CHAPTER-IV

SIMULTANEOUS ONE-SIDED CONFIDENCE INTERVALS
FOR THE ORDERED PAIRWISE DIFFERENCES OF
EXPONENTIAL LOCATION PARAMETERS

In this Chapter we consider \( k (k=3) \) exponential populations such that an observation from the \( i \)th population has probability density function (pdf)
\[
f(x|\mu_i, \theta_i) = \frac{1}{\theta_i} \exp\left(-\frac{(x-\mu_i)}{\theta_i}\right) I_{[\mu_i, \infty)}(x),
\]
where \( \mu_i > 0, \theta_i > 0 \) and \( I(.) \) is the indicator function, \( i=1, \ldots, k \). Test procedures for testing the null hypothesis \( H_0: \mu_1 = \ldots = \mu_k \) against the simple ordered alternative hypothesis \( H_1: \mu_1 < \ldots < \mu_k \), with at least one strict inequality, are proposed in two situations:
(i) \( \theta_1 = \ldots = \theta_k = \theta \) (unknown) and (ii) all \( \theta \)'s equal to unity. For some significance levels \( \alpha \in (0,1) \), exact critical points of each test procedure are tabulated for \( k=3, \ldots, 9 \) by solving two or three dimensional integral equations. Simultaneous one-sided confidence intervals for all ordered pairwise differences \( \mu_j - \mu_i \) (\( 1 \leq i < j \leq k \)) and all nonnegative contrasts of \( \mu \)'s, obtained by simple inversion of these test procedures, are discussed using these critical points. A multiple three-decision procedure, which maintains the probability of Type-III error at level \( \alpha \), is also proposed with the help of critical points of the test procedure. Chen (1982) proposed a test procedure for this problem in situation (i) and discussed simultaneous confidence intervals (SCIs) of linear contrasts of \( \mu \)'s. Our critical points are

The contents of this chapter have appeared as a research paper (refer Dhawan and Gill (1997)).
substantially smaller than the critical points proposed by Chen (1982). Statistical simulation, used to check the performance of the proposed critical points and the computation of powers, revealed that (i) the actual size levels of our critical points are almost conservative, i.e., are almost smaller than the nominal levels and (ii) the power of the proposed test relative to Chen's test is larger particularly for small sample sizes and tight slippage parameter configurations. An application of these results to Pareto family of distributions is also discussed.

4.1 INTRODUCTION

Let $X_{i,j}$, $j=1,...,n$ be the random sample of common size $n$ drawn from the $i$th exponential population, denoted by $\pi_i = E(\mu_i, \theta_i)$, with pdf $f(x|\mu_i, \theta_i) = 1/\theta_i \exp\{- (x-\mu_i)/\theta_i\} I_{[\mu_i, \infty]}(x)$, where $\mu_i > 0$, $\theta_i > 0$ and $I(.)$ is the indicator function, $i=1,...,k$. The problem of testing the null hypothesis $H_0: \mu_1=...=\mu_k$ against the simple ordered alternative hypothesis $H_1: \mu_1 \leq ... \leq \mu_k$ with at least one strict inequality, has attracted the attention of many researchers in the recent past (see e.g., Barlow et al. (1972) and Robertson et al. (1988)). For some related problems references may be made to Paulson (1941), Sukhatme (1936), Epstein and Tsao (1953), Hogg and Tenis (1963), and Chen (1982). All these test procedures except that of Chen (1982), though each being sensitive to configuration of the location parameters satisfying the order restrictions, suffer from the disadvantage of not having a convenient inversion to a set of simultaneous confidence
intervals (SCIs) for the useful contrasts of \( \mu_s \). In dose-response experiments the exponential distribution \( E(\mu, \theta) \) is used quite oftenly to model the ‘effective duration’ of a drug, where the location parameter \( \mu \) is referred to as the guaranteed effective duration and the scale parameter is called the mean effective duration in addition to \( \mu \). Also in Biological and Epidemiological studies the location parameter \( \mu \) is called the latency period (time between the infection and the onset of the disease) and scale parameter \( \theta \) is termed as mean duration of a disease in addition to the latency period. In reliability and engineering, location parameter is known as the guaranteed life time of a component and scale parameter is called the mean life in addition to guaranteed life. Many practical situations where it is known a priori that \( \mu_1 = \ldots = \mu_k \), arise in dose-response experiments when the \( k \) treatments consist of monotonically increasing levels of the dose of a certain drug, and it is postulated that the guaranteed effective period is a nondecreasing function of the dose-level.

Assuming all \( \theta \)'s to be equal but unknown, Chen (1982) used a new range statistic (NRS) 
\[
C_{\alpha}^{\alpha} \left( \frac{\sqrt{V}/2v}{\sqrt{R}} \right)^{-1}
\]
for this testing problem, where \( R \) is the range of \( k \) independent and identically distributed (iid) random variables from \( E(0,1) \) population and \( V \) is an independent chi-squared random variable with \( 2v \) degrees of freedom. There the author have tabulated critical points \( C_{k, \nu}^\alpha \) of the NRS and proposed SCIs for all the differences \( \mu_i - \mu_j \) \( (i \neq j) \) and for the linear contrasts \( \sum_{i=1}^{k} c_i \mu_i \) with \( \sum_{i=1}^{k} c_i = 0 \). In this chapter we
propose test statistics for testing $H_0$ against $H_1$ in two situations: (i) $\theta_1=\ldots=\theta_k=\theta$ (unknown) and (ii) $\theta_1,\ldots,\theta_k$ are each equal to unity. In the former case the test statistic is $W = \left( \max_{1 \leq i < j \leq k} n(Y_j - Y_i) / T \right)$ and in the later case it is $W_0 = \max_{1 \leq i < j \leq k} n(Y_j - Y_i)$, where $Y_i = \min X_{ij}$, $T = \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - Y_i) / \nu$ and $\nu = k(n-1)$. The motivation in using the statistics $W$ and $W_0$ to propose the test procedures is that they allow simple construction of a set of exact simultaneous one-sided confidence intervals of the form $(q_{ij},\infty)$ for all ordered pair wise differences of the location parameters $\mu_j - \mu_i$ ($1 \leq i < j \leq k$) with an extension to produce simultaneous one-sided confidence intervals for all 'nonnegative' contrasts $\mu_j - \mu_i$ ($1 \leq i < j \leq k$) where $\Sigma_{i=1}^{j} c_i = 0$ and $\Sigma_{i=1}^{j} c_i < 0$ for all $1 \leq i < j < k-1$. Calculation of exact critical points of the statistics $W$ and $W_0$ is more difficult with the complexity increasing as $k$ increases. However, for $3 \leq k \leq 9$ we have used the method of Hayter (1990) to find the critical points by solving no more than a three dimensional integral equation. Proposed test procedures, based on statistics $W$ and $W_0$, and some critical points for the cases $3 \leq k \leq 9$ and $\alpha \in (0,1)$ are given in Section 4.2. Exact one-sided SCIs for all the ordered pair wise differences $\mu_j - \mu_i$ ($1 \leq i < j \leq k$) as well as for all 'nonnegative' contrasts of $\mu$s are discussed in Section 4.3. These SCIs are seen to be the shortest possible confidence intervals of this kind than those obtained
under NRS of Chen (1982). A multiple three-decision procedure, which controls the probability of Type-III error at level $\alpha$, is proposed with the help of critical points of the test procedure in Section 4.4. In Section 4.5 some simulation results, used to check the performance ability of the critical points and the computation of powers, are presented. Finally an extension of these results to Pareto family is discussed in Section 4.6.

