CHAPTER III

SELECTION PROCEDURES FOR INCREASING FAILURE RATE AND INCREASING FAILURE RATE AVERAGE POPULATIONS

3.1 Introduction

Let $\mathcal{F}$ be the family of all absolutely continuous distribution functions with support $[0, \infty)$. Let us now define some terms to be used in this chapter.

**Definition 3.1.** A relation $\preceq$ on the space of distributions is a partial ordering if

- $F \preceq F$ for all distributions $F \in \mathcal{F}$
- $F \preceq G$, $G \preceq K$ implies $F \preceq K$ for all distributions $F$, $G$ and $K \in \mathcal{F}$.

(Note that $F \preceq G$ and $G \preceq F$ does not necessarily imply that $F = G$.)

**Definition 3.2.** Let $F \in \mathcal{F}$. The failure (hazard) rate corresponding to $F(x)$ and denoted by $r_F(x)$, is defined by

$$r_F(x) = \frac{f(x)}{1-F(x)}$$

for all $x > 0$ such that $F(x) < 1$. Here $f(\cdot)$ is the probability density function associated with $F(\cdot)$.

**Definition 3.3.** Let $r_F(x)$ be the hazard rate corresponding to $F \in \mathcal{F}$. The hazard function corresponding to $F(x)$ and denoted by $H(x)$, is defined by

$$H(x) = \int_0^x r_F(y) \, dy$$
Definition 3.4. A distribution function $F \in \Phi$ is said to be an increasing failure rate (IFR) distribution if the failure rate, $f(x)/[1-F(x)]$, is a non-decreasing function of $x$, for every $x > 0$ such that $F(x) < 1$.

Definition 3.5. A distribution function $F \in \Phi$ is said to be an increasing failure rate average (IFRA) distribution if $(1/x) \log [1-F(x)]$ is a non-increasing function of $x$, for every $x > 0$ such that $F(x) < 1$.

Definition 3.6. Let $F$ and $G \in \Phi$. $F$ is said to be star ordered with respect to $G$, and written as $F \preceq_s G$, if and only if $G^{-1}F(x)/x$ is non-decreasing in $x > 0$ on the support of $F$, or equivalently, if and only if $G^{-1}F(.)$ is star shaped on the support of $F$, that is,

$$G^{-1}F(\lambda x) \leq \lambda G^{-1}F(x)$$

for $0 \leq \lambda \leq 1$ and for all $x > 0$.

Definition 3.7. Let $F$ and $G \in \Phi$. $F$ is said to be convex ordered with respect to $G$, and written as $F \preceq_c G$, if and only if $G^{-1}F(.)$ is convex on the support of $F$, that is,

$$G^{-1}F[\lambda x + (1-\lambda)y] \leq \lambda G^{-1}F(x) + (1-\lambda) G^{-1}F(y)$$

for $0 \leq \lambda \leq 1$ and for all $x, y > 0$.

It can be easily verified that

(i) Star order and convex order are partial orders.

(ii) $F$ is an IFR distribution if and only if $F$ is convex ordered with respect to the standard exponential distribution (see, for example, Barlow, Marshall and Proschan (1963)).
(iii) \( F \) is an IFRA distribution if and only if \( F \) is star ordered with respect to the standard exponential distribution (see, for example, Birnbaum, Esary and Marshall (1966)).

(iv) \( F \prec G \) implies \( F \preceq G \). Hence the class of IFR distributions is contained in the class of IFRA distributions.

(v) Let \( H(\cdot) \) be the hazard function corresponding to \( F(\cdot) \). Then \( F(x) \) can be written as \( F(x) = 1 - \exp \{-H(x)\} \).

In Section 3.2 we give the formulation of the selection problem concerning IFR and IFRA distributions. In Section 3.3 we develop selection procedures for the problem when we have \( k \) populations whose distribution functions belong to the class of IFR distributions which differ only in their scale parameters. Section 3.4 deals with the selection problem concerning \( k \) IFRA populations which also differ only in their scale parameters. The procedures of Section 3.3 are based on the statistic of Epstein and Sobel (1953). This statistic has also been used by Patel (1976). The \( t \) populations associated with the \( t \) largest scale parameters are referred to as the \( t \) best populations. Now the goal is to select a subset of fixed size \( s \) of the \( k \) populations (IFR and IFRA, respectively, in Section 3.3 and 3.4) which contains at least \( c \) of the \( t \) best populations. The indifference zone formulation is used and it is shown that the infimum of the PCS for these populations is greater than or equal to the corresponding infimum of the PCS for exponential populations. Because of this fact we see that the values of the infima of the PCS, \( \psi(d^*,n) \) and \( Q_1(d^*,n) \), for exponential populations provide lower bounds for the PCS in the IFR and IFRA cases.
In literature, we find that the subset selection with respect to the largest quantile of order \(0 < \alpha < 1\) for certain restricted families (which include IFR and IFRA distributions) has been studied by Barlow and Gupta (1969). Their procedures are based on the sample quantiles (for the IFRA distributions) and the sample means (for IFR distributions). Patel (1967) considers the problem of selecting a subset containing the best one of several IFRA populations. Recently Patel (1976) has considered the selection problems relating to IFR populations which differ only in their scale parameter. He considers the following three problems: (i) selecting a best populations from the \(k\) IFR populations, (ii) selecting a subset containing a best population from the \(k\) IFR populations, and (iii) selecting a subset containing all populations from the set of \(k\) IFR populations 'better' than the 'control' population.

