Φ-REPRESENTATION OF SO-RINGS
§1.1. Congruence relations

In this section we obtain a necessary and sufficient condition for $R/\theta$ to be a so-ring. We begin with the following.

**Definition 1.1.1.** Let $((R^i, \Sigma^i, i^i) : i \in I)$ be a family of partial semirings. Then the partial semiring $(\prod R^i, \Sigma, \cdot)$ is called the *product partial semiring* where $\prod R^i$ is the cartesian product of the $R^i$'s, $\Sigma$ and $\cdot$ are defined as follows:

Let $(x_j : j \in J)$ be a family in $\prod R^i$. Then each $x_j = (x^i_j : i \in I)$, where $x^i_j \in R^i$. We say the family $(x_j : j \in J)$ is summable in $\prod R^i$ and we write $\Sigma_j(x_j : j \in J) = (\Sigma_j(x^i_j : j \in J) : i \in I)$ if for each $i \in I$, $(x^i_j : j \in J)$ is summable in $R^i$.

For any $(x_i : i \in I), (y_i : i \in I)$ in $\prod R^i, (x_i + y_i : i \in I) = (x^i + y^i : i \in I)$.

**Definition 1.1.2.** Let $R$ be a partial semiring and $\theta$ be an equivalence relation on $R$. Then $\theta$ is said to be a *partial semiring congruence* on $R$ if and only if $\theta$ is closed under the additive and multiplicative operations of the product partial semiring $R \times R$.

**Definition 1.1.3.** Let $(R, \Sigma, \cdot)$ be a partial semiring and $\theta$ be a partial semiring congruence on $R$. Then their quotient is the structure $(R/\theta, \Sigma', \cdot')$ where $R/\theta = \{x \mid x \in R\}$ (where $x$ is the equivalence class containing $x$ with respect to $\theta$), $\Sigma'$, $\cdot'$ are defined as follows:

The family $(\overline{x}_i : i \in I)$ is summable in $R/\theta$ if and only if the family $(x_i : i \in I)$ is summable in $R$ and we write $\Sigma \overline{x}_i = \overline{\Sigma x}_i$.

We define $\overline{x} \cdot \overline{y} = \overline{x \cdot y}$ for any $\overline{x}, \overline{y} \in R/\theta$.

The following example shows that $R/\theta$ need not be a partial semiring.

**Example 1.1.4.** We know that $(P(D), \Sigma, \cdot)$ is a partial semiring, where

$$\Sigma A_i = \begin{cases} \bigcup A_i, & \text{if } A_i \cap A_j = \emptyset \forall i \neq j \\ \text{undefined, otherwise} & \end{cases}$$
and $A \cdot B = A \cap B$. Take $D = \{x, y\}$. Then $\theta = \{(\emptyset, \emptyset), (\{x\}, \{x\}), (\{y\}, \{y\}), (D, D), (\{x\}, D), (D, \{x\}), (\Phi, \{y\}), (\{y\}, \Phi)\}$ is partial semiring congruence on $P(D)$. Now $P(D)/\theta = \{\overline{\Phi}, \{x\}\}$, where $\overline{\Phi} = \{\Phi, \{y\}\} = \overline{\{y\}}$, $\{x\} = \{\{x\}, D\} = \overline{D}$. Here $\{x\} + \{y\}$ is defined but $\{x, y\} + \{y\}$ is not defined, hence $\overline{\{x\}} + \overline{\{y\}}$ is not well defined. Hence $P(D)/\theta$ is not a partial semiring.

**Remark 1.1.5.** Let $\theta$ be a partial semiring congruence on a partial semiring $R$. Then a necessary and sufficient condition for $R/\theta$ to be partial semiring is that:

$$(y_i : i \in I)$$ is summable whenever $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

**Proof.** Suppose $R/\theta$ is partial-semiring, $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

Then $(\overline{x_i} : i \in I)$ is summable and $\overline{x_i} = \overline{y_i}$, $i \in I$.

$\Rightarrow (\overline{y_i} : i \in I)$ is summable.

Hence $(y_i : i \in I)$ is summable.

Conversely suppose the condition is true and let $(\overline{x_i} : i \in I)$, $(\overline{y_i} : i \in I)$ in $R/\theta$ such that $(\overline{x_i} : i \in I)$ is summable and $\overline{x_i} = \overline{y_i}$, $i \in I$.

Then $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

By condition, we get $(y_i : i \in I)$ is summable.

$\Rightarrow (\overline{y_i} : i \in I)$ is summable and $\Sigma \overline{x_i} = \Sigma \overline{y_i}$.

Hence $R/\theta$ is a partial semiring. \qed

The following is an example of a so-ring $R$ in which $\theta$ is a partial semiring congruence on $R$ such that $R/\theta$ is a partial semiring but not a so-ring.

**Example 1.1.6.** Let $R$ be the set of all real numbers with finite support addition and ordinary multiplication. Take $M = R \times R$ and define $\Sigma$ on $M$ as componentwise addition, $*$ as $(a, b) * (c, d) = (ac + bd, ad + bc)$. Then $(M, \Sigma, *)$ is a so-ring. Define $\theta$ on $M$ as $(a, b) \theta (c, d)$ if and only if $a + d = b + c$. 