4.2. PROPOSED TEST PROCEDURES

Using the notation given in the introduction, let the critical points $w_k, \alpha, \nu$ and $w_k, \alpha$ be defined by

$$P_0[W \geq w_k, \alpha, \nu] = \alpha$$  \hspace{1cm} (4.2.1)

and

$$P_0[W_0 \leq w_k, \alpha] = \alpha,$$  \hspace{1cm} (4.2.2)

where $P_0(\Lambda)$ indicates that probability of event $\Lambda$ is computed under $\mu_1=\ldots=\mu_k$. Test is reject $H_0$ at level $\alpha$ if and only if the underlying test statistic exceeds the corresponding critical point.

Now we will discuss the calculation of critical points $w_k, \alpha, \nu$ and $w_k, \alpha$.

4.2.1 CALCULATION OF CRITICAL POINTS

Let $Z_1, \ldots, Z_k$ be $k$ iid random variables from $E(0,1)$ population and define

$$a_k(x, y, c) = P\{x \leq Z_i < y, 1 \leq i \leq k; Z_j - Z_i \leq c, 1 \leq i < j \leq k\}.$$

Then for a given $\alpha \in (0,1)$ the critical point $w_k, \alpha, \nu = w$ can be found from the equation
\[
\int_0^\infty a_k(0, \omega, tw) f_\nu(t) dt = 1 - \alpha,
\]
where \( f_\nu(t) \) is the pdf of a gamma variate (with shape parameter \( \nu \)) divided by \( \nu \). The critical point \( w_k, \alpha = w_1 \) can be found from the equation
\[
a_k(0, \omega, w_1) = 1 - \alpha.
\]

The results of Appendix B of Hayter (1990), stated briefly in the following lemma, have been used to find convenient expression for the function \( a_k(0, \omega, c) \) with \( c > 0 \), so that they can be evaluated quickly when \( 3 \leq k \leq 9 \). For \( 10 \leq k \leq 19 \), the evaluation of critical points \( w_k, \alpha, \nu \) by this method require evaluation of four-dimensional (three-dimensional) integral equation, and have not been tried in this chapter.

**Lemma 4.2.1**: Let \( a_k(x,y,c) = a_k(x,y) \) and \( a_0(x,y) = 1 \).

Note that \( a_1(x,y) = e^{-x} e^{-y} = B(x,y) \) (say) for \( x \leq y \).

Then for any \( k, \in \mathbb{N}, r \leq k \), and \( x > 0, y > 0 \) with \( y - x = c \), we have
\[
a_k(x,y) = \int_{z=x}^{y-c} e^{-z} a_{k-1}(x,z+c) \times \\
\left[ \sum_{m=1}^{k} B^{r-m}(z,z+c) a_{m-1}(z,y) \right] dz + a_{k-1}(x,y) B^{r}(y-c,y).
\]

Proof of Lemma : Let \( Z_1, \ldots, Z_k \) be \( k \) iid standard exponential variates and define
\[
d_k(x,y;r,m) = p[x \leq Z_i \leq y, 1 \leq i \leq k, Z_j - Z_i \leq c, 1 \leq i < j \leq k; Z_m \leq Z_1, 1 \leq i \leq r].
\]

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It may be noted that \( a_k(x,y) = \sum_{m=1}^{k} d_k(x,y; r_m) \).

Now for \( 1 \leq m \leq r \), we have

\[
d_k(x,y;r,m) = \int_{z=x}^{y-c} e^{-z} a_{k-r}(x,z+c) B(z,z+c) a_{m-1}(z,y)\,dz + \int_{z=y-c}^{y} e^{-z} a_{k-r}(x,y) B_{r-1}(z,y)\,dz
\]

Above expression has been derived by conditioning on the value \( z \) of \( Z_m \). In the first integral, when \( x \leq z \leq y-c \), since \( Z_m \leq Z \) \((1 \leq m \leq r)\), the only condition on \( Z_{r+1}, \ldots, Z_k \) are that they should be in the interval \((x,Z_m+c)\) and satisfy the order requirement among themselves, which has been taken care of by term \( a_{k-r}(x,z+c) \). Furthermore, the variables \( Z_{m+1}, \ldots, Z_r \) must lie in the interval \([Z_m, Z_m+c]\), in which case the order restriction among themselves is necessarily satisfied, and hence they are counted for by the term \( B_{r-m}(z,z+c) \). Finally, regardless of the possible values of \( Z_{m+1}, \ldots, Z_k \), the only restriction on \( Z_1, \ldots, Z_{m-1} \) are that they should be in the interval \([Z_m,y]\) and satisfy the order requirement among themselves, which is accounted for by the term \( a_{m-1}(z,y) \). The second integral term is derived similarly but with \( y \) replacing \( z+c \) and \( a_{m-1}(z,y) = B_{r}(z,y) \), since \( y \geq z+c \). Therefore

\[
a_k(x,y) = \sum_{m=1}^{k} d_k(x,y;r,m)
\]
We have computed the values of $a_k(z,\infty) = a^{(z,\infty)}_k$ for $k=3(1)9$. The detailed computation is given only for $k=2,3$ and 4. For $k=2$ and $r=1$ we have

$$a_2(x,y) = \int_{z=x}^{y-c} e^{-z} a_2(x,z+c) \, dz + a_1(x,y) B(y-c,y).$$

Put $y = \infty$, we get

$$a_2(x,\infty) = \int_{x}^{\infty} e^{-z} a_1(x,z+c) \, dz + 0.$$
\[ a_2(z, \infty; c) = e^{2z} - \frac{1}{2} e^{2z - c} \]