These goals are rather restricted compared to our goal specified in the following formulation.

3.2. Formulation of the Problem

Let \(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k\) be \(k\) populations. Let the random variable \(X_i\) \((i=1,2,\ldots,k)\) associated with \(\mathcal{P}_i\) have a continuous (IFR or IFRA) distribution \(F_i(x) = F(x; \theta_i), \theta_i \in \Theta\), defined on \([0,\infty)\). We assume that for all \(i \ (i=1,2,\ldots,k)\) the functional form of \(F_i(x)\) is the same but unspecified, and they differ only in the scale parameters \(\theta_i\). \(\Theta\) is a scale parameter in the sense that the distribution of the random variable \(X/\Theta\)
is independent of $\theta$. Exponential, gamma, Weibull (with the shape parameter greater than or equal to unity) and chi-square are some examples of such distributions.

Let the ordered values of $\theta_i$ ($i=1,2,\ldots,k$) be denoted by $0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$. Since the distributions differ only in the scale parameters (other parameters, if any, are equal) we have

$$F(x;\theta_i) = F(x/\theta_i) \quad \text{and} \quad F(x;\theta_j) = F(x/\theta_j)$$

and, therefore, $\theta_i < \theta_j$ implies that $F(x;\theta_i) > F(x;\theta_j)$ for all $x$. Thus

$$F(x;\theta_1) > F(x;\theta_2) > \ldots > F(x;\theta_k)$$

for all $x$. Since $\theta_i < \theta_j$ implies $F(x;\theta_i) > F(x;\theta_j)$ it follows that

$$E_F(x;\theta_i)(x) < E_F(x;\theta_j)(x).$$

Hence the ordering of $\theta_i$'s is also equivalent to the ordering of the population means. The $t$ populations associated with $F(\cdot;\theta_{(k-t+1)}), F(\cdot;\theta_{(k-t+2)}), \ldots, F(\cdot;\theta_k)$ are called the $t$ best populations.

Let $\theta = (\theta(1), \theta(2), \ldots, \theta(k))$ denote a point in the parametric space $\Omega$. Let $d^*(d^* > 1)$ be a specified constant. The space $\Omega$ is partitioned into a preference zone $\Omega(d^*)$ given by

$$\Omega(d^*) = \{\theta : (\theta_{(k-t+1)}/\theta_{(k-t)}) > d^*\} \quad \ldots \quad (3.1)$$

and its complement $\overline{\Omega}(d^*)$, the indifference zone.
3.3. Selection Procedures for IFR Distributions

Let \( F_i(x) = F(x; \theta_i) \) be an IFR distribution which corresponds to the population \( \Pi_i (i=1,2,...,k) \). The problem is as formulated in the last section 3.2. The selection procedure \( R_1 \) described here is based on the statistic \( T \) defined as follows, in the life testing context: an experimenter waits till a fixed number \( r (1 < r < n) \) of the total \( n \) units on life test fail; so that for each of the \( k \) populations \( \Pi_i \) the \( r \) smallest observations \( X_{i1} \leq X_{i2} \leq \cdots \leq X_{ir}, i=1,2,...,k \), are available. The statistic \( T_{in} = T(X_{11},X_{12},...,X_{ir}) \) associated with the population \( \Pi_i \) is defined by

\[
T_{in} = \sum_{\alpha=1}^{r-1} X_{\alpha} + (n-r+1) X_{ir} \quad \ldots \quad (3.2)
\]

for \( i=1,2,...,k \). This is the well-known statistic introduced by Epstein and Sobel (1953) for exponential life testing problems. They have shown that \( T/r \) is a minimum variance unbiased (MVU) estimator for the mean \( \theta \) of an exponential distribution and that \( 2T/\theta \) has a chi-square distribution with \( 2r \) degrees of freedom.

Let the ordered \( T_{in}'s \) be denoted by \( T_{1n} \leq T_{2n} \leq \cdots \leq T_{kn} \).

**PROCEDEDE \( R_1 \):** Select the set of \( s \) populations corresponding to \( T_{(k-s+1)n}, T_{(k-s+2)n}, \ldots, T_{(k)n} \).

**Probability of Correct Selection and Solution:**

Let \( Y_i \) denote the statistic associated with the population having parameter \( \theta_{(i)} (i=1,2,...,k) \). That is, the set \( (Y_1,Y_2,...,Y_k) \) is some permutation of the set \( (T_{1n},T_{2n},...,T_{kn}) \) and \( Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k \) correspond to the \( t \) best populations.
Let
\[ n \geq \max \{r, \lfloor (r-1)\alpha^*/(\alpha^*-1)\rfloor \} \quad \ldots (3.3) \]
where \([x]\) is the smallest integer greater than or equal to \(x\).

