CHAPTER II

TESTING HOMOGENEITY OF SCALE PARAMETERS AGAINST ORDERED ALTERNATIVES USING HODGES-LEHMANN TYPE ESTIMATORS

In many practical situations the data from various independent experimental sources are modelled by members of symmetric family of distributions. The experimenter wants to make some statistical inferences regarding the parameters of these distributions using such data. For example in many agricultural experiments members of normal family are used to model the data obtained under different treatments. Similarly in production processes, where the product of a specified length/weight is produced, the members of the symmetric family of distributions are used to model the data. In this Chapter, we assume $k$ ($k \geq 2$) independent sources such that the observations from the $i$th source follow a symmetric distribution with unknown common point of symmetry $\mu$ ($-\infty < \mu < \infty$) and scale parameter $\theta_i$ ($\theta_i > 0$), $i=1, \ldots, k$. A class of distribution free tests for testing the homogeneity of scale parameters against simple ordered alternative is proposed using weighted linear combination of Hodges-Lehman (1963) type of estimators of ratio of consecutive scale parameters.

2.1 INTRODUCTION

Let $X_{i1}, \ldots, X_{in_i}$ be a random sample from a distribution with cumulative distribution function (cdf) $F_i(x) = F((x-\mu)/\theta_i)$ indexed by an unknown

The contents of this chapter have appeared as a research paper (refer Gill and Dhawan (1996)).
common point of Symmetry $\mu(-\infty<\mu<\infty)$ and unknown scale parameter $\theta_i$ ($\theta_1>0), i=1,...,k$. Here $F(.)$ is an absolutely continuous cdf such that $F(x)+F(-x)=1$, i.e., $F(.)$ is symmetric about zero with $f(.)$ as the corresponding probability density function (pdf). We want to test, at some appropriate level of significance $\alpha (0<\alpha<1)$, the null hypothesis:

$$H_0 : \theta_1 = \ldots = \theta_k \quad (2.1.1)$$

against the simple ordered alternative

$$H_1 : \theta_1 \leq \ldots \leq \theta_k \quad (2.1.2)$$

with at least one strict inequality.

This problem finds one of its applications in quality control where the aim of the analyst is to verify whether the subsequent repairs or renovations lead to more homogeneous quality product by taking $\theta_i$ as the spread in the quality product after $(k-i+1)^{th}$ repair or renovation, $i=1,...,k$. Similarly, in certain biological experiments, the distribution of the treatment effects do not differ in location, however, there may be differences in scale. In the past Govindarajulu and Haller (1977), Govindarajulu and Gupta (1978), Rao (1982), Kochar and Gupta (1986) and Kusum and Bagai (1988) have proposed tests for this problem under the assumptions that all the $k$-distributions have either the same (known or unknown) median or the same known quantile of order $p$, $0<p<1$. Kochar and Gupta (1986) assumed the populations to have the same point of symmetry. One may refer to Jonckheere (1954), Chacko (1963), Puri (1965), Tryon and Hettmansperger (1973) and Shanubhogue (1988) among others for testing $H_0:F_1(x)=\ldots=F_k(x)$ against the ordered stochastic ordering alternative,

$$H_2:F_1(x)\leq\ldots\leq F_k(x).$$
For the problem considered here we propose a class of distribution-free tests based on weighted linear combinations of Hodges-Lehmann (1963) type of estimators, given by Bhattacharyya (1977), of ratio of consecutive scale parameters. The proposed tests given in Section 2.2 are easy to apply as compared to most of the existing tests, since the Hodges-Lehmann (1963) type of estimators of ratio of scale parameters are easy to compute numerically as well as graphically (see Bhattacharyya (1977)). The distributions of test statistics are discussed in Section 2.3. The optimal members in this class of tests, obtained by finding the weighting coefficients which maximize the efficacy of the test in the proposed class when the scale parameters are assumed to be equally spaced and the sample sizes are all equal, are identified in Section 2.4. In Section 2.5 the proposed class of tests is compared with the other known tests in the Pitman asymptotic relative efficiency (ARE) sense and a numerical example, based on real life data, is given.

2.2 THE PROPOSED CLASS OF TESTS

First we consider the estimator of ratio of scale parameters, proposed by Bhattacharyya (1977), based on two samples and then extend it to the k-sample problem under consideration. The cdf of ith population is

\[ F_i(x) = F((x-\mu)/\theta_i) = H(x), \quad (2.2.1) \]

where \( H(.) \) is any absolutely continuous distribution function. If we let

\[ \beta_{ij} = \theta_j/\theta_i, \quad j \neq i, \]

then

\[ F_j(x) = H(x/\beta_{ij}). \]

Let \( \mathbf{X}_i = (X_{i1}, \ldots, X_{in_i})' \) be the vector of
observations from the ith distribution, i=1,...,k. Suppose that \( W(X_i, X_j) \) be the Ansari-Bradley (1960) or Siegel-Tukey (1960) or Sukhatme (1957) statistic for testing \( H_0: \theta_i = \theta_j \) against \( \theta_i < \theta_j \). Bhattacharyya (1977) have defined an equivalent version \( h(X_i^*, X_j^*) \) of \( W(X_i, X_j) \) and proposed the estimator \( T_{ij} \) of \( \beta_{ij} \) using the statistic \( h(X_i^*, X_j^*) \) on the lines similar to those of Hodges and Lehmann (1963). As an illustration, below we explain in brief the method of finding the estimator \( T_{ij} \) of \( \beta_{ij} \) when \( W(X_i, X_j) \) is Ansari-Bradley (1960) statistic.