For \( k=3 \) and \( r=2 \), we have

\[ a_3(x, y) = \int_{z=x}^{y-c} e^{-z} a_1(x, z+c) \times \left[ \sum_{m=1}^{2} B^{2-m}(z, z+c) a_{m-1}(z, y) \right] dz + a_1(x, y)B^2(y-c, y) \]

\[ = \int_{z=x}^{y-c} e^{-z} a_1(x, z+c) \left[ B(z, z+c) a_0(z, y) + a_1(z, y) \right] dz + a_1(x, y)B^2(y-c, y). \]

Put \( y=\infty \), we get

\[ a_3(x, \infty) = \int_{x}^{\infty} \left[ e^{-z} (e^{-x} - e^{-z - c}) \left( B(z, z+c) + a_1(z, \infty) \right) \right] dz \]

\[ = \int_{x}^{\infty} \left[ (e^{-z-x} - e^{-2z-c}) \left( 2e^{-z} - e^{-z-c} \right) \right] dz \]

\[ = \int_{x}^{\infty} \left( 2e^{-2z-x} - 2e^{-3z-c} - e^{-2z-x-c} + e^{-3z-2c} \right) dz \]
Now put $y = \infty$, we get

$$a(x, \infty) = \lim_{y \to \infty} \left[ e^{-2y-x} \left( -y+\infty \right) + \frac{2}{3} e^{-3y-c} \left( -y+\infty \right) + \frac{1}{2} e^{-2y-x-c} \left( -y+\infty \right) - \left( e^{-3y-c} - \frac{2}{3} e^{-3y-c} - \frac{1}{2} e^{-3y-c} + \frac{1}{3} e^{-3y-2c} \right) \right]$$

$$= e^{-3x} \left( -y+\infty \right) + \frac{2}{3} e^{-3y-c} \left( -y+\infty \right) + \frac{1}{2} e^{-2y-x-c} \left( -y+\infty \right)$$

$$= e^{-3x} - \frac{7}{6} e^{-3y-c} + \frac{1}{3} e^{-3y-2c} .$$

Therefore $a_3(z, \infty; c) = e^{-3z} - \frac{7}{6} e^{-3z-c} + \frac{1}{3} e^{-3z-2c} .$

For $k=4$ and $r=3$, we have

$$a_4(x, y) = \int_{z=x}^{y=c} e^{-z} a_1(x, z+c) \left[ \sum_{m=1}^{3} B^3 (z, z+c) a_{m-1}(z, y) \right] dz +$$

$$\int_{z=x}^{y=c} e^{-z} (e^{-x} - e^{-z-c}) \left( B^2 (z, z+c) a_0(z, y) + B(z, z+c) a_1(z, y) + a_2(z, y) \right) dz +$$

$$a_1(x, y) B^r (y-c, y).$$

Now put $y = \infty$, we get

$$a_4(x, \infty) = \int_{x}^{\infty} (e^{-x} - e^{-2y-c}) \left[ (e^{-2y} + e^{-2y-2c} - 2e^{-2y-c}) a_0(z, \infty) + (e^{-y} - e^{-y-c}) a_1(z, \infty) + a_2(z, \infty) \right] dz$$
\[
\begin{align*}
&= \int x^\infty (e^{-z-x} - e^{-2z-c}) \left( e^{-2z} + e^{-2z-2c} - 2e^{-2z-c} \right) dz \\
&\quad + e^{-2z} - e^{-2z-c} + e^{-2z} - \frac{e^{-2z-c}}{2} \right) dz \\
&= \int \left. x^3 \left( e^{-z-x} - e^{-2z-c} \right) \left( 3e^{-2z} - \frac{7}{2} e^{-2z-2c} + e^{-2z-2c} \right) \right|_{x}^{\infty} dz \\
&= \int \left. 3e^{-3z-x} - 3e^{-4z-c} - \frac{7}{2} e^{-3z-x-c} + \frac{7}{2} e^{-4z-2c} + \\
&\quad e^{-3z-x-2c} - e^{-4z-3c} \right|_{x}^{\infty} dz \\
&= -\left. e^{-3z-x} + \frac{3}{4} e^{-4z-c} + \frac{7}{6} e^{-3z-x-c} - \frac{7}{8} e^{-4z-2c} + \frac{1}{2} e^{-3z-x-2c} \right|_{x}^{\infty} \\
&\quad - \frac{1}{3} e^{-4z-3c} + \frac{1}{4} e^{-4z-3c} \\
&= e^{-4x} - \frac{3}{4} e^{-4x-c} - \frac{7}{6} e^{-4x-c} + \frac{7}{8} e^{-4x-2c} + \frac{1}{3} e^{-4x-2c} - \\
&\quad \frac{1}{4} e^{-4x-3c} \\
&= e^{-4x} - \frac{23}{12} e^{-4x-c} + \frac{29}{24} e^{-4x-2c} - \frac{1}{4} e^{-4x-3c}.
\end{align*}
\]

Therefore,

\[ a_4(z,\omega; c) = e^{-4z} - \frac{23}{12} e^{-4z-c} + \frac{29}{24} e^{-4z-2c} - \frac{1}{4} e^{-4z-3c} \]

Similarly the values of \( a_k(z,\omega; c) \) for \( k=5(1)9 \) are given below:

\[ a_5(z,\omega; c) = e^{-5z} - \frac{163}{60} e^{-5z-c} + \frac{329}{120} e^{-5z-2c} - \]