32
Now we prove that $\theta$ is a partial semiring congruence on $M$: For any $a, b \in R$, $(a, b)\theta(a, b)$. Let $(a, b)\theta(c, d)$. Then $a + d = b + c$. $\Rightarrow c + b = d + a$ and hence $(c, d)\theta(a, b)$. Let $(a, b)\theta(c, d)$ and $(c, d)\theta(e, f)$. Then $a + d = b + c$ and $c + f = d + e$.

$\Rightarrow a + d + c + f = b + c + d + e$. $\Rightarrow a + f = b + e$ and hence $(a, b)\theta(e, f)$. Hence $\theta$ is an equivalence relation on $M = R \times R$.

Let $((a_i, b_i) : i \in I)$, $((c_i, d_i) : i \in I)$ be summable families such that $(a_i, b_i)\theta(c_i, d_i)$, $i \in I$. Then $a_i + d_i = b_i + c_i$, $i \in I$. $\Sigma(a_i + d_i) = \Sigma(b_i + c_i)$.

$\Rightarrow \Sigma a_i + \Sigma d_i = \Sigma b_i + \Sigma c_i$. $\Rightarrow (\Sigma a_i, \Sigma b_i)\theta(\Sigma c_i, \Sigma d_i)$ and hence $[\Sigma(a_i, b_i)]\theta[\Sigma(c_i, d_i)]$.

Let $(a, b)\theta(c, d)$ and $(e, f)\theta(g, f)$. Then $a + d = b + c$ and $e + h = f + g$. Consider $ae + bf + ch + dg = be + ce + de + af + df - cf + cf + cg - ce + de + dh - df = be + af + cg + dh$.

$\Rightarrow (ae + bf, af + be)\theta(cg + dh, ch + dg)$ and hence $[(a, b) * (e, f)]\theta[(c, d) * (g, h)]$. Hence $\theta$ is a partial semiring congruence on $M$.

Clearly $M/\theta$ is a partial semiring. Since $(1, 1) = (2, 2) = (1, 2) + (1, 0)$ and $(1, 2) = (1, 1) + (0, 1)$, we have $(1, 2) \leq (1, 1)$ and $(1, 1) \leq (1, 2)$ but $(1, 1) \neq (1, 2)$.

Hence $M/\theta$ is not a so-ring.

**Definition 1.1.7.** A partial semiring congruence $\theta$ on a so-ring $R$ is said to have the diagonal property if it satisfies the following:

$$a\theta(b + k) \text{ and } b\theta(a + h) \Leftrightarrow a\theta b \forall a, b, h, k \in R.$$ 

**Theorem 1.1.8.** Let $\theta$ be a partial semiring congruence on a so-ring $R$ such that $R/\theta$ is a partial semiring. Then $R/\theta$ is a so-ring if and only if $\theta$ has the diagonal property.

**Proof.** Suppose $R/\theta$ is a so-ring. Then

$$x\theta(y + k) \& (x + h)\theta y \Leftrightarrow \bar{x} = \bar{y} + \bar{k} \& \bar{x} + \bar{h} = \bar{y} \Leftrightarrow \bar{y} \leq \bar{x} \& \bar{x} \leq \bar{y} \Leftrightarrow \bar{x} = \bar{y} \Leftrightarrow x\theta y.$$ 

Hence $\theta$ has the diagonal property.

Conversely suppose $\theta$ has the diagonal property. Then

$$\bar{x} \leq \bar{y} \& \bar{y} \leq \bar{x} \Leftrightarrow \bar{x} + \bar{h} = \bar{y} \& \bar{y} + \bar{k} = \bar{x} \text{ for some } h, k \in R \Leftrightarrow (x + h)\theta y \& x\theta(y + k)$$ 

$\Leftrightarrow x\theta y \Leftrightarrow \bar{x} = \bar{y}$. Hence $R/\theta$ is a so-ring. 

\[\Box\]
Definition 1.1.9. A partial semiring congruence \( \theta \) on a so-ring \( R \) is said to be a congruence relation on \( R \) if and only if \( \theta \) has the diagonal property.

Remark 1.1.10. \( \text{Con} R \), the set of all congruence relations on a so-ring \( R \) forms a complete lattice with \( 0_R \) and \( 1_R \), the smallest and largest congruences, respectively.

Definition 1.1.11. Let \( (R, \Sigma, \cdot), (R_1, \Sigma_1, \ast) \) be partial semirings. Then a mapping \( f : R \to R_1 \) is said to be a homomorphism if it satisfies the following

(i). whenever \( (x_i : i \in I) \) is summable in \( R \), then \( (f(x_i) : i \in I) \) is summable in \( R_1 \)
and \( f(\Sigma x_i) = \Sigma_1 f(x_i) \)
(ii). \( f(x \cdot y) = f(x) \ast f(y) \) for all \( x, y \) in \( R \).

Remark 1.1.12. If \( f : R \to R_1 \) is a homomorphism then \( f(0_R) = 0_{R_1}, f(1_R) = 1_{R_1} \).