The combined sample of \( X_i \)'s and \( X_j \)'s is arranged in increasing order of magnitude and ranked in the following manner

\[
(n_i + n_j)/2, \ldots, 2112, \ldots, (n_i + n_j)/2.
\]

For simplicity, we shall take both \( n_i \) and \( n_j \) to be even. Some modification will be necessary if this is not the case. According to the above ranking scheme the test statistic \( W(X_i, X_j) \) is the sum of the ranks of \( X_j \)'s. This has been shown to be asymptotically normally distributed through the use of a theorem of Chernoff and Savage (1958).

Let \( M_{ij} \) be the sample median of combined ith and jth sample. Let \( X_{i\alpha}^* = X_{i\alpha} - M_{ij} \) \( (\alpha=1, \ldots, n_i) \) and \( X_{j\beta}^* = X_{j\beta} - M_{ij} \) \( (\beta=1, \ldots, n_j) \). For the adjusted observations \( X_{i\alpha}^* \) \( (\alpha=1, \ldots, n_i) \) and \( X_{j\beta}^* \) \( (\beta=1, \ldots, n_j) \), the pair \( (X_{i\alpha}^*, X_{j\beta}^*) \) is said to form a relevant pair.
when $X_{i\alpha}^*$ and $X_{j\beta}^*$ are either both positive or both negative. Corresponding to each relevant pair find the ratio $X_{j\beta}^*/X_{i\alpha}^*$. Then the estimator $T_{ij}$ of $\beta_{ij} = \theta_j/\theta_i$ is the median of these ratios, that is,

$$T_{ij} = \text{med} \left( \frac{X_{j\beta}^*}{X_{i\alpha}^*} \right),$$

(2.2.2)

where $X_{j\beta}^*$ and $X_{i\alpha}^*$ in $(X_{j\beta}^*/X_{i\alpha}^*)$ form a relevant pair.

In case of Ansari-Bradley statistic Bhattacharyya (in his Theorem 2.2.1) has shown that $h(X_i,X_j) = W(X_i,X_j)$, whereas in Section (2.3.1) he used the Hodges-Lehmann (1963) technique to derive the estimator $T_{ij}$ from the statistic $h(X_i,X_j)$. It may be noted that: (i) $\min W(X_i,X_j) = n_j(n_j+2)/4$ and is obtained when all $X_j$'s take consecutive middle ranks and (ii) the value of $h(.,.)$ remains same whether $X_i$, $X_j$ or $X_i^*$, $X_j^*$ are used since Ansari-Bradley ranks are not altered by subtracting the combined sample median $M_{ij}$.

In the foregoing discussion we have explained the method of finding $T_{ij}$ when $h(.,.)$ is an equivalent version of Ansari-Bradley statistic. For the estimator $T_{ij}$ of $\beta_{ij}$ based on equivalent versions of two-sample Seigel-Tukey (1960) and Sukhatme (1957) statistics we refer to Bhattacharyya (1977).

Let $n_{ij} = n_i + n_j$ and define the statistic $W_{ij}$ as

$$W_{ij} = \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \sqrt{n_{ij}} (T_{ij} - 1),$$

(2.2.3)
For testing $H_0$ against $H_1$ we propose the class of statistics

$$A_k = \sum_{i=1}^{k-1} a_i W_{i,i+1},$$

(2.2.4)

where $a_1, \ldots, a_{k-1}$ are some real constants to be chosen suitably and corresponding to each set of values of $a=(a_1, \ldots, a_{k-1})'$ we get a member of the proposed class of test statistics. Under $H_1$, $W_{i,i+1}$ is expected to take large values for each $i = 1, \ldots, k-1$. Thus, the large values of $A_k$ are significant for testing $H_0$ against $H_1$. In case all $a_i$s are negative, small values of $A_k$ are significant for testing $H_0$ against $H_1$.

2.3. DISTRIBUTION OF $A_k$

We will derive the distribution of $A_k$ when $h(X_i, X_j)$ is an equivalent version of two sample linear rank statistic (e.g Ansari-Bradley (1960) or Siegel-Tukey (1960)) which converges in quadratic mean to a square integrable function $J(u)$ ($0<u<1$). Under the scale model considered here assume that $F(.)$ has finite unknown variance, finite Fisher information with respect to the location and

$$I(f) = \int_{-\infty}^{\infty} (-1-x \frac{f'(x)}{f(x)})^2 f(x) dx < \infty, \quad (2.3.1)$$

where $f(.)$ is the pdf corresponding to the cdf $F(.)$. In the following lemma we state a result of Bhattacharyya (1977) which establishes the asymptotic equivalence, in distribution, of $W_{ij}$ and $h(X_i, X_j)$.