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\[ \frac{73}{60} e^{-5z-3c} + \frac{1}{5} e^{-5z-4c} \]

\[ a_6(z,w;c) = e^{-6z} - \frac{71}{20} e^{-6z-c} + \frac{901}{180} e^{-6z-2c} - \]
\[ \frac{2521}{720} e^{-6z-3c} + \frac{437}{360} e^{-6z-4c} - \frac{1}{6} e^{-6z-5c} \]

\[ a_7(z,w;c) = e^{-7z} - \frac{617}{140} e^{-7z-c} + \frac{10141}{1260} e^{-7z-2c} - \]
\[ \frac{39271}{5040} e^{-7z-3c} + \frac{5311}{1260} e^{-7z-4c} - \frac{169}{140} e^{-7z-5c} + \]

\[ \frac{1}{7} e^{-7x-6c} \]

\[ a_8(z,w;c) = e^{-8z} - \frac{1479}{280} e^{-8z-c} + \frac{119999}{10080} e^{-8z-2c} - \]
\[ \frac{49843}{3360} e^{-8z-3c} + \frac{148283}{13440} e^{-8z-4c} - \frac{9869}{2016} e^{-8z-5c} + \]

\[ \frac{1343}{1120} e^{-8z-6c} - \frac{1}{8} e^{-8z-7c} \]

\[ a_9(z,w;c) = e^{-9z} - \frac{15551}{2520} e^{-9z-c} + \frac{167327}{10080} e^{-9z-2c} - \]
\[ \frac{2305753}{90720} e^{-9z-3c} + \frac{2929523}{120960} e^{-9z-4c} - \]
\[ \frac{444601}{30240} e^{-9z-5c} + \frac{503543}{90720} e^{-9z-6c} - \frac{3001}{2520} e^{-9z-7c} + \]

\[ \frac{1}{9} e^{-9z-8c} \]

Table I contains values of critical points \( w_k, \alpha = w_1 \) with \( \alpha = .01, .05 \) and .1. These critical points have been obtained by solving the equations \( a_k(0, w_1) = 1-\alpha \), using bisection method, for \( k=3, \ldots, 9 \).
Table I

Critical Points \( w, k, \alpha \) for \( \alpha = .01, .05 \) and \( .10 \)

<table>
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<tr>
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<th>( \alpha = .01 )</th>
<th>( .05 )</th>
<th>( .10 )</th>
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<tr>
<td>9</td>
<td>6.4221</td>
<td>4.8009</td>
<td>4.0928</td>
</tr>
</tbody>
</table>

The critical points \( w, k, \alpha, \nu = w \) are computed by solving the following equations for \( k = 3(1)9 \):

\[
\int_0^{\infty} a_k(0, m, tw) g_\nu(t) dt = 1 - \alpha
\]

where \( g_\nu(t) = \frac{e^{-\nu t} \nu^\nu t^{\nu-1}}{\Gamma(\nu)} \) is the pdf of gamma variate divided by its shapes parameter \( \nu \).

Now

\[
\int_0^{\infty} a_k(0, m, tw) g_\nu(t) dt = \frac{\nu^\nu}{\Gamma(\nu)} \int_0^{\infty} a_k(0, m, tw) e^{-\nu t} t^{\nu-1} dt.
\]

Thus for \( k = 3 \)

\[
\int_0^{\infty} a_3(0, m, tw) g_\nu(t) dt =
\]

\[
= \frac{\nu^\nu}{\Gamma(\nu)} \int_0^{\infty} (1 - \frac{7}{6} e^{-tw} + \frac{1}{3} e^{-2tw}) e^{-\nu t} t^{\nu-1} dt
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^{\infty} (1 - \frac{7}{6} e^{-xw/\nu} + \frac{1}{3} e^{-2xw/\nu}) e^{-x} \frac{x^{\nu-1}}{\nu} dx \quad | \text{put } \nu t = x
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^{\infty} (1 - \frac{7}{6} e^{-xw/\nu} + \frac{1}{3} e^{-2xw/\nu}) e^{-x} x^{\nu-1} dx
\]

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\[ \int_{0}^\infty e^{-x} x^{\nu-1} \, dx = \frac{1}{\Gamma(\nu)} \int_{0}^\infty e^{-\frac{w}{\nu}} e^{-x} x^{\nu-1} \, dx + \frac{1}{3 \Gamma(\nu)} \int_{0}^\infty e^{-x(1 + \frac{2w}{\nu})} x^{\nu-1} \, dx \]

\[ = 1 - \frac{7}{6 \nu} \int_{0}^\infty e^{-x(1+w/\nu)} x^{\nu-1} \, dx + \frac{1}{3 \Gamma(\nu)} \int_{0}^\infty e^{-x(1+2w/\nu)} x^{\nu-1} \, dx \]

\[ = 1 - \frac{7}{6} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{1}{3} \left( \frac{\nu}{\nu + 2w} \right)^\nu. \]

Therefore,

\[ \int_{0}^\infty a_3(0, \infty, tw) g_\nu(t) \, dt = 1 - \alpha, \text{ gives} \]

\[ 1 - \frac{7}{6} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{1}{3} \left( \frac{\nu}{\nu + 2w} \right)^\nu = 1 - \alpha \quad \text{or} \]

\[ \alpha - \frac{7}{6} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{1}{3} \left( \frac{\nu}{\nu + 2w} \right)^\nu = 0. \quad (4.2.3) \]

For \( k=4 \), we have

\[ \int_{0}^\infty a_4(0, \infty, tw) g_\nu(t) \, dt \]

\[ = \int_{0}^\infty \left( 1 - \frac{23}{12} e^{-tw} + \frac{29}{24} e^{-2tw} - \frac{1}{4} e^{-3tw} \right) \frac{\nu^\nu}{\Gamma(\nu)} e^{-\nu t} t^{\nu-1} \, dt \]

put \( \nu t = x \), we get

\[ \int_{0}^\infty a_4(0, \infty, tw) g_\nu(t) \, dt \]

\[ = \frac{\nu^\nu}{\Gamma(\nu)} \int_{0}^\infty \left( 1 - \frac{23}{12} e^{-xw/\nu} + \frac{29}{24} e^{-2xw/\nu} - \frac{1}{4} e^{-3xw/\nu} \right) \times \]
\[
e^{-x} \frac{x^{\nu-1}}{\nu} \, dx
\]

\[
= \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-x} x^{\nu-1} \, dx - \frac{23}{12 \Gamma(\nu)} \int_0^\infty e^{-x(1+w/\nu)} x^{\nu-1} \, dx + \\
\frac{29}{24 \Gamma(\nu)} \int_0^\infty e^{-x(1+2w/\nu)} x^{\nu-1} \, dx - \frac{1}{4 \Gamma(\nu)} \int_0^\infty e^{-x(1+3w/\nu)} x^{\nu-1} \, dx
\]

\[
= 1 - \frac{23}{12} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{29}{24} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{1}{4} \left( \frac{\nu}{\nu + 3w} \right)^\nu.
\]