Definition 1.1.13. The homomorphism \( p_i : (\prod_{i \in I} R_i, \Sigma, \cdot) \to (R_i, \Sigma_i, \cdot_i) \) defined by \( x \mapsto x^i \) is called the \( i \)-th projection map.

§1.2. Structure theorem

In this section we show that any so-ring is a subdirect product of subdirectly irreducible so-rings.

Definition 1.2.1. Let \( \{R_i \mid i \in I\} \) be a family of so-rings. Then a subset \( R \subset \prod_{i \in I} R_i \)
is said to be a subdirect product of so-rings \( R_i, i \in I \) if it satisfies the following

(i). \( R \) is subso-ring of \( \prod_{i \in I} R_i \)
(ii). all the projections of \( \prod_{i \in I} R_i \) are onto.

Definition 1.2.2. A so-ring \( R \) is said to be subdirectly irreducible if and only if

\[ \bigcap_{i \in I} \theta_i = 0_R \Rightarrow \theta_i = 0_R \text{ for some } i \in I. \]
Lemma 1.2.3. Let $R$ be a so-ring and $\theta$ be a congruence on $R$. Then there is a one to one correspondence between congruence relations of $R$ containing $\theta$ and congruence relations of $R/\theta$.

Proof. Let $\phi$ be a congruence of $R$ containing $\theta$.

Define a relation $\phi/\theta$ on $R/\theta$ by $[a][\phi/\theta][b]$ if and only if $a\phi b$, where $[a]$ denotes the equivalence class containing $a$ relative to $\theta$.

Since $\phi$ is congruence on $R$, $\phi/\theta$ is an equivalence relation on $R/\theta$.

Now we prove $\phi/\theta$ is congruence relation on $R/\theta$:

Let $([a_i] : i \in I)$ and $([b_i] : i \in I)$ be summable families in $R/\theta$ such that $[a_i][\phi/\theta][b_i]$, $i \in I$.

Then $(a_i : i \in I)$ and $(b_i : i \in I)$ are summable families in $R$ such that $a_i\phi b_i$, $i \in I$.

$\Rightarrow (\Sigma a_i)\phi(\Sigma b_i) \Rightarrow \Sigma [a_i][\phi/\theta][\Sigma b_i]$. Hence $\Sigma [a_i][\phi/\theta][\Sigma b_i]$.

Let $[a][\phi/\theta][b]$ and $[c][\phi/\theta][d]$. Then $a\phi b$ and $c\phi d$. $\Rightarrow (ac)\phi(bd)$.

$\Rightarrow ([a][c][\phi/\theta])([b][d])$.

Hence $\phi/\theta$ is a partial semiring congruence on $R/\theta$.

Now $[a][\phi/\theta][b][h]$ and $[a][\phi/\theta][b][k]$ for some $h, k \in R$.

$\Leftrightarrow a\phi(b + h)$ and $(a + k)\phi b \Leftrightarrow a\phi b \Leftrightarrow [a][\phi/\theta][b]$.

Hence $\phi/\theta$ is a congruence relation on $R/\theta$.

Let $\phi'$ be a congruence relation on $R/\theta$.

Define a relation $\phi_b$ on $R$ by $a\phi b$ if and only if $[a][\phi'][[b]]$.

Since $\phi'$ is congruence on $R/\theta$, $\phi_b$ is an equivalence relation on $R$.

Now we prove $\phi_b$ is a congruence relation on $R$ containing $\theta$:

Let $(a_i : i \in I)$ and $(b_i : i \in I)$ be summable families in $R$ such that $a_i\phi b_i$, $i \in I$.

Then $([a_i] : i \in I)$ and $([b_i] : i \in I)$ are summable families in $R/\theta$ such that $[a_i][\phi'][[b_i]]$, $i \in I$. $\Rightarrow \Sigma [a_i][\phi'][[b_i]]$ and hence $(\Sigma a_i)\phi(\Sigma b_i)$.
Let \( a \phi b \) and \( c \phi d \). Then \([a]^\theta \phi'[b]^\theta\) and \([c]^\theta \phi'[d]^\theta\).

\[\Rightarrow [ac]^\theta \phi'[bd]^\theta\] and hence \((ac)\phi(bd)\).

Hence \( \phi \) is partial semiring congruence on \( R \).

Now \( a \phi b + h \) and \((a + k) \phi b\) for some \( h, k \in R \) \( \iff [a]^\theta \phi'[b] + [h]^\theta\)
and \([a]^\theta + [k]^\theta\) \( \phi'[b]^\theta \iff a \phi b\).

Let \( a \theta b \). Then \([a]^\theta = [b]^\theta \) \( \Rightarrow [a]^\theta \phi'[b]^\theta \Rightarrow a \phi b\). Hence \( \theta \subseteq \phi \).

Hence \( \phi \) is a congruence relation on \( R \) containing \( \theta \).

Let \( \phi \) be a congruence relation of \( R \) containing \( \theta \).