**Lemma 2.3.1** : Let the true value of $\beta_{ij}$ be 1 and $\mu_{ij}$ = mean of $h(X_i, X_j)$ under $\theta_i=\theta_j$, then
\[
\lim_{n_i j \to \infty} P_{\beta ij} = \left[ \frac{1}{n_i} + \frac{1}{n_j} \right] \left\{ \left( t_{ij} - \beta ij \right) \sqrt{\frac{n_i j}{n_j}} - c_{ij} \right\} = 0
\]

\[
= \lim_{n_i j \to \infty} P_{\beta ij} = \left[ \frac{1}{n_i} + \frac{1}{n_j} \right] \left\{ h(X_i, X_j) - \mu ij \right\} = 0,
\]

where \( X_{ia} (a=1, \ldots, n_i) \) have distribution function \( H(\cdot) \)

and \( X_\beta (\beta=1, \ldots, n_j) \) have distribution function

\[
H \left[ x \left( 1 + \frac{c_{ij}}{\sqrt{n_i j}} \right)^{-1} \right].
\]

**Proof:** In Section 4, Bhattacharyya (1977) have proved that

\[
P[h(X_i, c_{ij}X_j) < \mu ij] \leq P[T_{ij} < \mu ij] \leq P[h(X_i, c_{ij}X_j) \leq \mu ij],
\]

where \( \mu ij \) is the mean of \( h(X_i, X_j) \) under \( \theta_i = \theta_j \). Thus, under \( \theta_i = \theta_j \)

\[
\lim_{n_i j \to \infty} P_{\beta ij} = 1 \left[ \sqrt{\frac{n_i j}{n_i j}} \left( t_{ij} - \beta ij \right) \leq c_{ij} \right]
\]

\[
= \lim_{n_i j \to \infty} P_{\beta ij} = 1 \left[ T_{ij} \leq \left( c_{ij} + \sqrt{\frac{n_i j}{n_i j}} \beta ij \right)/\sqrt{\frac{n_i j}{n_i j}} \right]
\]

\[
= \lim_{n_i j \to \infty} P_{\beta ij} = 1 \left[ h(X_i, ((c_{ij} + \sqrt{\frac{n_i j}{n_i j}})X_j)/\sqrt{\frac{n_i j}{n_i j}}) \leq \mu ij \right]
\]

\[
= \lim_{n_i j \to \infty} P_{\beta ij} = \sqrt{\frac{n_i j}{n_i j}} / (c_{ij} + \sqrt{\frac{n_i j}{n_i j}}) \left[ h(X_i, X_j) \leq \mu ij \right]
\]

\[
= \lim_{n_i j \to \infty} P_{\beta ij} = 1 \left[ h(X_i, X_j) \leq \mu ij \right].
\]

This completes the proof of the lemma.

Now let \( V_{ij} = \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \left[ h(X_i, X_j) - \mu ij \right] \) and
$S_{ij}$ be the statistic obtained from $V_{ij}$ by replacing $h(X^*_i, X^*_j)$ and $\mu_{ij}$ by their respective score values. Then

$$
\lim_{n_{ij} \to \infty} P_{\beta_{ij}} = 1 \left[ \left( \frac{1}{n_i} + \frac{1}{n_j} \right) \sqrt{\frac{n_{ij}}{n_i n_j}} \left( T_{ij} - 1 \right) - c_{ij} \right] = 0
$$

$$
= \lim_{n_{ij} \to \infty} P_{\beta_{ij}} = 1 \left[ S_{ij} = 0 \right]. \quad (2.3.2)
$$

It follows from (2.3.2) that the random variables $W_{ij}$ and $S_{ij}$ are asymptotically equal in distribution under $\theta_{i} = \theta_{j}$ (i.e. $c_{ij} = 0$) as well as under the sequence of local alternatives (i.e. $c_{ij}/\sqrt{n_{ij}} \to 0$ as $n_{ij} \to \infty$) and thus have the same moments (by Remark 1.3.11 of Randles and Wolfe (1979)) with respect to these parametric configurations. Moreover, nonparametric estimators are adaptations of relevant test statistics and the Pitman Asymptotic Relative Efficiency (ARE) of such an estimator is the same as that of the corresponding test statistic. This implies that same conclusion will hold by using either the linear combination $a'S$ or $a'W$, where $S' = (S_{12}, S_{23}, \ldots, S_{k-1,k})$ and $W' = (W_{12}, W_{23}, \ldots, W_{k-1,k})$. Therefore, in the sequel, for brevity sake, we derive the distribution and the optimal weighting coefficients using the linear combination $a'S$ instead of $a'W$ since the same results will hold for both the linear combinations.