Therefore,

\[
\int_0^\infty a_4(0, w, tw) g_\nu(t) \, dt = 1 - \alpha, \quad \text{gives}
\]

\[
1 - \frac{23}{12} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{29}{24} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{1}{4} \left( \frac{\nu}{\nu + 3w} \right)^\nu = 1 - \alpha
\]

or

\[
\alpha = \frac{23}{12} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{29}{24} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{1}{4} \left( \frac{\nu}{\nu + 3w} \right)^\nu = 0.
\]

(4.2.4)

For \( k = 5 \), the integral equation

\[
\int_0^\infty a_5(0, w, tw) g_\nu(t) \, dt = 1 - \alpha,
\]

where \( a_5(0, w, tw) =
\]

\[
1 - \frac{163}{60} e^{-tw} + \frac{329}{120} e^{-2tw} - \frac{73}{60} e^{-3tw} + \frac{1}{5} e^{-4tw},
\]

\[\text{gives}
\]

\[
\alpha = - \frac{163}{60} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{329}{120} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{73}{60} \left( \frac{\nu}{\nu + 3w} \right)^\nu
\]

\[+ \frac{1}{5} \left( \frac{\nu}{\nu + 4w} \right)^\nu = 0.
\]

(4.2.5)

For \( k = 6 \), the integral equation
\[
\int_0^\infty a_6(0, \omega, tw) g_\omega(t) \, dt = 1 - \alpha,
\]
where
\[
a_6(0, \omega, tw) = 1 - \frac{71}{20}e^{-tw} + \frac{901}{180}e^{-2tw} - \frac{2521}{720}e^{-3tw} + \frac{437}{360}e^{-4tw} - \frac{1}{6}e^{-5tw},
\]
becomes
\[
\alpha = \frac{71}{20} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{901}{180} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{2521}{720} \left( \frac{\nu}{\nu + 3w} \right)^\nu + \frac{437}{360} \left( \frac{\nu}{\nu + 4w} \right)^\nu - \frac{1}{6} \left( \frac{\nu}{\nu + 5w} \right)^\nu = 0.
\]

(4.2.6)

For \( k = 7 \), the integral equation
\[
\int_0^\infty a_7(0, \omega, tw) g_\omega(t) \, dt = 1 - \alpha,
\]
where
\[
a_7(0, \omega, tw) = 1 - \frac{617}{140}e^{-tw} + \frac{10141}{1260}e^{-2tw} - \frac{39271}{5040}e^{-3tw} + \frac{5311}{1260}e^{-4tw} - \frac{169}{140}e^{-5tw} + \frac{1}{7}e^{-6tw},
\]
becomes
\[
\alpha = \frac{617}{140} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{10141}{1260} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{39271}{5040} \left( \frac{\nu}{\nu + 3w} \right)^\nu + \frac{5311}{1260} \left( \frac{\nu}{\nu + 4w} \right)^\nu - \frac{169}{140} \left( \frac{\nu}{\nu + 5w} \right)^\nu + \frac{1}{7} \left( \frac{\nu}{\nu + 6w} \right)^\nu = 0.
\]

(4.2.7)

For \( k = 8 \), the integral equation
\[
\int_0^\infty a_8(0, \omega, tw) g_\omega(t) \, dt = 1 - \alpha,
\]
where
\[
a_8(0, \omega, tw) = 1 - \frac{1479}{280}e^{-tw} + \frac{119999}{10080}e^{-2tw} - \frac{49843}{3360}e^{-3tw} + \frac{148283}{13440}e^{-4tw} - \frac{9869}{2016}e^{-5tw} + \frac{49843}{3360}e^{-6tw} - \frac{148283}{13440}e^{-7tw} + \frac{9869}{2016}e^{-8tw},
\]
\[
\frac{1343}{1120} e^{-6tw} - \frac{1}{8} e^{-7tw},
\]
becomes
\[
\alpha - \frac{1479}{280} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{119999}{10080} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \frac{49843}{3360} \left( \frac{\nu}{\nu + 3w} \right)^\nu + \\
\frac{148283}{13440} \left( \frac{\nu}{\nu + 4w} \right)^\nu - \frac{9869}{2016} \left( \frac{\nu}{\nu + 5w} \right)^\nu + \\
\frac{1343}{1120} \left( \frac{\nu}{\nu + 6w} \right)^\nu - \frac{1}{8} \left( \frac{\nu}{\nu + 7w} \right)^\nu = 0. \quad (4.2.8)
\]
Lastly for \( k = 9 \), the integral equation
\[
\int_0^\infty a_g(0,\omega, tw) g_{\nu}(t) \, dt = 1 - \alpha,
\]
where \( a_g(0,\omega, tw) = 1 - \frac{15551}{2520} e^{-tw} + \frac{167327}{10080} e^{-2tw} - \\
\frac{2305753}{90720} e^{-3tw} + \frac{2929523}{120960} e^{-4tw} - \frac{444601}{30240} e^{-5tw} + \\
\frac{503543}{90720} e^{-6tw} - \frac{3001}{2520} e^{-7tw} + \frac{1}{9} e^{-8tw},
\]
gives
\[
\alpha - \frac{15551}{2520} \left( \frac{\nu}{\nu + w} \right)^\nu + \frac{167327}{10080} \left( \frac{\nu}{\nu + 2w} \right)^\nu - \\
\frac{2305753}{90720} \left( \frac{\nu}{\nu + 3w} \right)^\nu + \frac{2929523}{120960} \left( \frac{\nu}{\nu + 4w} \right)^\nu - \\
\frac{444601}{30240} \left( \frac{\nu}{\nu + 5w} \right)^\nu + \frac{503543}{90720} \left( \frac{\nu}{\nu + 6w} \right)^\nu - \\
\frac{3001}{2520} \left( \frac{\nu}{\nu + 7w} \right)^\nu + \frac{1}{9} \left( \frac{\nu}{\nu + 8w} \right)^\nu = 0. \quad (4.2.9)
\]
The equations (4.2.3) to (4.2.9) have been solved, using bisection method, for \( w \), for selected values of \( \alpha \) and \( \nu \). Tables II, III and IV contain these values of
critical points \( w_{k, \alpha, \nu} = w \) for \( \nu = 5(5)50, (10)100, (20)180, \infty \) and \( \alpha = .01, .05 \) and .1 respectively. In each case values are given for \( k = 3, \ldots, 9 \).