Then \( \phi / \theta \) is a congruence relation on \( R/\theta \).

\[\Rightarrow (\phi / \theta) / \theta \) is a congruence relation on \( R / \theta \).

Note that \( a \phi b \iff [a]^\theta (\phi / \theta)[b]^\theta \iff a(\phi / \theta) b\).

Hence \((\phi / \theta) / \theta = \phi \).

Let \( \phi' \) be a congruence relation on \( R / \theta \).

Then \( \phi \) is a congruence relation of \( R \) containing \( \theta \).

\[\Rightarrow (\phi \theta) / \theta \) is a congruence relation of \( R / \theta \).

Note that \([a]^\theta \phi'[b]^\theta \iff a \phi b \iff [a]^\theta (\phi \theta)[b]^\theta\).

Hence \((\phi \theta) / \theta = \phi' \).

Hence the lemma. \( \square \)

**Lemma 1.2.4.** Let \( \{\theta_i \mid i \in I\} \) be a family of congruences of a so-ring \( R \) such that 
\[\bigcap_{i \in I} \theta_i = 0_R\]. Then \( R \) is isomorphic to a subdirect product of so-rings \( R / \theta_i, i \in I \).

**Proof.** Define a mapping \( f : R \rightarrow \prod_{i \in I} R / \theta_i \) by \( f(a) = f_a \forall a \in R \),

where \( f_a \in \prod_{i \in I} R / \theta_i \) and \( i f_a = [a]^{\theta_i}, i \in I \). Take \( M = \{f_a \mid a \in R\} \).

Now we prove \( f \) is a one-one homomorphism.

Let \( (x_j : j \in J) \) be a summable family in \( R \).
Then for any $i \in I$, $i f_{(x_j)} = [\Sigma_j x_j]^\theta_i = \Sigma_j [x_j]^\theta_i = i(\Sigma_j f_{x_j})$.

Hence $f(\Sigma_j x_j) = \Sigma_j f(x_j)$.

For any $x, y \in R$ and for any $i \in I$, $i f_{xy} = [xy]^\theta_i = [x]^\theta_i [y]^\theta_i = i f_x \cdot i f_y = i(f_x f_y)$.

Hence $f(xy) = f(x)f(y)$.

Let $x, y \in R$ such that $f(x) = f(y)$.

Then $i f_x = i f_y \forall i \in I. \Rightarrow [x]^\theta_i = [y]^\theta_i \forall i \in I. \Rightarrow x(\bigcap_{i \in I} \theta_i) y. \Rightarrow x = y$.

Hence $f : R \rightarrow \prod_{i \in I} R/\theta_i$ is a one-one homomorphism.

Hence $f : R \rightarrow M$ is an isomorphism.

Now we prove $M$ is subdirect product of $R/\theta_i, i \in I$:

It can be noted that $M$ is subso-ring of $\prod_{i \in I} R/\theta_i$.

Now $p_i(M) = \{p_i(f_x) \mid f_x \in M\} = \{i f_x \mid x \in R\} = \{[x]^\theta_i \mid x \in R\} = R/\theta_i, i \in I$.

Hence $M$ is a subdirect product of so-rings $R/\theta_i, i \in I$.

Hence the lemma.

\[\text{Lemma 1.2.5.}\]

Let $\{\theta_i \mid i \in I\}$ be a simply ordered family of congruences of a so-ring $R$. Then $\bigcup(\theta_i : i \in I) = \bigvee(\theta_i : i \in I)$.

**Proof.** We know that $\bigvee(\theta_i : i \in I) = \bigcap \{\theta' \mid \theta' \in \text{Con}R, \theta' \supseteq \bigcup \theta_i\}$.

$\Rightarrow \bigcup(\theta_i : i \in I) \subseteq \bigvee(\theta_i : i \in I)$.

Let $a(\bigvee \theta_i)b$. Then $\exists a = z_0, ..., z_{n-1}, z_n = b$ and $\theta_1, ..., \theta_{n-1}, \theta_n \in \{\theta_i \mid i \in I\}$ such that $z_{i}(\theta_{i+1})z_{i+1}$ for all $0 \leq i < n$.

Since $\{\theta_i \mid i \in I\}$ is simply ordered, $\exists \theta \in \{\theta_i \mid i \in I\} \ni \theta_i \subseteq \theta \forall 1 \leq i \leq n$.

$\Rightarrow z_i \theta z_{i+1}$ for all $0 \leq i < n$ and $\theta \in \text{Con}R$.

$\Rightarrow z_0 \theta z_n \Rightarrow a \theta b \Rightarrow a(\bigcup \theta_i)b$.

Hence $\bigvee(\theta_i : i \in I) \subseteq \bigcup(\theta_i : i \in I)$.

Hence the lemma.
Lemma 1.2.6. Let \( R \) be a so-ring and \( a \neq b \). Then there is a congruence relation \( \theta(a, b) \) on \( R \) such that \( a \not\equiv b(\theta(a, b)) \) (\( a \) is not congruent to \( b \) under \( \theta(a, b) \)) and \( \theta(a, b) \) is maximal with respect to this property.