**Note:** In lemma 2.3.1, we have discussed the asymptotic distribution when $h(X_i, X_j)$ is an equivalent version of two sample linear rank statistic (e.g. Ansari-Bradley (1960) or Siegel Tukey (1960)). If we rank the combined $i$th and $j$th sample observations from smallest
to largest, then in case of Ansari-Bradley test another equivalent form of statistic \( W(X_i, X_j) \) is
\[
\sum_{\beta=1}^{n_j} \left| R_{j\beta} - \frac{n_{ij} + 1}{2} \right|,
\]
where \( R_{j\beta} \) is the rank of \( X_{j\beta} \) in the combined \( i \)th and \( j \)th samples and the scores, converging to a square integrable function \( J(u) \) (\( 0 < u < 1 \)), corresponding to this statistic are
\[
a_{n_{ij}}(R_{j\beta}) = \left| \frac{R_{j\beta}}{n_{ij} + 1} - \frac{1}{2} \right|. \]
Now under the relationship
\[
h(X_i, X_j) = W(X_i, X_j) - \min W(X_i, X_j), \]
for Ansari-Bradley statistic, the score values of \( h(X_i, X_j) \) and \( W(X_i, X_j) \) are equal since \((\min W(X_i, X_j))/(n_{ij} + 1) = \) (value of \( W(X_i, X_j) \) when \( X_{j\beta} \)s take consecutive middle ranks) \( / (n_{ij} + 1) \to 0 \) as \( n_{ij} \to \infty \). Therefore, for Ansari-Bradley statistic the score value of
\[
h(X_i, X_j) = \sum_{\beta=1}^{n_j} a_{n_{ij}}(R_{j\beta}) \]
and its expected value under \( \theta_i = \theta_j \) is
\[
\theta_i = \theta_j \text{ is } \left( \frac{n_j}{n_{ij}} \right) \left[ \sum_{\alpha=1}^{n_i} a_{n_{ij}}(R_{i\alpha}) + \sum_{\beta=1}^{n_j} a_{n_{ij}}(R_{j\beta}) \right].
\]
This implies that \( S_{ij} = \sum_{\beta=1}^{n_j} \left( a_{n_{ij}}(R_{j\beta}) - \tilde{a}_{n_{ij}} \right) \), where
\[
\tilde{a}_{n_{ij}} = \left( \frac{1}{n_{ij}} \right) \left[ \sum_{\alpha=1}^{n_i} a_{n_{ij}}(R_{i\alpha}) + \sum_{\beta=1}^{n_j} a_{n_{ij}}(R_{j\beta}) \right].
\]
The asymptotic distribution of \( S \) follows from the results of Koziol and Ried (1977), stated in the following lemmas:

**Lemma 2.3.2**: Under \( \theta_1 = \ldots = \theta_k \) and as \( n_i, \ldots, n_k \to \infty \) in such a way that \( n_i/N \to p_i \) (\( 0 < p_i < 1 \)), \( i=1, \ldots, k \), the
random vector \((\sqrt{N}/\sigma_N)S\) is asymptotically normally distributed with mean vector 0 and covariance matrix 
\[
\Sigma=((\sigma_{ij})) , \text{ where}
\]
\[
\sigma_{ij} = \begin{cases} 
\frac{1}{p_i} + \frac{1}{p_{i+1}} & , i=j=1, \ldots, k-1 \\
-\frac{1}{p_{i+1}} & , j=i+1, i=1, \ldots, k-2 \\
-\frac{1}{p_i} & , j=i-1, i=2, \ldots, k-1 
\end{cases} \quad (2.3.3)
\]

\[
N = \sum_{i=1}^{k} n_i, \quad \sigma_N^2 = \frac{1}{N-1} \sum_{\beta=1}^{N} \left( a_N(\beta) - \bar{a}_N \right)^2 
\]

\[
\rightarrow \int_0^1 (J(u)-\bar{J})^2 du, \quad \bar{a}_N \to \bar{J} = \int_0^1 J(u) du \text{ and } a_N(\beta) \text{ is the score corresponding to the } \beta\text{th observation in the combined sample. When the sample sizes are all equal, that is, } p_1=\ldots=p_k=1/k, \text{ the covariance matrix } 
\Sigma=((\sigma_{ij}^*)) \text{ given by (2.3.3) becomes } 
\Sigma^*=(\sigma_{ij}^*), \text{ where}
\]
\[
\sigma_{ij}^* = \begin{cases} 
2k & \text{for } i=j=1, \ldots, k-1 \\
-k & \text{for } j=i+1, i=1, \ldots, k-2 \\
-k & \text{for } j=i-1, i=2, \ldots, k-1 
\end{cases} \quad (2.3.4)
\]

**Lemma 2.3.3:** For the sequence of local alternatives 
\[
\theta^*_N = (\theta^*_1N, \ldots, \theta^*_{kN})' \text{ for } \theta \text{ such that as } \min_{1\leq i \leq k} n_i \to \infty, \\
\max_{1\leq i \leq k} (\theta^*_iN - \bar{\theta}_N)^2 \to 0, \quad n_i/N \to p_i \quad (0<p_i<1) \text{ for } i=1, \ldots, k \\
\text{and} \quad \left[ \int (f) \sum_{i=1}^{k} n_i (\theta^*_1N - \bar{\theta}_N)^2 \right] \to d, \quad 0<d=\infty, \text{ the random vector } 
(\sqrt{N}/\sigma_N)S \text{ is asymptotically normally distributed with the same covariances as in (2.3.3) but with limiting means}
\]
\[
(\sqrt{N}/\sigma_N)E(S_{ij})=(\sqrt{N}/\sigma_N)A_{1j}(N)\int_0^1 J(u)J_{sc}(u,f) du, \quad (2.3.5)
\]
where $\Lambda_{ij}(N) = \log \left( \frac{\theta_{ijN}}{\theta_{i1N}} \right)$.

\[
J_{sc}(u, f) = \left[ -1 - P^{-1}(u) \frac{f'(P^{-1}(u))}{f(P^{-1}(u))} \right]
\]

$(0<u<1)$ and $\theta_N = \frac{1}{N} \sum_i \theta_{i1N}$.

Since $a'S$ is a linear combination of the components of $S$, the proof of the following theorem follows from the transformation theorem (see Serfling 1980, p.122).