Thus to carry out the selection procedure, we specify \(\alpha^*\) and \(P^*\) first, find \(r\) satisfying requirement (3.18) specified below and then decide the value of \(n\) satisfying (3.3). In fact, the properties of the selection procedure don't depend on \(n\), the notional sample size but only on \(r\), the actual number of observations made. The PCS using the selection procedure \(R_q\) is given by

\[
P(CS : \Theta, R_q) = P\{\text{c-th largest of } (Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k) > (s-c+1) \text{-th largest of } (Y_1, Y_2, \ldots, Y_{k-t}) : \Theta, R_q\} \]

Let the distribution function of \(Y_1\) be \(G_1(x) = G(x; \Theta_1)\). Since \(F_1(x) = F(x; \Theta_1)\) differ only in the scale parameters \(\Theta_1\), it follows that \(T_i/\Theta_i, i = 1, 2, \ldots, k\) are independent and identically distributed random variables, say, with distribution \(G(\cdot; \cdot)\). Also

\[
G(x; \Theta_i) = P\{T_i \leq x\} = P\{(T_i/\Theta_i) \leq (x/\Theta_i)\} = G(x/\Theta_i) \quad \text{for } i = 1, 2, \ldots, k, \text{ and, therefore } \Theta_1 \leq \Theta_2 \leq \ldots \leq \Theta_k \text{ implies that } G(x; \Theta_1) \geq G(x; \Theta_2) \geq \ldots \geq G(x; \Theta_k). \]

Hence, the family of distribution functions of \(Y_i\)'s is a SI family of distributions. Therefore, by Lemma 2.1, the infimum of the
PCS occurs when the configuration of \( \theta_1 \)'s is

\[
\theta_1 = \theta_2 = \cdots = \theta_{k-t} = \theta^* \quad \text{(say)}
\]

and

\[
\theta_{k-t+1} = \theta_{k-t+2} = \cdots = \theta_k = \theta^* \quad \text{(say)}
\]  

\[
\theta^* = \theta^* \theta^*
\]

where \( \theta^* = \theta^* \theta^* \).

Under the least favourable configuration (3.4), the random variables \( Y_1, Y_2, \ldots, Y_{k-t} \) are i.i.d. with distribution \( G(x; \theta) \) and the random variables \( Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k \) are i.i.d. with distribution \( G(x; \theta^*) \). Thus we have, under the configuration (3.4), that

\[
\inf_{\theta \in \Theta} \mathbb{P}(G : \theta, \bar{y})
\]

\[
= \mathbb{P}\{\text{c}th \text{ largest of } (Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k) \\
\geq (s-c+1)st \text{ largest of } (Y_1, Y_2, \ldots, Y_{k-t})\}
\]

\[
= \mathbb{P}\{\text{c}th \text{ largest of } (\theta^* Y'_{k-t+1}, \ldots, \theta^* Y'_{k}) \\
\geq (s-c+1)st \text{ largest of } (\theta Y'_1, \theta Y'_2, \ldots, \theta Y'_{k-t})\}
\]

\[
= \mathbb{P}\{\text{c}th \text{ largest of } (a^* Y'_{k-t+1}, \ldots, a^* Y'_{k}) \\
\geq (s-c+1)st \text{ largest of } (Y'_1, Y'_2, \ldots, Y'_{k-t})\}
\]

\[
\text{where}
\]

\[
Y'_i = \begin{cases} 
Y_i / \theta & , \ i=1,2,\ldots,k-t \\
Y_i / \theta^* & , \ i=k-t+1,\ldots,k
\end{cases}
\]

\[
(3.5)
\]

and, therefore, \( Y'_1 (i=1,2,\ldots,k) \) are i.i.d. random variables having a distribution independent of the parameter \( \theta \).

Let \( X_{11}, X_{12}, \ldots, X_{1n} \) denote the random sample from the population having parameter \( \theta_1 \) and let
Therefore, from (3.6) and the definition of $Y_i$, 

$$y_i = \sum_{\alpha=1}^{r-1} w_{i\alpha}^{(n-r+1)} w_i^{(r)}$$

for $i = 1, 2, \ldots, k$.

Suppose that the hazard function corresponding to $F_1(x) = F(x; \theta_1)$ is $H_1^*(x)$ and, therefore, we can write

$$F_1(x) = 1 - \exp \{-H_1^*(x)\} \quad \text{(3.7)}$$

Also from the fact that a differentiable function $\mu(x)$ is convex if and only if its derivative $\mu'(x)$ is increasing in $x$, it follows that $F_1(x)$ is an IFR distribution if and only if its hazard function $H_1^*(x)$ is convex in $x$.

Let $H(x) = H_1(x)$ ($i=1,2,\ldots,k$) be the common hazard function of distributions corresponding to the random variables $X_i/\theta_i$ ($i=1,2,\ldots,k$). Then $H(x)$ is independent of the parameters $\theta_i$ ($i=1,2,\ldots,k$) and is also increasing and convex in $x$. Since $H(x)$ is an increasing function of $x$, we have from (3.5) that