Proof. For any \( a, b \in R \) such that \( a \neq b \), define \( C = \{ \phi \in \text{Con}R \mid a \neq b(\phi) \} \).

Since \( a \neq b(0_R) \), we have \( 0_R \in C \).

Let \( \{ \phi_i \mid i \in I \} \) be a simply ordered family in \( C \) and take \( \theta = \bigvee (\phi_i : i \in I) \).

Then by lemma 1.2.5, \( \theta = \bigvee (\phi_i : i \in I) = \bigcup (\phi_i : i \in I) \).

Since \( a \neq b(\phi_i) \) \( \forall i \in I \), we have \( a \neq b(\theta) \).

\( \Rightarrow \theta = \bigcup (\phi_i : i \in I) \) is an upper bound of \( \{ \phi_i \mid i \in I \} \) in \( C \).

Hence by Zorn’s lemma, \( C \) has a maximal element, say \( \theta(a, b) \).

Hence the lemma.

Lemma 1.2.7. Let \( R \) be a so-ring. Then \( R \) is subdirectly irreducible if and only if \( \text{Con}R \) has one and only one atom which is contained in every congruence relation other than \( 0_R \), the zero congruence.

Proof. Suppose \( R \) is subdirectly irreducible and \( \text{Con}R \) has more than one atom.

Then \( \text{Con}R \setminus \{0_R\} \) has no least element. \( \Rightarrow \bigcap (\text{Con}R \setminus \{0_R\}) = 0_R \).

\( \Rightarrow \theta_i = 0_R \) for some \( i \in I \), where \( \theta_i \in \text{Con}R \setminus \{0_R\} \), a contradiction.

Hence \( \text{Con}R \) has one and only one atom which is contained in every congruence relation other than \( 0_R \).

Conversely suppose that \( \text{Con}R \) has one and only one atom which is contained in every congruence relation other than \( 0_R \).

Let \( \{ \theta_i \mid i \in I \} \) be a family of congruences on \( R \) \( \exists \bigcap \theta_i = 0_R \).

Suppose \( \theta_i \neq 0_R \) for every \( i \in I \). Then \( \bigcap \theta_i = \theta' \) for some \( \theta' \neq 0_R \), a contradiction.

Hence \( A \) is subdirectly irreducible.
**Theorem 1.2.8.** Any so-ring is a subdirect product of subdirectly irreducible so-rings.

**Proof.** Let \( R \) be a so-ring. Consider the family of congruences \( C = \{ \theta(a, b) \mid a \neq b \} \) as constructed in the lemma 1.2.6.

Suppose \( x \equiv y(\bigcap_{a \neq b} \theta(a, b)) \). Then \( x \equiv y(\theta(a, b)) \ \forall a \neq b \).

Suppose \( x \neq y \). Then \( x \equiv y(\theta(x, y)) \), a contradiction.

Hence \( \bigcap_{a \neq b} \theta(a, b) = 0_R \).

By lemma 1.2.4, \( R \) is isomorphic to a subdirect product of so-rings \( R/\theta(a, b), a \neq b \).

Now we prove \( R/\theta(a, b), a \neq b \) is subdirectly irreducible.

Let \([\theta(a, b)]\) denote the set of congruences of \( R \) containing \( \theta(a, b) \).

Denote \( \psi(a, b) \) be the smallest congruence such that \( a \equiv b \).

Then \( \theta(a, b) \subseteq \psi(a, b) \lor \theta(a, b) \).

Suppose \( \theta(a, b) = \psi(a, b) \lor \theta(a, b) \). Then \( \psi(a, b) \subseteq \theta(a, b) \).

\( \Rightarrow a \equiv b(\theta(a, b)) \) for \( a \neq b \), a contradiction.

Hence \( \theta(a, b) \subset \psi(a, b) \lor \theta(a, b) \).

\( \Rightarrow \psi(a, b) \lor \theta(a, b) \in [\theta(a, b)] \) and \( \psi(a, b) \lor \theta(a, b) \neq \theta(a, b) \).

Let \( \phi \) be another congruence in \([\theta(a, b)]\) other than \( \theta(a, b) \).

Then \( a \equiv b(\phi) \). \( \Rightarrow \psi(a, b) \subseteq \phi \). Hence \( \psi(a, b) \lor \theta(a, b) \subseteq \phi \).

Therefore \( \psi(a, b) \lor \theta(a, b) \) is the only atom which is contained in every congruence in \([\theta(a, b)]\) other than \( \theta(a, b) \).

By lemma 1.2.3, there is a one to one correspondence between congruences of \( R/\theta(a, b), a \neq b \) and \([\theta(a, b)]\).

\( \Rightarrow \text{Con}(R/\theta(a, b)) \) has one and only one atom which is contained in every congruence other than \( \theta(a, b) \) (the zero congruence of \( R/\theta(a, b) \)).

By lemma 1.2.7, \( R/\theta(a, b), a \neq b \) is subdirectly irreducible.