**Theorem 2.3.1:** Under $H_0$ and as $\min(n_1, \ldots, n_k) \to \infty$, $n_i/N \to p_i$ $(0<p_i<1)$, $i=1,\ldots,k$, the asymptotic distribution of $(\sqrt{N}/\sigma_N)a'S$ and hence that of $(\sqrt{N}/\sigma_N)a'W$ is normal with mean zero and variance $a'Sa = 2k \left[ \sum_{i=1}^{k-1} a_i^2 + \sum_{i=1}^{k-2} a_i a_{i+1} \right]$ when $p_i = 1/k$, $i=1,\ldots,k$.

### 2.4. Optimal Choice of Weights

Here we consider the problem of finding the optimal weights $a_i$s so that the test $A_K$ has maximum efficacy for the sequence of Pitman type of alternatives

$H_i: F_i(x) = F \left( (x-\mu) \left( \sigma + \frac{\delta_i}{\sqrt{N}} \right)^{-1} \right)$ $i=1,\ldots,k$, \hspace{1cm} (2.4.1)

where $\sigma$ and $\delta_i$ are real positive constants. We assume without loss of generality that $\mu=0$ and $\sigma=1$, since all relative orderings and hence $A_K$ remains invariant if the variables are all added by the same constant and multiplied by the same positive constant. Furthermore, for the efficiency comparison, we consider the equal sample size case, that is, $p_i = 1/k$, $i=1,\ldots,k$ and the equally spaced alternatives of the type $\delta_i = i\delta$, $\delta>0.$
for $i=1,\ldots,k$. Thus the alternative $H_N$ becomes

$$H_N: F_1(x) = F \left[ x \left( 1 + \frac{i \delta}{\sqrt{N}} \right)^{-1} \right] \quad (2.4.2)$$

The following theorem due to Koziol and Ried (1977) gives the asymptotic distribution of $S$ under the sequence of local alternatives $\{H'_N\}$.

**Theorem 2.4.1:** Let $X_{i\alpha}$ be independent random variables with cumulative distribution function $F_1(x)$, where $F_1s$ are given by (2.4.2), $\alpha=1,\ldots,n_i$; $i=1,\ldots,k$. If the scores corresponding to $h(X_{i\alpha}, X_j)$ converge in quadratic mean to a square integrable function $J(u)$, $0<u<1$, then the limiting distribution of $(\sqrt{N}/\sigma_N)S$ is $k$-dimensional multivariate normal with mean vector $\omega = E_{k-1}$ and covariance matrix $\Sigma = (\sigma_{ij})$, where $\Sigma$ is given by (2.3.3), $E_{k-1} = [1]_{k-1 \times 1}$ (a $(k-1)$-dimensional vector with each component unity) and $\omega = (\sqrt{N}/\sigma_N)(\delta/\sqrt{N}) \int_{-\phi}^{\phi} J(u) J_{sc}(u,f) du$.

**Proof:** The sequence of local alternatives $\{H'_N\}$ satisfy the conditions of Lemma 2.3.3, therefore, the random vector $(\sqrt{N}/\sigma_N)S$ is asymptotically normally distributed with covariance matrix (2.3.3) and components of mean vector as

$$E[(\sqrt{N}/\sigma_N)S_{i,i+1}] = \left( \sqrt{N}/\sigma_N \right) \Delta_{i,i+1}(N) \int_0^1 J(u) J_{sc}(u,f) du$$

$$= \left( \sqrt{N}/\sigma_N \right) \left( \delta/\sqrt{N} \right) \int_0^1 J(u) J_{sc}(u,f) du,$$
since $\Lambda_{i,j}^{i+1}(N) = \log \left[ \frac{\theta_{i+1}(N)}{\theta_i(N)} \right]$

$$= \log \left[ \frac{(1+(i+1)/N)^{i+1}}{(1+i/\sqrt{N})^{i+1}} \right] = \frac{\delta}{\sqrt{N}}$$ for large values of $N$. Here terms involving higher powers of $N$ in the denominator are neglected and $a_N = b_N$ means $a_N/b_N \to 1$ as $N \to \infty$. Recall that in Lemma 2.3.1 it is seen that $W_{ij}$ and $S_{ij}$ have the same moments under $\theta_i = \theta_j$ as well as under the sequence $\{H_N\}$, thus

$$E \left[ \left( \frac{1}{\sigma_N} \right) \Lambda_{i,j}^{i+1} \right]$$

$$= \left( \frac{1}{\sigma_N} \right) \left( \frac{\delta}{\sqrt{N}} \right) \int_0^1 J(u)J_{sc}(u,f) du \quad (2.4.3)$$

Now we will state a result in the following lemma (see Rao (1973) p.60), useful in finding optimal weights which maximize the efficacy.

**Lemma 2.4.1:** Let $A$ be a positive definite $m \times m$ matrix and $U \in \mathbb{E}^m$, i.e., $U$ is an $m \times 1$ vector, then

$$\sup_{X \in \mathbb{E}^m} \frac{(U'X)^2}{X'A^{-1}X} = U' A^{-1} U,$$

and the supremum is attained at $X=X^*$, where

$$X^* = A^{-1} U.$$

The optimum weights $a_i$'s, which maximize the efficacy of the members of the proposed class can be obtained with the help of following theorem:

**Theorem 2.4.2:** Under the sequence of local alternatives $\{H_N\}$, satisfying the conditions of Lemma 2.3.3 and the scores corresponding to $h(X_i,X_j)$ converging in quadratic mean to a square integrable function $J(u)$, $(0 < u < 1)$, the Pitman efficacy of the
test $A_k$ is maximized when
\[ a_i = \frac{i(k-i)}{2k}, \quad i=1,...,k-1. \] (2.4.4)