$$\inf_{\theta \in \Omega} P(CS: \theta, R_1) = \{c \text{ th largest of } \left( \frac{d^*_1}{n} y_{k+t+1}^*, \ldots, \frac{d^*_k}{n} y_k^* \right) \}
> (s-c+1)^{st} \text{ largest of } \left( \frac{y_1^*}{n}, \frac{y_2^*}{n}, \ldots, \frac{y_{k-t}^*}{n} \right)$$
By using either the definition of a convex function or applying Lemma 4.1 of Barlow and Proschen (1966) with constants 
\[ a_1 = a_2 = \ldots = a_{r-1} = \frac{1}{n}, a_r = \frac{(n-r+1)}{n}, a_{r+1} = \ldots = a_n = 0, \]
we get, that is,
\[ H\left\{ \sum_{\alpha=1}^{n} a_\alpha w_i^{\alpha} \right\} \leq \sum_{\alpha=1}^{n} a_\alpha H(w_i^{\alpha}) \]
that is,
\[ H\left\{ \frac{1}{n} \left( \sum_{\alpha=1}^{r-1} w_i^{\alpha} + (n-r+1) w_i^{r} \right) \right\} \leq \frac{1}{n} \left\{ \sum_{\alpha=1}^{r-1} H(w_i^{\alpha}) + (n-r+1) H(w_i^{r}) \right\} \quad (3.9) \]

Using (3.9) in (3.8), we get,
\[ \inf_{\theta \in \mathbb{S}} \mathbb{P}(CS : \Theta, R)^{(\alpha^*}) \]
\[ > P\{c^{th} \text{ largest of } \left[ H\left( \frac{1}{n} \sum_{\alpha=1}^{r-1} w_i^{\alpha} + (n-r+1) w_i^{r} \right) \right], \]
\[ > \ldots, H\left( \frac{1}{n} \sum_{\alpha=1}^{r-1} w_i^{\alpha} + (n-r+1) w_i^{r} \right) \right\}] \]
\[ > (s-c+1)^{st} \text{ largest of } \left[ \frac{1}{n} \sum_{\alpha=1}^{r-1} H(w_i^{\alpha}) + (n-r+1) H(w_i^{r}) \right], \]
\[ > \ldots, \frac{1}{n} \sum_{\alpha=1}^{r-1} H(w_i^{\alpha}) + (n-r+1) H(w_i^{r}) \right\}] \quad (3.10) \]
Also applying Lemma 4.3 of Barlow and Proschan (1966) with constants \( b_1 = b_2 = \ldots = b_{r-1} = d^* / n, \) \( b_r = \left\{ \frac{(n-r+1)d^*}{n} \right\} > 1, \) (by (3.3)), \( b_{r+1} = \ldots = b_n = 0, \) we get

\[
H\{\sum_{\alpha=1}^{n} b_{\alpha} W_{1\alpha}^r\} > \sum_{\alpha=1}^{n} b_{\alpha} H(W_{1\alpha}^r)
\]

that is,

\[
H\{\frac{d^*}{n} \sum_{\alpha=1}^{r-1} W_{1\alpha}^r + (n-r+1) W_{1\alpha}^r\}
\]

\[
> \frac{d^*}{n} \sum_{\alpha=1}^{r-1} H(W_{1\alpha}^r) + (n-r+1) H(W_{1\alpha}^r)
\]  \( \ldots (3.11) \)

Using (3.11) in (3.10), we get

\[
\inf_{\theta \in \Theta} \mathcal{P}(CS : \theta, R_1)
\]

\[
> P\{s^{th} \text{ largest of} \left[ \frac{d^*}{n} \sum_{\alpha=1}^{r-1} H(W_{k-t+1,\alpha}^r) + (n-r+1) H(W_{k-t+1,\alpha}^r) \right], \ldots, \frac{d^*}{n} \sum_{\alpha=1}^{r-1} H(W_{k-r,\alpha}^r) + (n-r+1) H(W_{k-r,\alpha}^r) \}
\]

\[
> (s-c+1)^{st} \text{ largest of} \left[ \frac{1}{n} \sum_{\alpha=1}^{r-1} H(W_{k-t,\alpha}^r) + (n-r+1) H(W_{k-t,\alpha}^r) \right], \ldots, \frac{1}{n} \sum_{\alpha=1}^{r-1} H(W_{k,\alpha}^r) + (n-r+1) H(W_{k,\alpha}^r) \} \]  \( \ldots (3.12) \)

We also note that the variable \( Z_{1\alpha} = H(W_{1\alpha}^r) \) is the \( \alpha \)th order statistic in a random sample of size \( r \) from the standard exponential distribution, \( i = 1, 2, \ldots, k, \) \( \alpha = 1, 2, \ldots, r. \) This is because of the relation (3.7) between the distribution function and the hazard function. Now define

\[
Z_{1} = \sum_{\alpha=1}^{r-1} Z_{1\alpha} + (n-r+1) Z_{1r}
\]  \( \ldots (3.13) \)

Then \( Z_{1} (i=1, 2, \ldots, k) \) has a chi-square distribution with \( 2r \) degrees of freedom (see, Epstein and Sobel (1953)). Now using the definition of \( Z_{1\alpha} \) in (3.13) we get
for $i = 1, 2, \ldots, k$ and, therefore, from (3.12), it follows that

$$
\inf_{\theta \in \Theta} \min_{d^*}(P(\theta : \Theta, z)) > P\{c^{th} \text{ largest of } (\frac{a}{n} Z_{k-t+1}, \ldots, \frac{d^*}{n} Z_{k-t}) \}
\geq (s-c+1)^{st} \text{ largest of } (\frac{1}{n} Z_{1}, \ldots, \frac{1}{n} Z_{k-t})
\geq (s-c+1)^{st} \text{ largest of } (2Z_{1}, \ldots, 2Z_{k-t})
\geq (s-c+1)^{st} \text{ largest of } (2Z_{1}, \ldots, 2Z_{k-t}) \} \quad (3.14)
$$