Hence the theorem. \( \square \)
§1.3. \( \Phi \)-representation of so-rings

Walendziak [35] studied the \( \Phi \)-representation of algebras as a common generalization of subdirect and direct product of algebras. In this section we show that \( \langle (R_i : i \in I), f \rangle \) is a \( \phi \)-representation of \( R \) if and only if \( 0_R = \prod_{\Phi}(\theta_i : i \in I) \) where \( R_i = R/\theta_i, i \in I \).

**Definition 1.3.1.** Let \( \{R_i | i \in I\} \) be a family of so-rings and \( R \) be a subset of \( \prod_{i \in I} R_i \), and let \( \phi \in ConR \). Then \( R \) is said to be a \( \phi \)-product of the so-rings \( R_i, i \in I \) if it satisfies the following

(i). \( R \) is a subdirect product of the so-rings \( R_i, i \in I \)

(ii). for every \( x = \langle x_i : i \in I \rangle \in A^I, \) if \( (x_i, x_j) \in \phi \) \( \forall i, j \in I \) then \( < x_i(i) : i \in I > \in A \). 

**Example 1.3.2.** Take \( R_1 = \{0, 1\} \). Define \( \Sigma \) as

\[
\Sigma x_i = \begin{cases} 
  x_j, & \text{if } x_i = 0 \ \forall i \neq j \text{ for some } j \\
  1, & \text{if } x_h = 1, x_k = 1 \text{ for some } h, k, x_i = 0 \ \forall i \neq h, k \\
  \text{undefined, otherwise.}
\end{cases}
\]

and ‘·’ as \( 0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0, 1 \cdot 1 = 1 \). Then \( (R_1, \Sigma, \cdot) \) is a so-ring.

Take \( R_2 = \{0, a, 1\} \). Define \( \Sigma \) as

\[
\Sigma x_i = \begin{cases} 
  x_j, & \text{if } x_i = 0 \ \forall i \neq j \text{ for some } j \\
  a, & \text{if } x_h = x_k = a \text{ for some } h, k, x_i = 0 \ \forall i \neq h, k \\
  1, & \text{if } x_h = 1, x_k = a \text{ or } 1 \text{, for some } h, k, x_i = 0 \ \forall i \neq h, k \\
  \text{undefined, otherwise.}
\end{cases}
\]

and ‘·’ as \( 0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = a \cdot 0 = 0 \cdot a = 0, a \cdot a = a \cdot 1 = 1 \cdot a = a, 1 \cdot 1 = 1 \). Then \( (R_2, \Sigma, \cdot) \) is a so-ring. Also \( R_1 \times R_2 \) is a so-ring. Take \( R = \{< 0, 0 >, < 1, 0 >, < 1, a >, \ldots \} \). 

40
Remark 1.3.3. Let \( R \) and \( R_i (i \in I) \) be so-rings. Then

(i). \( R \) is a subdirect product of \( R_i \) if and only if \( R \) is a 0\(_R\)-product of \( R_i, i \in I \).

(ii). \( R \) is a 1\(_R\)-product of \( R_i, i \in I \) if and only if \( R = \prod_{i \in I} R_i \).

Proof.

(i). Suppose \( R \) is a 0\(_R\)-product of so-rings \( R_i, i \in I \). Then \( R \) is a subdirect product of so-rings \( R_i, i \in I \).

Conversely suppose \( R \) is a subdirect product of \( R_i, i \in I \).

Let \( \bar{x} = \langle x_i : i \in I \rangle \in R^I \ni (x_i, x_j) \in 0_R \forall i, j \in I \).

\[ \Rightarrow \langle x_i(i) : i \in I \rangle = x_i \in R. \]

Hence \( R \) is a 0\(_R\)-product of so-rings \( R_i, i \in I \).

(ii). Suppose \( R \) is a 1\(_R\)-product of so-rings \( R_i, i \in I \). Then \( R \subseteq \prod_{i \in I} R_i \).

Let \( x \in \prod_{i \in I} R_i \Rightarrow x = \langle x_i : i \in I \rangle \in \prod_{i \in I} R_i \) where \( x_i \in R_i, i \in I \).

Since \( p_i |_R: R \to R_i \) is a surjective homomorphism, \( \exists a_i \in R \ni p_i |_R (a_i) = x_i, i \in I. \)
⇒ ∃ a =< a_i : i ∈ I >∈ R_i ⊃ a_i(i) = x_i ∀i ∈ I.

Since 1_R = R_i, (a_i, a_j) ∈ 1_R ∀i, j ∈ I.

⇒ < a_i(i) : i ∈ I >∈ R. ⇒ x =< x_i : i ∈ I >∈ R.

Hence R = \prod_{i \in I} R_i.

Conversely suppose that R = \prod_{i \in I} R_i.

Then R is clearly a subdirect product of R_i, i ∈ I.

Let \( x =< x_i : i ∈ I >∈ R_i \) for some x_i ∈ R_i ∀i ∈ I.

⇒ < x_i(i) : i ∈ I >∈ \prod_{i \in I} R_i = R.