**Proof**: From Theorem 2.3.1, Lemma 2.3.3 and the expression (2.4.3) it follows that $(\sqrt{N}/\sigma_N)A_k$ is asymptotically normally distributed with mean $\omega \sum_{i=1}^{k-1} a_i$ and variance $\omega^2 \Sigma$, where $\omega = (a_1',...,a_{k-1}')'$. If we let $\lambda = \delta/\sqrt{N}$, then the efficacy of $A_k$ (see Govindarajulu and Gupta (1978) and Rao (1982)) is given by

\[ \epsilon(A_k) = \left[ \frac{\frac{d}{d\lambda} H_N' \left( (\sqrt{N}/\sigma_N)A_k \right) }{N \omega^2 \Sigma} \right]^2 \]

\[ = \left( H^2 \left( E_{k-1} a \right)^2 / (\sigma_N^2 a' \Sigma^* a) \right), \]

where $H = \int_0^1 J(u) J_{sc}(u,f) du$. Now maximizing $\epsilon(A_k)$ with respect to $a$ is equivalent to maximizing $(E_{k-1} a)^2 / a' \Sigma^* a$ with respect to $a$. But the value of $a$ which maximizes $(E_{k-1} a)^2 / a' \Sigma^* a$ (by lemma 2.4.1 stated above) is
\[ a = (\Sigma^*)^{-1} E_{k-1} \] (2.4.5)

and
\[ \sup_a \epsilon(A_k) = (H/\sigma_N^2) E_{k-1} (\Sigma^*)^{-1} E_{k-1} \] (2.4.6)

since $\Sigma^*$ is a $(k-1) \times (k-1)$ matrix and $E_{k-1} = [1]_{(k-1) \times 1}^T$, i.e., $E_{k-1}$ is a $(k-1)$ dimensional vector with each component unity. Now $(\Sigma^*)^{-1}$ can easily be written from $\Sigma^*$ (using standard text book results, e.g., Graybill...
(1969)) as follows

\[
(\Sigma^*)^{-1} = ((\sigma^{*}_{ij})),
\]

where

\[
\sigma^{*}_{ij} = \begin{cases} 
  i(k-j)/k^2 & \text{if } isj \\
  j(k-i)/k^2 & \text{if } i \neq j.
\end{cases}
\]  

(2.4.7)

It follows from (2.4.5) to (2.4.7) that for the choice of \(a_i\) as

\[
a_i^* = \frac{i(k-i)}{2k}, \quad i=1,\ldots,k-1
\]

and

\[
\mathcal{E}_{k-1}(\Sigma^*)^{-1} \mathcal{E}_{k-1} = (k^2-1)/12
\]

the efficacy of the optimum test statistic say

\[
A_k^* = \sum_{i=1}^{k-1} a_i^* W_{i,i+1} \text{ is}
\]

\[
e(\Lambda_k^*) = \left(\frac{(H/\sigma_n)^2(k^2-1)}{12} \right)/12. \quad (2.4.8)
\]

2.5. ASYMPTOTIC RELATIVE EFFICIENCIES

In this Section, the asymptotic relative efficiencies of the \(A_k^*\) test statistics relative to some other known nonparametric tests for this testing problem are derived. For the ARE computations, we restrict to the case when the estimators \(T_{ij}\) are derived from an equivalent version \(h(X_1, X_2)\) of Ansari-Bradley (1960) statistic for which \(J(u) = |u-1/2|, \sigma_n^2 \approx 1/48\) and \(H = \int_{-\infty}^{\infty} |x| f^2(x) \, dx\).

Kochar and Gupta (1986) proposed a class of test statistics \(W_{c,d}\) based on the linear combination of consecutive two sample U-statistics \(U_{i,i+1}\).
i = 1, ..., k - 1. They defined the statistics \( U_{i, i+1} \) via two sample kernel \( \phi(X_{i\alpha_1}, ..., X_{i\alpha_c}; X_{j\beta_1}, ..., X_{j\beta_d}) \)

\[
U_{i, i+1} = \begin{cases} 
1 & \text{if } \min(X_{i\alpha_1}, ..., X_{i\alpha_c}) > \min(X_{j\beta_1}, ..., X_{j\beta_d}) \\
\text{and } \max(X_{i\alpha_1}, ..., X_{i\alpha_c}) < \max(X_{j\beta_1}, ..., X_{j\beta_d}) 
\end{cases} \\
-1 & \text{if } \min(X_{i\alpha_1}, ..., X_{i\alpha_c}) < \min(X_{j\beta_1}, ..., X_{j\beta_d}) \\
\text{and } \max(X_{i\alpha_1}, ..., X_{i\alpha_c}) > \max(X_{j\beta_1}, ..., X_{j\beta_d}) \\
0 & \text{otherwise},
\]

where \( c \) and \( d \) are two fixed integers such that \( 2s(c, d) = \min(n_1, ..., n_k) \), \( \alpha_1, ..., \alpha_c \) are \( c \) distinct integers chosen from \( (1, ..., n_i) \) and \( \beta_1, ..., \beta_d \) are \( d \) distinct integers chosen from \( (1, ..., n_j) \). The test was, reject \( H_0 \) for large values of \( W_{c, d}^* \). The efficacy derived by them with these optimum weighting coefficients is