Now $2Z_{1}, \ldots, 2Z_{k-t}$ are i.i.d. random variables distributed like $G_0(\cdot)$ and the random variables $2^a Z_{k-t+1}$, $\ldots, 2^a Z_{k}$ are i.i.d. random variables distributed like $G_0^*(\cdot)$, where

$$
G_0(x) = \int_0^x \frac{(1/2)^r e^{-(1/2)y}y^{r-1}}{(r-1)!} \ dy \quad (3.15)
$$

and

$$
G_0^*(x) = \int_0^x \frac{(1/2^a)^r e^{-(1/2^a)y}y^{r-1}}{(r-1)!} \ dy \quad (3.16)
$$

Therefore, by Theorem 2.1 we see that the right hand side of (3.14) is equal to

$$
Q^*(a^*, r) = \frac{s-c}{(t-c, c-1)} \sum_{\alpha=0}^{s-c} \binom{k-t}{\alpha} \int_0^\infty [G_0(x)]^{k-t-\alpha} [1-G_0(x)]^\alpha 
\cdot [G_0^*(x)]^{t-c} [1-G_0^*(x)]^{c-1} g_0^*(x) \cdot (3.17)
$$

where $G_0(x)$ and $G_0^*(x)$ are given by (3.15) and (3.16) respectively.
Hence
\[
\inf_{\varepsilon, R_1} P(CS : \varepsilon, R_1) > \varphi_1(d^*, r) = \varphi(d^*, r)
\]
where \(\varphi(d^*, r)\) is the same as defined in (2.11) with \(n\) replaced by \(r\). For given \(k, d^*\) and \(P^*\), the solution to the problem is obtained by first finding the smallest integer \(r\) which satisfies
\[
\varphi(d^*, r) > P^* \quad \cdots (3.18)
\]
and then finding \(n\) from (3.3). It is to be noted that \(\varphi(d^*, r)\) does not depend upon \(n\). The tables given for the exponential case are used to find \(r\) directly from (3.18). (In the tables reading \(r\) in place of \(n\).)

**Selection of IFR populations associated with small**

**Values of the Scale Parameter:**

If we are interested in selecting a subset of fixed size \(s\) from the \(k\) IFR populations that includes at least \(c\) of these \(t\) best populations which correspond to \(\varepsilon(1), \ldots, \varepsilon(t)\), then we use the selection procedure \(R_2\).

**PROCEDURE \(R_2\)** Select the set of \(s\) populations associated
\[
T(1)n, T(2)n, \ldots, T(s)n^*
\]
Here we take the preference zone to be the set
\[
\bigcup_{1} (d^*) = \{\varepsilon : (t+1) / \varepsilon(t) \geq d^*\}
\]
where \(d^* > 1\). On similar lines, as above, we can show that in this case
\[
\inf_{\varepsilon, R_2} P(CS : \varepsilon, R_2) > \varphi_1(d^*, r)
\]
where

\[ q_1(q^*, r) = \frac{t!}{(t-c)!} \sum_{c=0}^{s-c} \left( k-t \right)^c \int_0^\infty [G_0^*(x)]^c [1-G_0^*(x)]^{k-t-c} \]

\[ \times \left( G_0(x) \right)^{c-1} \left[ 1-G_0(x) \right]^{t-c} dG_0(x), \]

where \( G_0(*) \) and \( G_0^*(*) \) are defined by (3.15) and (3.16) respectively.

3.4 Selection Procedures for IFRD Distributions

Let \( F_j(x) = F(x; \theta_j) \) be an IFRD distribution which corresponds to the population \( \mathcal{P}_i \), \( i = 1, 2, \ldots, k \). Here also the framework formulated in section 3.2 is used. Let \( X_{11}, X_{12}, \ldots, X_{1n} \) be independent observations from the population \( \mathcal{P}_i \), \( i = 1, 2, \ldots, k \) and let \( T_{jn} = T(X_{j1}, X_{j2}, \ldots, X_{jn}) \), \( i = 1, 2, \ldots, k \), be an appropriate set of independent statistics. Let \( T(1)n \leq T(2)n \leq \cdots \leq T(k)n \) be the ordered \( T_{jn} \)’s. Let \( G_0(*) | \theta_i \) be the distribution function of \( T_{jn} \), \( i = 1, 2, \ldots, k \). We use the procedure \( R_1 \) to select a subset of \( s \) populations from the \( k \) IFRD populations which includes at least \( c \) of the \( t \) best populations, that is, those associated with \( \theta_i \) \( (k-t+1), \theta_i (k-t+2), \ldots, \theta_i (k) \).

We now show that the infimum of the PCS over the preference zone \( \Omega \) for the IFRD populations is greater than or equal to the corresponding infimum of the PCS for exponential populations provided the statistic \( T \) satisfies certain conditions.