### Table II

**Critical Points \( w_{k, \alpha, \nu} \) for \( \alpha = .01 \)**

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( k \to 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
</tr>
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<tbody>
<tr>
<td>15</td>
<td>5.5902</td>
<td>6.2762</td>
<td>6.7706</td>
<td>7.1545</td>
<td>7.4737</td>
<td>7.7420</td>
<td>7.9905</td>
</tr>
<tr>
<td>20</td>
<td>5.3655</td>
<td>5.9977</td>
<td>6.4520</td>
<td>6.8030</td>
<td>7.0936</td>
<td>7.3373</td>
<td>7.5581</td>
</tr>
<tr>
<td>45</td>
<td>5.0158</td>
<td>5.5687</td>
<td>5.9592</td>
<td>6.2603</td>
<td>6.5064</td>
<td>6.7167</td>
<td>6.8988</td>
</tr>
<tr>
<td>80</td>
<td>4.9003</td>
<td>5.4264</td>
<td>5.7994</td>
<td>6.0856</td>
<td>6.3185</td>
<td>6.5144</td>
<td>6.6850</td>
</tr>
<tr>
<td>90</td>
<td>4.8839</td>
<td>5.4055</td>
<td>5.7774</td>
<td>6.0618</td>
<td>6.2921</td>
<td>6.4861</td>
<td>6.6551</td>
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<tr>
<td>120</td>
<td>4.8519</td>
<td>5.3687</td>
<td>5.7307</td>
<td>6.0125</td>
<td>6.2394</td>
<td>6.4300</td>
<td>6.5959</td>
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<td>140</td>
<td>4.8381</td>
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<td>5.7137</td>
<td>5.9916</td>
<td>6.2169</td>
<td>6.4061</td>
<td>6.5707</td>
</tr>
<tr>
<td>180</td>
<td>4.8200</td>
<td>5.3292</td>
<td>5.6868</td>
<td>5.9642</td>
<td>6.1866</td>
<td>6.3745</td>
<td>6.5373</td>
</tr>
<tr>
<td>( \infty )</td>
<td>4.8135</td>
<td>5.3209</td>
<td>5.6780</td>
<td>5.9537</td>
<td>6.1773</td>
<td>6.3635</td>
<td>6.5257</td>
</tr>
</tbody>
</table>

### Table III

**Critical Points \( w_{k, \alpha, \nu} \) for \( \alpha = .05 \)**

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>( k \to 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 6 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
<th>( 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4.3061</td>
<td>5.2039</td>
<td>5.8716</td>
<td>6.4022</td>
<td>6.8414</td>
<td>7.2181</td>
<td>7.7604</td>
</tr>
<tr>
<td>10</td>
<td>3.6656</td>
<td>4.3356</td>
<td>4.8189</td>
<td>5.2058</td>
<td>5.5210</td>
<td>5.7886</td>
<td>6.0814</td>
</tr>
<tr>
<td>ν</td>
<td>k→3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>----</td>
<td>------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>5</td>
<td>3.0774</td>
<td>3.8412</td>
<td>4.4070</td>
<td>4.8558</td>
<td>5.2267</td>
<td>5.5432</td>
<td>6.0522</td>
</tr>
<tr>
<td>25</td>
<td>2.5464</td>
<td>3.0820</td>
<td>3.4631</td>
<td>3.7627</td>
<td>4.0054</td>
<td>4.2102</td>
<td>4.4213</td>
</tr>
<tr>
<td>30</td>
<td>2.5268</td>
<td>3.0542</td>
<td>3.4308</td>
<td>3.7228</td>
<td>3.9606</td>
<td>4.1611</td>
<td>4.3647</td>
</tr>
<tr>
<td>35</td>
<td>2.5129</td>
<td>3.0345</td>
<td>3.4063</td>
<td>3.6941</td>
<td>3.9279</td>
<td>4.1263</td>
<td>4.3247</td>
</tr>
<tr>
<td>45</td>
<td>2.4945</td>
<td>3.0084</td>
<td>3.3740</td>
<td>3.6568</td>
<td>3.8686</td>
<td>4.0803</td>
<td>4.2721</td>
</tr>
<tr>
<td>50</td>
<td>2.4880</td>
<td>2.9993</td>
<td>3.3628</td>
<td>3.6438</td>
<td>3.8722</td>
<td>4.0643</td>
<td>4.2538</td>
</tr>
</tbody>
</table>

Table IV

Critical Points \( w_{k,\alpha,\nu} \) for \( \alpha = .10 \)

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In the sequel we restrict the discussion for the case of equal but unknown scale parameters. The results can be adapted to the case (ii), that is, known scale parameters on parallel lines by using statistic $W_0$ and the critical points of Table-1.

4.3 ONE SIDED SIMULTANEOUS CONFIDENCE INTERVALS

The test procedure given in equation (4.2.1) can be inverted to produce $1-\alpha$ level simultaneous one-sided confidence intervals for the ordered pairwise differences $\mu_j - \mu_i$ $(1 \leq i < j \leq k)$ using standard technique (see, e.g., Randles and Wolf, 1979, Ch. 6; Miller, 1981, Ch.4, Hayter, 1990) as

$$1-\alpha = P_0 \left[ \max_{1 \leq i < j < k} \frac{n(Y_j - Y_i)}{1} \leq w_{k,\alpha,\nu} \right]$$

$$= P_{\frac{Y_j - Y_i - (\mu_j - \mu_i)}{\sqrt{T/n}} \leq w_{k,\alpha,\nu}, 1 \leq i < j \leq k}$$

$$= P_{\mu_j - \mu_i \geq q_{ij}, 1 \leq i < j \leq k}, \quad (4.3.1)$$
where \( q_{ij} = (Y_j - Y_i) - \frac{T}{n} w_k, \alpha, \nu \) and \( \mu = (\mu_1, \ldots, \mu_k)^T \).

It may be noted that SCIs given in (4.3.1) are:
(i) not based on the assumption \( \mu_1, \ldots, \mu_k \) and (ii) valid only if this ordering is specified independently without any examination of the data. The detailed discussion and improvements of such SCIs under the prior information \( \mu_1, \ldots, \mu_k \) can be found in Hayter (1990).