\[
e(W_{c, d}^*) = \frac{(k - 1)G_m^2}{12 \rho_m}, \tag{2.5.1}
\]

where

\[
G_m = \int_{-\infty}^{\infty} \left[ \frac{m-2}{2} \text{e}^{m-1} - \left( \frac{m-2}{2} \text{e}^{-m} \right)^{m-2} \right] \text{e}^{2}(x) \text{d}x,
\]

\[
\rho_m = \frac{1}{(m-1)^2} \left[ \frac{1}{2m-1} - \frac{2}{m^2} + \frac{(m-1)!}{2m-1} \right]
\]

and \( m = c + d \) with \( c \geq 2 \) and \( d \geq 2 \) as the sub-sample sizes from the two samples respectively. From (2.5.1) and (2.4.8) we find that the asymptotic relative efficiency of \( A_k^* \) relative to \( W_{c, d}^* \) is
The AREs of $A_{k}^{*}$ test, when the estimators of the ratio of scale parameters derived from the equivalent version of Ansari-Bradley statistic, relative to $W_{c,d}^{*}$ test for some standard symmetric distributions are given in Table-I.

**Table-I**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double</td>
<td>.864</td>
<td>.849</td>
<td>.831</td>
<td>.855</td>
<td>.803</td>
<td>.802</td>
</tr>
<tr>
<td>Exponential</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal</td>
<td>.800</td>
<td>.776</td>
<td>.749</td>
<td>.723</td>
<td>.688</td>
<td>.668</td>
</tr>
<tr>
<td>Logistic</td>
<td>.837</td>
<td>.820</td>
<td>.800</td>
<td>.784</td>
<td>.766</td>
<td>.762</td>
</tr>
</tbody>
</table>

Rao (1982) proposed $K_{2}$-test statistic based on linear combinations of the generalized form of Sugiura (1965) type U-statistic for this problem. The efficacy of Rao’s test with the optimum weighting coefficients (see Kochar and Gupta (1986)) is

$$e(k_{2}) = \frac{(k_{2}^{2}-1)(2k-1)^{2} \sigma_{2k}^{2}}{12 A(2k-1,2k-1)}.$$  \(2.5.2\)

where

$$A(2k-1,2k-1) = \frac{(2k-1)^{2}}{2k^{2}(4k-1)} - \frac{1}{2k^{2}} + \frac{2[(2k-1)!]^{2}}{(4k-1)!}.$$  \(2.5.3\)

Now from (2.4.8) and (2.5.2) the ARE of $A_{k}^{*}$ test relative to $K_{2}$-test is

$$\text{ARE}(A_{k}^{*}, K_{2}) = \frac{\sigma_{N}^{2}}{\sigma_{2k}^{2}} \frac{A(2k-1,2k-1)}{(2k-1)^{2}} \frac{A(2k-1,2k-1)}{\sigma_{2k}}.$$
The AREs (2.5.3) computed for some standard symmetric distributions are given in Table-II.

Table-II
Asymptotic Relative Efficiency of $A_k^*$ Relative to $k$

<table>
<thead>
<tr>
<th>Distribution</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double</td>
<td>.864</td>
<td>.831</td>
<td>.808</td>
<td>.801</td>
<td>.804</td>
<td>.812</td>
<td>.824</td>
<td>.837</td>
</tr>
<tr>
<td>Exponential</td>
<td>.800</td>
<td>.749</td>
<td>.704</td>
<td>.677</td>
<td>.662</td>
<td>.665</td>
<td>.671</td>
<td>.683</td>
</tr>
<tr>
<td>Normal</td>
<td>.830</td>
<td>.801</td>
<td>.773</td>
<td>.763</td>
<td>.725</td>
<td>.771</td>
<td>.783</td>
<td>.799</td>
</tr>
<tr>
<td>Logistic</td>
<td>.830</td>
<td>.801</td>
<td>.773</td>
<td>.763</td>
<td>.725</td>
<td>.771</td>
<td>.783</td>
<td>.799</td>
</tr>
</tbody>
</table>

Kusum and Bagai (1988) considered the same problem with the assumption that the underlying distributions differing in scale parameters have the same known quantile of order $p$ ($0 < p < 1$), not necessarily equal to $1/2$. There the authors used the statistic $T^*_k$ which is the linear combination of the $k$ consecutive two sample U-statistics $U_{i,i+1}$ proposed by Deshpande and Kusum (1984) in the context of two sample scale problem, based on the kernel

$$\phi(x,y) = \begin{cases} 
1 & \text{if } |x| \leq |y| \text{ and } xy \geq 0 \\
0 & \text{if } xy < 0 \\
-1 & \text{if } |x| > |y| \text{ and } xy > 0.
\end{cases}$$

where $x(y)$ is an observation from the $i$th ($(i+1)^{th}$) distribution $i=1,...,k-1$ and common known quantile of order $p$ is zero. The test was, reject $H_0$ for large values of $T^*_k$. The efficacy of $T^*_k$ derived by the authors is
When $p=1/2$, the efficacy (2.5.4) becomes

$$e(T_k^*) = \frac{k^2-1}{4(k^2-1)} H^2. \quad (2.5.5)$$

From (2.4.8) and (2.5.5) we see that the ARE of $A_k^*$ relative to $T_k^*$ is one when $p=1/2$, that is, when all the distributions have the same known point of symmetry.