Probability of Correct Selection and Solution:

Let \( Y_1 \) be the statistic \( T_{jn} \) based on the sample from the population with the parameter \( \theta_i \) \( (i = 1, 2, \ldots, k) \) and let
\[ Y(1) \leq Y(2) \leq \cdots \leq Y(k) \] be the ordered \( Y_i \)'s. Since the set 
\((Y_1, Y_2, \ldots, Y_k)\) is some permutation of the set 
\((T_{1n}, T_{2n}, \ldots, T_{kn})\), it follows that 
\[ Y_i = T_i \quad \text{for} \quad i = 1, 2, \ldots, k. \] 
The procedure \( R_1 \) which is defined in terms of the statistics \( T_{in} \)'s, therefore, 
based on the statistics \( Y_i \). We make the following three 
assumptions regarding the statistics \( Y_i \).

**Assumption 3.1.** The statistic \( Y_i \) is a non-decreasing function of the observations from the population with the parameter \( \theta(1) \), that is,

\[ Y_i(z_1, z_2, \ldots, z_n) \leq Y_i(w_1, w_2, \ldots, w_n) \]

if \( w_i > z_i \) for \( i = 1, 2, \ldots, n \) where \( z_1, z_2, \ldots, z_n \) and \( w_1, w_2, \ldots, w_n \)
represent some two random samples from the population which 
corresponds to the parameter \( \theta(1) \).

**Assumption 3.2.** The distribution \( G_n(\cdot; \theta(1)) \) of \( Y_i \) is an absolutely continuous distribution.

**Assumption 3.3.** The family \( \mathcal{F} = \{ G_n(\cdot; \theta(1)) : \theta(1) \in \Theta \} \)
of the distribution functions is a SI family for each positive integer \( n \).

[The assumptions 3.2 and 3.3 are the same as the Assumptions 2.1 and 2.2 made in Chapter II.]

The PCS for the selection procedure \( R_1 \) is given by

\[ P(\text{CS} : \Theta, R_1) = P\{ \text{c-th largest of } (Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k) \]
\[ > (s-c+1) \text{-th largest of } (Y_1, Y_2, \ldots, Y_{k-t}) : \Theta, R_1 \} \]
where $Y_1, Y_2, \ldots, Y_k$ are independent random variables such that the distribution function of $Y_i$ is $G_i(\cdot; \theta_i)$, $i = 1, 2, \ldots, k$.

Under the Assumptions 3.2 and 3.3, by Lemma 2.1, it follows that the infimum of the PCS occurs when the configuration of the parameters $\theta_i$'s is

$$\theta(1) = \theta(2) = \cdots = \theta(k-t) = \theta \quad \text{(say)}$$

and

$$\theta(k-t+1) = \theta(k-t+2) = \cdots = \theta(k) = \theta^* \quad \text{(say)} \quad \ldots \quad (3.19)$$

where $\theta^* = \theta^* \theta$.

Now under the configuration (3.19), $Y_1, Y_2, \ldots, Y_{k-t}$ are i.i.d. random variables with distribution $G_n(\cdot; \theta^*)$ and each is based on a sample of size $n$ from $F(x) = F(x; \theta)$ and $Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k$ are i.i.d. random variables with distribution $G_n(\cdot; \theta^*)$ and each is based on a sample of size $n$ from $F^*(x) = F(x; \theta^*)$. Thus we have

$$F^*(x) = F(x/\theta^*) \quad \ldots \quad (3.20)$$

and

$$\inf_{\theta \in \Omega} F(\cdot; \theta, \theta^*) = \frac{t!}{(t-c)(c-1)!} \sum_{\alpha=0}^{t-c} \left( \begin{array}{c} k-t \alpha \\ \alpha \end{array} \right) \int_0^\infty [G_n(x; \theta)]^{k-t-\alpha} [1 - G_n(x; \theta)]^\alpha [G_n(x; \theta^*)]^{t-c} [1 - G_n(x; \theta^*)]^{c-1} 4G_n(x; \theta^*) \quad \ldots \quad (3.21)$$

We now generalize a result of Bejsum (1969) which is used to prove that the infimum of the PCS (3.21) for IFRA populations is greater than or equal to the corresponding infimum of the PCS for the exponential populations.
Lemma 3.1. Let \( u_1, u_2, \ldots, u_M \) and \( v_1, v_2, \ldots, v_N \) be two independent random samples from the populations with continuous distribution functions \( F(.) \) and \( F^*(.) \) (these need not be the same \( F(.) \) and \( F^*(.) \) as defined by (3.20) above). Let \( \mathcal{P}(u, v) \) be a function such that

1. \( \mathcal{P}(u_1, u_2, \ldots, u_M; v_1, \ldots, v_N) \)

\[ \leq \mathcal{P}(u_1, \ldots, u_M; v_1^*, \ldots, v_N^*) \]

if \( v_j^* \geq v_j \), \( j = 1, 2, \ldots, N \) and

2. for every increasing function \( g(.) \),

\[ \mathcal{P}(u_1, \ldots, u_M; v_1, \ldots, v_N) = \mathcal{P}(g(u_1), \ldots, g(u_M); g(v_1), \ldots, g(v_N)) \]

Let \( F^*(x) = F(x/d^*) \), \(-\infty < x < \infty\), \( d^* > 1 \). If \( F \leq K \) (that is, \( K^{-1}F(x)/x \) is non-decreasing on the support of \( F \)), \( K^*(x) = K(x/d^*) \) and \( K(x) < 1 \) for each \( x < \infty \), or \( F(x) < 1 \) for each \( x < a \), then

\[ \mathcal{E}_{F, F^*} \mathcal{P}(u, v) \leq \mathcal{E}_{F, F^*} \mathcal{P}(u, v) \quad \ldots \quad (3.22) \]

for each \( d^* > 1 \).