Hence R is a 1_R-product of so-rings R_i, i ∈ I.

\[ \text{Theorem 1.3.4.} \]

Let \{ R_i | i ∈ I \} be a family of so-rings, let R be a subso-ring of \( \prod_{i \in I} R_i \) and let \( \phi ∈ ConR \). For i ∈ I, let \( \theta_i \) be the kernel of the projection at i, restricted to R. If R is a \( \phi \)-product of so-rings R_i, i ∈ I, then

(i). \( 0_R = \bigcap_{i \in I} \theta_i \)

(ii). for every \( \bar{x} = (x_i : i ∈ I) ∈ R_i \) if \( (x_i, x_j) ∈ \phi \) ∀i, j ∈ I, then \( ∃ x ∈ R \exists (x, x_i) ∈ \theta_i, ∀i ∈ I \)

(iii). \( R/\theta_i ≅ R_i ∀i ∈ I \).

Proof.

(i). Let \( (x, y) ∈ \bigcap_{i \in I} \theta_i \) for some x, y ∈ R. Then \( p_i |_{R_i} (x) = p_i |_{R_i} (y) ∀i ∈ I \).

⇒ \( p_i(x) = p_i(y) ∀i ∈ I \). ⇒ x = y.

Hence \( 0_R = \bigcap_{i \in I} \theta_i \).

(ii). Let \( \bar{x} = (x_i : i ∈ I) ∈ R_i \) if \( (x_i, x_j) ∈ \phi \) ∀i, j ∈ I.

Then \( x =< x_i(i) : i ∈ I >∈ R. ⇒ x(i) = x_i(i), x_i, x ∈ R_i, ∀i ∈ I. \)

⇒ \( p_i |_{R_i} (x) = p_i |_{R_i} (x_i) ∀i ∈ I \). ⇒ \( (x, x_i) ∈ \theta_i ∀i ∈ I. \)

Hence \( ∃ x ∈ R \exists (x, x_i) ∈ \theta_i ∀i ∈ I. \)
(iii). Define $f : R/\theta_i \to R_i$ by $[a]_{\theta_i} \mapsto a(i)$.

For any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$, $[a]_{\theta_i} = [b]_{\theta_i} \iff (a, b) \in \theta_i$

$\iff p_i \mid_R (a) = p_i \mid_R (b) \iff a(i) = b(i) \iff f([a]_{\theta_i}) = f([b]_{\theta_i})$.

Hence $f$ is well defined and one-one.

For any $a_i \in R_i$, $\exists a \in R \ni a(i) = a_i$.

$\Rightarrow \exists [a]_{\theta_i} \in R/\theta_i \ni f([a]_{\theta_i}) = a(i) = a_i$.

Hence $f$ is onto.

Let $([x_j]_{\theta_i} : j \in I)$ be a summable family in $R/\theta_i$.

Then $f(\Sigma_{j \in I}[x_j]_{\theta_i}) = f((\Sigma_j x_j)_{\theta_i}) = (\Sigma_j f([x_j]_{\theta_i}))$.

For any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$, $f([a]_{\theta_i} \cdot [b]_{\theta_i}) = f([a \cdot b]_{\theta_i}) = (a \cdot b)(i)$

$= a(i) \cdot b(i) = f([a]_{\theta_i}) \cdot f([b]_{\theta_i})$ and hence $f$ is a homomorphism.

Hence $R/\theta_i \cong R_i, \; i \in I$.

**Definition 1.3.5.** Let $R$ be a so-ring and let $\phi \in \text{Con}R$. For any system $\{\theta_i : i \in I\}$ of congruences of $R$, we write $0_R = \prod_{\phi}(\theta_i : i \in I)$ if and only if the conditions (i) and (ii) of theorem 1.3.4 are satisfied.

**Remark 1.3.6.** Let $R$ be a so-ring and let $\{\theta_i : i \in I\}$ be a system of congruences of $R$. Then

(i). $0_R = \prod_{0_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i \in I} \theta_i$

(ii). $0_R = \prod_{1_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i \in I} \theta_i$ and for every $(x_i : i \in I) \in R^I$,

$\exists x \in R \ni \exists (x, x_i) \in \theta_i \forall i \in I$.

**Proof.**

(i). Suppose $0_R = \prod_{0_R}(\theta_i : i \in I)$ then $0_R = \bigcap(\theta_i : i \in I)$.

Conversely suppose that $0_R = \bigcap(\theta_i : i \in I)$.

Let $\triangledown = < x_i : i \in I > \in R^I \ni (x_i, x_j) \in 0_R \forall i, j \in I$. 

43
\[\Rightarrow x_i = x_j \quad \forall i, j \in I. \Rightarrow \exists x = x_i \in R \ni (x, x_i) \in \theta_i \quad \forall i \in I.\]

Hence \(0_R = \prod_{0_R}(\theta_i : i \in I).\)

(ii). \(0_R = \prod_{1_R}(\theta_i : i \in I) \iff 0_R = \bigcap_{i \in I} \theta_i\) and for every

\[\exists = \langle x_i : i \in I \rangle \in R^{1}, \exists x \in R \ni (x, x_i) \in \theta_i \quad \forall i \in I. \quad \square\]

**Definition 1.3.7.** Let \(f : R \to R'\) be an epimorphism of so-rings \(R, R'\) and \(\alpha\) be any congruence relation on \(R\). Then we define \(f(\alpha) = \{(f(x), f(y)) \mid (x, y) \in \alpha\}\).