From Table-I we see that for large values of $m=c+d$, the ARE of our test relative to the test of Kochar and Gupta (1986) decreases marginally. However, the large values of $m$ not only makes the computation of Kochar and Gupta’s statistic tedious but at the same time one is unable to decide about the values of $c$ and $d$. The AREs of our test relative to Rao’s (1982) test, given in Table-II, decrease first in $k$ and then go on increasing for $k \geq 7$ since in the ARE expression (2.5.3) the decrease in factor $A(2k-1,2k-1)/(2k-1)^2$ is more than that of $G_{2k}^2$ for $k<6$ but as this trend is reversed for $k \geq 7$. Thus, for large values of $k$ we recommend the use of the proposed test as compared to Rao’s test. Also from the ARE tables we see that the proposed test performs better for heavy tailed symmetric distributions.

The AREs given in Table-I & II, computed for the particular case of estimators based on the equivalent version of Ansari-Bradley (1960) statistic, are less because of the performance of Ansari-Bradley statistic. Furthermore, the ARE of our test relative to the test of Kusum and Bagai (1988) is one when in the latter test the common known quantile of the
underlying distributions is their common point of symmetry. The proposed test is very simple to implement as compared to the test of Kochar and Gupta (1986) or Rao (1982) because the estimators of ratio of scale parameters are easy to compute. Graphic method is also available for their computation (see Bhattacharya (1977)). Now we will illustrate the implementation of the proposed test, with optimum weights, through a numerical data taken from Gibbons, Olkin and Sobel (1977, p153).

Example: For three different gasoline blends, three different drivers were assigned randomly to three different brand-new automobiles (of identical make and model) on each of ten different runs over a specified course. The following data are miles per gallon of the gasoline blends on each of the 10 runs.

<table>
<thead>
<tr>
<th>Blends</th>
<th>miles per gallon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi_1 )</td>
<td>8.7 8.9 10.1 11.3 9.7</td>
</tr>
<tr>
<td></td>
<td>9.9 10.6 10.2 9.3 11.1</td>
</tr>
<tr>
<td>( \Pi_2 )</td>
<td>9.3 8.6 8.5 10.5 10.8</td>
</tr>
<tr>
<td></td>
<td>9.2 11.3 11.5 9.7 ( \approx )</td>
</tr>
<tr>
<td>( \Pi_3 )</td>
<td>7.8 8.2 10.5 15.1 11.5</td>
</tr>
<tr>
<td></td>
<td>10.8 9.5 11.9 13.5 7.7</td>
</tr>
</tbody>
</table>

It is known that the average mileage per gallon is same for all the blends. The experimenter wants to test that the blend \( \Pi_i \) gives least variation in mileage per gallon than the blend \( \Pi_{i+1} \), \( i=1,2 \).

Let \( \theta_i \) be the parameter representing the spread in the gasoline consumption in miles per gallon of gasoline blend \( \Pi_i \), \( i=1,2,3 \). We want to test the null hypothesis \( H_0: \theta_1=\theta_2=\theta_3 \) against the alternative
\( H_1: \theta_1 \leq \theta_2 \leq \theta_3 \) with at least one strict inequality.

For notational convenience we represent the observations from \( \Pi_1, \Pi_2 \) and \( \Pi_3 \) by \( x, y \) and \( z \) respectively in order to compute the estimator \( T_{12} \) (or \( T_{23} \)) of \( \theta_2/\theta_1 \) (or \( \theta_3/\theta_2 \)) using equivalent version of Ansari-Bradley statistic. The \( x \) and \( y \) observations arranged in the increasing order after subtracting their common median 9.8 are

\[
\begin{align*}
&y 
&x 
&y 
&x 
&y 
&x 
&y
 
&-1.3 
&-1.2 
&-1.1 
&-0.9 
&-0.9 
&-0.6 
&-0.5 
&-0.5 
&-0.1 
&-0.1
\end{align*}
\]

The six negative \( y \) values and four negative \( x \) values form 24 ratios of the form \(-y/(-x)\). Similarly, there will be 24 ratios of the form \( y/x \) corresponding to four positive \( y \) values and six positive \( x \) values. The estimate \( T_{12} \) of \( \theta_2/\theta_1 \) is the median of these 48 ratios and is given by

\[ T_{12} = 1.3205. \]

On similar lines the estimate \( T_{23} \) of \( \theta_3/\theta_2 \), computed from these data, is

\[ T_{23} = 1.675. \]

Here \( k=3 \), \( n_1=n_2=n_3=10 \), \( N=30 \) and \( a_1^* = a_2^* = 1/3 \). Also

\[
W_{12} = (2\sqrt{2n} (T_{12}-1))/n = .2866, \quad W_{23} = .6037, \quad \sigma_2 W_{23} = 2/3 \quad \text{and} \quad \sigma_2^2 \to \int_0^1 (J(u)-\overline{J})^2 du = 1/48 \quad \text{since} \quad J(u) = |u-1/2|.
\]

Now \( A_k = a_1^* W_{12} + a_2^* W_{23} = .2967 \) and by Theorem 2.3.1,
\[ Z = \left( \frac{N}{\sigma_N^2 \sum_{i} a_i^* \Sigma a_i^*} \right)^{1/2} \Lambda_N^* \] is standard normal under \( H_0 \).

Here the calculated value of \( Z = 13.7893 \), being significantly large, leads to the rejection of \( H_0 \).