Proof: Let \( u_i^* = K^{-1}[F(u_i)] \), \( i = 1, 2, \ldots, M \)

\[ v_j^* = K^{-1}[F(v_j)] \quad j = 1, 2, \ldots, N \]

Since \( F \) is star ordered with respect to \( K \), \( K^{-1}F \) is a non-decreasing function. Therefore, by property (ii) of \( \mathcal{P} \), we have
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\[ \mathcal{G}(u_1, \ldots, u_M; v_1, \ldots, v_N) \]

\[ = \mathcal{G}(u^*_1, \ldots, u^*_M; v^*_1, \ldots, v^*_N) \quad \ldots (3.23) \]

Let

\[ v^*_j = d^* K^{-1}[F(v_j/d^*)], \quad j = 1, 2, \ldots, N \]

Then the definition of the star shaped ordering implies that (because \( d^* \geq 1 \)),

\[ v^*_j = d^* K^{-1}[F(v_j/d^*)] \leq K^{-1}[F(v_j)] = v_j \]

provided \( v_j \) and \( v_j/d^* \) are in \( I_F = \{ x : 0 < F(x) < 1 \} \). Since

\[ P\{v_j/d^* \leq z\} = F(v_j \leq d^* z) = F^*(z d^*) = F(z), \]

it follows that the distribution of \( v_j/d^* \) is the same as that of \( u_j \) and therefore \( v_j/d^* \) is in \( I_F \) with probability 1. If \( v_j \) is not in \( I_F \), then \( F(v_j) = 1 \) and \( v^*_j = K^{-1}(1) = \infty \), under the additional condition: \( K(x) < 1 \) for each \( x < \infty \). On the other hand, if \( F(x) < 1 \) for each \( x < \infty \), then \( v_j \) is in \( I_F \) a.s.

(almost surely). Hence in all cases

\[ v^*_j \geq v^*_j \quad \text{a.s.} \]

Also by property (i) of the function \( \mathcal{G} \), we have

\[ \mathcal{G}(u^*_1, \ldots, u^*_M; v^*_1, \ldots, v^*_N) \]

\[ \leq \mathcal{G}(u^*_1, \ldots, u^*_M; v^*_1, \ldots, v^*_N) \quad \ldots (3.24) \]
Now (3.23) and (3.24) imply that

\[ \mathcal{P}(u_1^*, \ldots, u_M^* ; v_1^*, \ldots, v_N^*) \]

\[ \leq \mathcal{P}(u_1, u_2, \ldots, u_M ; v_1, \ldots, v_N) \quad \cdots (3.26) \]

Also

\[ P[u_1^* \leq x] = P[K^{-1}F(u_1) \leq x] = P[F(u_1) \leq K(x)] = K(x) \]

and

\[ P[v_j^* \leq x] = P[a^*k^{-1}[F(v_j/a^*)] \leq x] \]

\[ = P[F(v_j/a^*) \leq K(x/a^*)] \]

\[ = K(x/a^*) = K^*(x) \]

Therefore, the distribution of \( u_1^* \) is \( K(u) \) and that of \( v_j^* \) is \( K^*(v) \). Hence taking expectations on both sides of (3.25), we get

\[ E_{K, K^*}(\mathcal{P}(u, v)) \leq E_{F, F^*}(\mathcal{P}(y, z)) \]

as was to be proved.

**Theorem 3.1.** The infimum of the PCS for IF3) populations given by (3.21) is greater than or equal to the corresponding infimum of the PCS for exponential populations provided the order of \( Y_i \)'s is preserved if the observations are operated upon by any increasing function \( g(.) \), that is,

\[ Y_i(g(x_{i1}^t), \ldots, g(x_{i_1}^t)) \leq Y_j(g(x_{j1}^t), \ldots, g(x_{j_1}^t)) \]

if

\[ Y_i(x_{i1}^t, \ldots, x_{i_1}^t) \leq Y_j(x_{j1}^t, \ldots, x_{j_1}^t) \]

for \( i, j = 1, 2, \ldots, k \).