The SCIs for the ordered pair wise differences given in equation (4.3.1) will be of most interest to the experimenter, as they enable him to declare which dose levels (components) have unequal guaranteed effective period (guaranteed life time) following rejection of the null hypothesis. If the experimenter is expecting differences in \( \mu_1, \ldots, \mu_k \) in conformity with the ordering \( \mu_1, \ldots, \mu_k \), then the one-sided simultaneous confidence intervals given by equation (4.3.1) are of course more sensitive to the differences of this kind than the corresponding confidence intervals provided by the Chen's (1982) test procedure. This follows by observing that the critical points of the proposed one-sided test \( w_k, \alpha, \nu \) are substantially smaller than the corresponding critical points \( c_k, \alpha, \nu \) of the Chen's (1982) test procedure. Some illustrative values of the ratio \( t_k, \alpha, \nu = w_k, \alpha, \nu / c_k, \alpha, \nu \), representing the ratio of the length of the one-sided confidence interval for \( \mu_j - \mu_i \) derived from the proposed test statistic \( W \) to the length of the confidence interval for \( \mu_j - \mu_i \) derived from the Chen's (1982) NRS, are given in Table V for \( \alpha = 0.05 \). We conclude that substantial saving can be made.
by using proposed test if the experimenter believes that \( \mu_s \) are ordered and consequently wants to concentrate on detecting difference in a certain direction.

<table>
<thead>
<tr>
<th>Table V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values of ratio ( t_{k, \alpha, \nu} ) for ( \alpha = .05 )</td>
</tr>
<tr>
<td>k --&gt;</td>
</tr>
<tr>
<td>n=11</td>
</tr>
<tr>
<td>n=21</td>
</tr>
</tbody>
</table>

Finally, following Hayter (1990, appendix-A) (also see Hochberg and Tamhane, 1987, p.81) the simultaneous one-sided confidence intervals given in equation (4.3.1) can be extended to get simultaneous one-sided confidence intervals for nonnegative contrasts of \( \mu_s \) as

\[
P_{\mu} \left[ \sum_{i=1}^{k} c_i \mu_i \leq \sum_{i=1}^{k} c_i Y_i - \frac{1}{2} \frac{T}{n} w_{k, \alpha, \nu} ; \ c \in c_k^* \right] = 1-\alpha, \tag{4.3.2}\]

where \( c_k^* = \left[ c \in \mathbb{R}^k ; \sum_{i=1}^{k} c_i = 0 ; \sum_{i=1}^{k} c_i \geq 0, 1 \leq j \leq k-1 \right] \).

Again these confidence intervals do not depend on an assumption that \( \mu_1 = \ldots = \mu_k \).

In the following Section we propose a multiple three-decision procedure for the two sided pair wise comparison of exponential population in terms of their location parameters to decide, which populations are better than which others and which populations are worse than others.

4.4 THREE-DECISION PROCEDURE

The population \( E(\mu_1, \theta) \) is called better
(worse) than population $E(\mu_j, \theta)$ if $\mu_i - \mu_j > 0$ ($\mu_i - \mu_j < 0$).

A Type-III error occurs when a decision procedure classifies the population $E(\mu_i, \theta)$ better than population $E(\mu_j, \theta)$ when it is worse and vice-versa.

A multiple three-decision procedure for two-sided pair-wise comparisons of location parameters, to decide which populations are better then which others and which population are worse than others, is follows:

(i) infer $E(\mu_j, \theta)$ better than $E(\mu_i, \theta)$ (i.e., $\mu_j - \mu_i > 0$)

if $(n(Y_i - Y_j)/T) > w_k, \alpha, \nu$

(ii) infer $E(\mu_j, \theta)$ worse than $E(\mu_i, \theta)$ (i.e., $\mu_i - \mu_j > 0$)

if $(n(Y_i - Y_j)/T) < -w_k, \alpha, \nu$

(iii) make no decision on $E(\mu_i, \theta)$ and $E(\mu_j, \theta)$

if $-w_k, \alpha, \nu \leq (n(Y_i - Y_j)/T) \leq w_k, \alpha, \nu$.

(4.4.1)

Here $w_k, \alpha, \nu$ satisfies the equation (4.2.1). These critical points maintain Type-III error at level $\alpha$ as explained below:

\[P[\text{no Type-III error}] = P[n(Y_i - Y_j)/T \geq -w_k, \alpha, \nu \ \forall \ i, j \text{ with } \mu_j - \mu_i > 0 \text{ and} \]

\[n(Y_i - Y_j)/T \leq w_k, \alpha, \nu \ \forall \ i, j \text{ with } \mu_j - \mu_i < 0 \]

\[P \left[ \frac{(Y_j - Y_i) - (\mu_j - \mu_i)}{T/n} \geq -w_k, \alpha, \nu \ \forall \ i, j \text{ with } \mu_j - \mu_i > 0 \right. \]

and \[\frac{(Y_j - Y_i) - (\mu_j - \mu_i)}{T/n} \leq w_k, \alpha, \nu \]
\[
\frac{(\mu_i - \mu_j)}{\sqrt{n}} \quad \forall \ i, j \text{ with } \mu_j - \mu_i < 0
\]

\[
= P \left[ \frac{(Y_i - Y_j) - (\mu_i - \mu_j)}{\sqrt{n}} \leq w, \alpha, \nu^+ + \frac{(\mu_i - \mu_j)}{\sqrt{n}} \quad \forall \ i, j \text{ with } \mu_i - \mu_j < 0 \right]
\]

\[
\text{and } \frac{(Y_j - Y_i) - (\mu_j - \mu_i)}{\sqrt{n}} \leq w, \alpha, \nu^+ 
\]

\[
\frac{(\mu_i - \mu_j)}{\sqrt{n}} \quad \forall \ i, j \text{ with } \mu_j - \mu_i < 0
\]

\[
= P \left[ \frac{(Y_j - Y_i) - (\mu_j - \mu_i)}{\sqrt{n}} \leq w, \alpha, \nu^+ + \frac{(\mu_i - \mu_j)}{\sqrt{n}} \quad \forall \ i, j \text{ with } \mu_j - \mu_i > 0 \right]
\]

\[
\geq P \left[ \frac{(Y_i - Y_j) - (\mu_i - \mu_j)}{\sqrt{n}} \leq w, \alpha, \nu \quad \forall \ 1 \leq i < j \leq k \right]
\]

\[
= P \left[ \max_{1 \leq i < j \leq k} \frac{(Y_i - Y_j) - (\mu_i - \mu_j)}{\sqrt{n}} \leq w, \alpha, \nu \right]
\]

\[
= 1 - P_{H_0} \left[ W \geq w, \alpha, \nu \right]
\]

\[
= 1 - \alpha.
\]

Hence, the procedure (4.4.1) controls the probability of Type-III error at level \( \alpha \).

