \quad \quad V_{11} = U_s U_1 = U_s \\
&\quad \quad \quad S_1 \\
&\quad \quad \quad (a_1) = \text{Incident shock}
\end{align*}

Reflected shock

\begin{align*}
U_r &\quad (2) \\
U_2 &\quad (5) \\
&\quad \quad \quad U_s = 0 \\
&\quad \quad \quad S_r \\
&\quad \quad \quad (b_1) \\
V_{12} = U_r U_2 &\quad (2) \\
\quad &\quad (5) \\
&\quad \quad \quad V_{12} = U_r U_s = U_r \\
&\quad \quad \quad S_r \\
&\quad \quad \quad (b_2 = \text{reflected shock fixed})
\end{align*}

(a_i) and (b_i) are in laboratory fixed coordinates (U_r is the velocity of the reflected normal shock wave in laboratory fixed coordinates)

(a_i) and (b_i) are in shock fixed coordinates (A subscript '1' or '2' refers to incident or reflected shock fixed coordinates respectively)

Fig.8. 4 Flow situation before and after the reflection of a shock wave from a rigid wall. Region 5 is behind the reflected shock wave.
Fig. 8.4 shows the relevant particle-flow velocities associated with the reflection of a normal shock wave in a uniform tube, from an end wall. When the normal shock front $S_1$, (Fig. 8.4a), traveling at the velocity $U_s$ relative to the tube, hits the end wall, it is reflected as the normal shock front $S_R$ (Fig 8.4b) which travels back into region (2) with a diminished speed $U_R$. The gas flows into this front with a relative velocity $V_{r_1}$, where $V_{r_1} = U_R + U_2$. Since after a normal reflection the particle velocity relative to the wall must be zero ($U_5 = 0$), the gas gives up the whole of its kinetic energy on passing through the front $S_R$ into the region of the reflected normal shock, thus increasing the properties of state of the gas in this region, above those in the incident region. The reflected shock thus can be envisaged as an extending column which advances out from the end wall of the tube and contains a static gas with high temperature, density, and pressure.

Referring to Fig. 8.4, the flow velocities relative to the reflected normal shock front may be written as:

\[ V_{r_1} = U_R \]  
\[ V_{r_2} = U_R + U_2 \]  

Thus, the Mach number of the gas ahead of the reflected shock is defined by:

\[ M_{R_2} = \frac{V_{r_2}}{a_2} \]  

NOTE: $M_{R_2} \neq M_R$

The conservation of mass for the flow through the reflected wave is written as:

\[ \rho_2 V_{r_2} = \rho_5 V_{r_5} \]  

Substitution of Equations 8.32 and 8.33 into Equation 8.35 yield:
\[ \rho_2(U_R + U_2) = \rho_5 U_R \]  
(8.36)

Simplifying equation 8.36 and rewritten in the form:

\[ U_2 = \left( U_R + U_2 \right) \left[ 1 - \frac{\rho_2}{\rho_5} \right] \]  
(8.37)

Multiplying and dividing the right hand side of equation (8.31) by \( a_2 \) and making use of the definition of \( M_{R2} \), the above equation is written as:

\[ U_2 = a_2 M_{R2} \left[ 1 - \frac{\rho_2}{\rho_5} \right] \]  
(8.38)

\( \rho_2/\rho_5 \) may be found by relating reflected shock to equation (8.12):

\[ \frac{\rho_2}{\rho_5} = \frac{(\gamma - 1) M_{R2}^2 + 2}{(\gamma + 1) M_{R2}^2} \]  
(8.39)

Thus substitution of equation (8.38) into (8.39) and using (8.19) gives the relation:

\[ U_2 = \frac{2a_2}{\gamma + 1} \left[ M_{R2} - \frac{1}{M_{R2}} \right] = \frac{2a_1}{\gamma + 1} \left[ M_s - \frac{1}{M_s} \right] \]  
(8.40)

Using the relation that \( a^2 = \gamma RT \) and the equation of state gives:

\[ \left( \frac{a_2}{a_1} \right)^2 = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2} \]  
(8.41)

Substituting for \( \rho_1/\rho_2 \) from equation (8.6) the above equation is written as
From the normal shock relations the ratio of pressure is written as:

\[
p_2 = \frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1} \frac{p_1}{p_2}
\]  \hspace{1cm} (8.43)

and the reflected shock pressure ratio becomes:

\[
p_5 = \frac{2\gamma M_{r_2}^2 - (\gamma - 1)}{\gamma + 1} \frac{p_2}{p_5}
\]  \hspace{1cm} (8.44)

Solving Equations (8.40), (8.41), (8.42), (8.43), and (8.44) for the pressure ratio \( p_5 / p_2 \):

\[
\frac{p_5}{p_2} = \frac{\gamma + 1}{\gamma - 1} \frac{1 + \frac{\gamma + 1}{\gamma - 1} \frac{p_1}{p_2}}{1 + \frac{\gamma + 1}{\gamma - 1} \frac{p_1}{p_2}}
\]  \hspace{1cm} (8.45)

The temperature ratio is found by using the equation of state to obtain:

\[
\frac{T_5}{T_2} = \frac{p_5}{p_2} \frac{\rho_2}{\rho_5}
\]  \hspace{1cm} (8.46)

where \( \rho_2 / \rho_5 \) is given by Equation (8.39) and \( p_5 / p_2 \) by Equation (8.45)
Equations (8.39) - (8.46) can be combined to obtain an equation for the pressure and temperature behind the reflected shock, in terms of the incident Mach number and initial pressure and temperature;

\[
\frac{p_s}{p_i} = \left[ \frac{2\gamma M_s^2 - (\gamma - 1)}{\gamma + 1} \right] \left[ \frac{(3\gamma - 1)M_s^2 - 2(\gamma - 1)}{(\gamma - 1)M_s^2 + 2} \right] \quad (8.47)
\]

\[
\frac{T_s}{T_i} = \frac{(2(\gamma - 1)M_s^2 + (3 - \gamma))(3\gamma - 1)M_s^2 - 2(\gamma - 1))}{(\gamma + 1)^2 M_s^2} \quad (8.48)
\]

Expansion Wave Reflection: The expansion wave is reflected from the driver (or top) section of the shock tube. The flow will be isentropically expanded between Regions 3 and 5.