**Proof:** Under the least favourable configuration (3.19), \( Y_1, Y_2, \ldots, Y_{k-1} \) are based on independent samples (each of size \( n \))
from the distribution \( F(x) = F(x; \theta) \) and \( Y_{k-t+1}, \ldots, Y_k \) are based on independent samples (each of size \( n \)) from the distribution \( F^*(x) = F(x; \theta^*) = F(x/d^*) \). Define the function \( \mathcal{Q} \) as follows.

\[
\mathcal{Q}(x_{11}, \ldots, x_{1n}; \ldots; x_{k1}, \ldots, x_{kn})
\]

\[
= \begin{cases} 
1, & \text{if } c^{th} \text{ largest of } (Y_{k-t+1}, \ldots, Y_k) \\
\text{elsewise} & > (s-c+1)^{st} \text{ largest of } (Y_1, Y_2, \ldots, Y_{k-t}) \\
0, & \text{otherwise} 
\end{cases} \quad \ldots (3.26)
\]

Then

\[
\inf_{\theta \in \Theta} \mathcal{Q}(CS: \theta, R_k) = E_{F^*}(\mathcal{Q}) \quad \ldots (3.27)
\]

and the function \( \mathcal{Q} \) depends on \( M = (k-t)n \) independent observations from \( F(x) \) and \( N = n^* \) independent observations from \( F^*(x) \). Let the observations from \( F(x) \) be denoted by \( u_1, u_2, \ldots, u_M \) and those from \( F^*(x) \) by \( v_1, v_2, \ldots, v_N \). Thus

\[
\mathcal{Q}(x_{11}, \ldots, x_{1n}; \ldots; x_{k1}, \ldots, x_{kn})
\]

\[
= \mathcal{Q}(u_1, \ldots, u_M; v_1, \ldots, v_N) \quad \ldots (3.28)
\]

The function \( \mathcal{Q} \) defined by (3.26) and (3.28) satisfies the requirements of Lemma 3.1. Hence, if we take \( K(.) \) to be star-ordered with respect to \( K(*. \), then

\[
E_{K, K^*}(\mathcal{Q}) \leq E_{F, F^*}(\mathcal{Q}) \quad \ldots (3.29)
\]

where \( K^*(x) = K(x/d^*) \).

Also we know that \( F \) is an IFRA distribution if and only if \( F \) is star-ordered with respect to the standard exponential
distribution. Hence taking $K$ to be the standard exponential distribution and $F$ to be any IFR distribution it follows from (3.27) and (3.29) that the infimum of the PCS for IFR populations is greater than or equal to the corresponding infimum of the PCS when the statistics $Y_1, Y_2, \ldots, Y_{k-t}$ are based on independent samples (each of size $n$) from $K(x) = 1 - \exp(-x)$ and $Y_{k-t+1}, \ldots, Y_k$ are based on independent samples (each of size $n$) from $K^*(x) = 1 - \exp\left(-x/a^*_t\right)$. This completes the proof of the theorem.

**Example 3.1.** Take the statistics $T_{in}$ to be

$$T_{in} = \max_{1 \leq i \leq n} X_{ij}.$$  

It is seen that conditions (i) and (ii) of Lemma 3.1 are satisfied if the function $\mathcal{Q}$ defined by (3.26) depends on the data only through these statistics. Then for the exponential case, the statistics $Y_1, Y_2, \ldots, Y_{k-t}$ are i.i.d. random variables with distribution

$$G(x) = [1 - \exp(-x)]^n$$

and $Y_{k-t+1}, Y_{k-t+2}, \ldots, Y_k$ are i.i.d. random variables with distribution

$$G^*(x) = G(x/a^*) = [1 - \exp\left(-x/a^*_t\right)]^n$$

and, therefore, in the exponential case, using Theorem 2.1, we get...
by substituting \( \exp(-x/d^*)=y \), Probability obtained from (3.30) is a lower bound to the actual PCS for IFRA populations when the selection procedure \( R_1 \) is based on the statistic \( T_n = \max_{1 \leq j \leq n} X_j \) and hence the selection procedure \( R_1 \) may be said to be a conservative procedure.

**Remark 3.1.** The case when the \( t \) best IFRA populations are the populations associated with \( \theta(1), \theta(2), \ldots, \theta(t) \), can be dealt with on similar lines by using the selection procedure \( R_2 \).

**Remark 3.2.** The selection procedure given by Patel (1967) is based on the number of failures obtained in the time interval \([0,T]\) for each population and not on the times at which the failure occurred.

The procedures developed by Patel (1976) and also our selection procedures \( R_1 \) and \( R_2 \) of Section 3.3 make use of the actual failure times using a specific statistic of Epstein and Sobel (1953). The experimenter stops the moment \( r \) items out of \( n \)
units on life test for each population fail and the value of $r$ is to be determined from (3.3). If $r$ is taken to be equal to $n$, then the inequality (3.3) reduces to $d^* > n$ for all $n$, which is impossible. So these procedures cannot be used for $r = n$. Also from the inequality (3.3) it follows that $d^* > n/(n-r+1)$. So $d^*$ can be taken near to 1 if $r$ is small. As $r$ increases, $n/(n-r+1)$ also increases and so in that case $d^*$ will have to be considerably large. Such difficulties don't appear in the procedures of Section 3.4.

Barlow and Gupta (1969) have developed selection procedures for the IFRA and IFR distributions using the subset selection approach. The selection procedures for the IFRA distributions are for the selection of the population with the smallest (largest) $x$-quantile and are based on order statistics. For the case of IFR distributions, which differ only in location, these selection procedures are based on the sample means. However, the lower bound to the infimum of the PCS obtained by them, for the IFR case, does not depend on the sample size $n$, it being the same as would be obtained by using a sample of size 1.

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