The validity of the critical points given in Tables I-IV and power computations have been carried out through statistical simulation as explained in the following Section.

4.5 STATISTICAL SIMULATION

In order to see how well the critical points \( w, \alpha, \nu = w \) and \( w, \alpha = w \) perform, some simulated
values of 
\[ P_0(W_{ik}, \alpha, \nu) \]
and 
\[ P_0(W_{ik}, \alpha) \]
are given in Tables VI and VII, respectively, for 
k=3,\ldots,9 \text{ and } \alpha=.01, .05 \text{ and .1. For each combination of } n, k \text{ and } \alpha \text{ the entries in these tables are proportion of the number of times the test statistic exceeds the corresponding critical point among } 2\times10^4 \text{ simulations and rounded to four decimal places. Random samples of size } n=6, 11 \text{ and } 21 \text{ are considered. Simulation was carried out in batches of } 10^4 \text{ repetitions. We did not notice much difference among the respective simulated values of both the batches and thus it was decided to stop after } 2\times10^4 \text{ repetitions. It is apparent from Tables VI and VII that the actual size levels are almost conservative, that is, are smaller than the nominal level.}

Simulated powers of the proposed test statistic \( W \) and Chen’s statistic \( C \) based on 10,000 simulations are given in Table VIII. These simulated values indicate that the power of the proposed test is larger than the chen’s test particularly for the small sample sizes and tight slippage parameter configurations. The entries in Table VIII are proportions of the number of times the respective test statistic exceeding the corresponding critical points among 10,000 repetitions by generating fresh random samples in each repetition with indicated parameteric configurations.
### Table VI

Simulated Error Rates of the Critical Points \( w_{k, \alpha, \nu} = w \) for Models with sample sizes \( n \).

<table>
<thead>
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<th>( n ) = 6</th>
<th>( \alpha )</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
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<td>.0105</td>
<td>.0098</td>
<td>.0102</td>
<td>.0093</td>
<td>.0104</td>
<td>.0107</td>
<td>.0094</td>
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</tr>
<tr>
<td>.05</td>
<td>.0499</td>
<td>.0499</td>
<td>.0489</td>
<td>.0496</td>
<td>.0503</td>
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<td>.10</td>
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<td>.0998</td>
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<td>.0993</td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n ) = 11</th>
<th>( \alpha )</th>
<th>k</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tr>
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### Table VII

Simulated Error Rates of the Critical Points \( w_{k, \alpha, \nu} = w \) for Models with sample sizes \( n \).

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* simulated powers of C.
For unbalanced models the obvious modification of statistic $W$ is

$$W_1 = \max_{1 \leq i < j \leq k} (\frac{Y_i - Y_j}{\sqrt{\frac{n_i}{n_i + 1/n_j}}})$$

where $T_i = \sum_{i=1}^{k} n_i \sum_{j=1}^{n_i} (X_{ij} - Y_i) / \nu$ and $\nu = \sum_{i=1}^{k} (n_i - 1)$.

In this case evaluation of critical points $w_{k,a,\nu}$ (say) from equation $P_0(W_1 = w_k,a,\nu) = \alpha$ by numerical integration is quite cumbersome even with small number of populations $k$. However, for the unbalanced models (and even for balanced models) where the exact critical points are not available, we recommend the use of simulation techniques. Simulation enables us critical points evaluation quickly and accurately for any finite number of populations $k$ and any sets $\{n_1, ..., n_k\}$ of sample sizes. The statistic $W_1$ can take only finite values, so its distribution function can be estimated by simulating a large number of independent realizations of the statistic $W_1$. Thus, estimation of the tail area, and hence the required critical points, can be done quite accurately and efficiently.

4.6 APPLICATION TO PARETO MODEL

Pareto distribution with pdf

$$f(x|a,c) = \frac{ac^a}{x^{a+1}}, \quad 0 < c \leq x, \quad a > 1$$

denoted by $P(a,c)$, is used to describe the distribution of personal income, the distribution of city population sizes and stock price fluctuations. The parameter $c$ is called minimum income, minimum population size and minimum stock price in these three
situations respectively.

Let \( P(a,c_i), i=1,\ldots,k \) be \( k(k+3) \) Pareto populations. Based on random samples \( X_{ij}, j=1,\ldots,n \) of common size \( n \), \( i=1,\ldots,k \), the critical constants \( w_{k,a,\nu} \) can be used to test the null hypothesis \( \mathcal{H}_0: c_1 = \cdots = c_k \) against the simple ordered alternative hypothesis \( \mathcal{H}_1: c^*_1 = \cdots = c^*_k \), with at least one strict inequality. The test procedure can then be inverted to obtain simultaneous one-sided confidence intervals for all ordered pair wise ratios \( c_j/c_i^* \), \( 1 \leq i < j \leq k \).

We note that

(i) \( Y_i = \min_{1 \leq j \leq n} X_{ij} \) follows Pareto distribution \( P(n,a,c_i), i=1,\ldots,k \) and

(ii) \( Z_i = \log Y_i \) follows exponential distribution \( \mathcal{E}(\log c_i, \frac{1}{na}), i=1,\ldots,k \). With \( \mu_i = \log c_i, i=1,\ldots,k \)

the statistic \( \hat{w} = \max_{1 \leq i < j \leq k} na(Z_j - Z_i) \) has the same distribution as \( W \), where

\[
\hat{a} = \frac{\nu}{k \sum_{i=1}^{n} \sum_{j=1}^{n} \log(X_{ij}/Y_i)}
\]

\( \nu = k(n-1) \) and \( \hat{a} \) has the pdf of a gamma variate (with shape parameter \( \nu \)) divided by \( \nu \). Therefore, test is reject \( \mathcal{H}_0 \) at level \( \alpha \) if

\[
\max_{1 \leq i < j \leq k} na(Z_j - Z_i) \geq w_{k,a,\nu}.
\]

Also using the argument of Section 4.3, we can write
where \( c = (c_1, \ldots, c_k)^t \in \mathbb{R}_+^k \) (the positive part of the \( k \)-dimensional Euclidean space). If Pareto distribution \( P(c_i, a) \) is used to describe the income distribution of \( i \)-th family, \( i=1, \ldots, k \) then the simultaneous confidence intervals given in (4.6.1) enable the experimenter to decide which families have unequal minimum income levels following rejection of the null hypothesis.