**Remark 1.3.8.** \(f(\alpha)\) is a congruence relation on \(R'\).

**Proof.** Note that \(f(\alpha) = \{(f(x), f(y)) \mid (x, y) \in \alpha\}\).

It can be easily obtained that \(f(\alpha)\) is an equivalence relation on \(R'\).

Let \((f(x_i) : i \in I)\) and \((f(y_i) : i \in I)\) be summable families such that

\[(f(x_i), f(y_i)) \in f(\alpha) \quad \forall i \in I.\]

\[\Rightarrow (x_i, y_i) \in \alpha \quad \forall i \in I. \Rightarrow (\Sigma x_i, \Sigma y_i) \in \alpha.\]

\[\Rightarrow (f(\Sigma x_i), f(\Sigma y_i)) \in f(\alpha). \Rightarrow (\Sigma f(x_i), \Sigma f(y_i)) \in f(\alpha).\]

Let \((f(x), f(y))\) and \((f(x'), f(y'))\) be \(f(\alpha)\). Then \((x, y)\) and \((x', y')\) are in \(\alpha\).

\[\Rightarrow (x', y') \in \alpha. \Rightarrow (f(x)f(x'), f(y)f(y')) \in f(\alpha).\]

Now let \(k', h' \in R'\).

Since \(f\) is an epimorphism, \(\exists k, h \in R \ni f(k) = k'\) and \(f(h) = h'\).

Now \((f(x), f(y) + k') \in f(\alpha)\) and \((f(x) + h', f(y)) \in f(\alpha)\)

\[\iff (f(x), f(y) + f(k)) \in f(\alpha) \text{ and } (f(x) + f(h), f(y)) \in f(\alpha)\]

\[\iff (f(x), f(y + k)) \in f(\alpha) \text{ and } (f(x + h), f(y)) \in f(\alpha)\]

\[\iff (x, y + k) \in \alpha \text{ and } (x + h, y) \in \alpha\]

\[\iff (x, y) \in \alpha \iff (f(x), f(y)) \in f(\alpha).\]

Hence \(f(\alpha)\) is a congruence relation on \(R'\). \(\square\)
Lemma 1.3.9. Let \( R \) and \( R' \) be so-rings and \( \phi, \theta_i (i \in I) \) be congruences of \( R \). If \( f \) is an isomorphism from \( R \) onto \( R' \), then

\[
0_R = \prod_{\phi}(\theta_i : i \in I) \text{ if and only if } 0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I).
\]

Proof. Suppose \( 0_R = \prod_{\phi}(\theta_i : i \in I) \).

By remark 1.3.8, \( f(\phi), f(\theta_i)(i \in I) \) are congruence relations on \( R' \).

Let \((f(x), f(y)) \in \bigcap_{i \in I} f(\theta_i)\). Then \((x, y) \in \bigcap_{i \in I} \theta_i = 0_R\).

\[
\Rightarrow x = y. \Rightarrow f(x) = f(y).
\]

Hence \( \bigcap_{i \in I} f(\theta_i) = 0_{R'} \).

Let \( \overline{y} = (y_i : i \in I) \in R'^I \ni (y_i, y_j) \in f(\phi) \forall i, j \in I \).

Since \( f \) is onto, \( \exists \overline{x} = (x_i : i \in I) \in R^I \ni f(x_i) = y_i \forall i \in I \& (x_i, x_j) \in \phi \forall i, j \in I \).

\[
\Rightarrow \exists x \in R \ni (x, x_i) \in \theta_i \forall i \in I.
\]

\[
\Rightarrow \exists y = f(x) \in R' \ni (y, y_i) \in f(\theta_i) \forall i \in I.
\]

Hence \( 0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I) \).

Conversely suppose \( 0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I) \).

Let \((x, y) \in \bigcap_{i \in I} \theta_i\).

Then \((f(x), f(y)) \in \bigcap_{i \in I} f(\theta_i) = 0_{R'}\).

\[
\Rightarrow f(x) = f(y). \Rightarrow x = y.
\]

Hence \( \bigcap_{i \in I} \theta_i = 0_R \).

Let \( \overline{x} = (x_i : i \in I) \in R^I \ni (x_i, x_j) \in \phi \forall i, j \in I \).

Then \( \overline{y} = (f(x_i) : i \in I) \in R'^I \ni (f(x_i), f(y_i)) \in f(\phi) \forall i, j \in I \).

\[
\Rightarrow \exists y \in R' \ni (y, f(x_i)) \in f(\phi) \forall i \in I.
\]

\[
\Rightarrow \exists x \in R \ni f(x) = y \& (x, x_i) \in \phi \forall i \in I.
\]

Hence \( 0_R = \prod_{\phi}(\theta_i : i \in I) \).
**Theorem 1.3.10.** Let $R$ be a so-ring, $\phi \in \text{Con}R$ and $\{\theta_i \mid i \in I\}$ be a system of congruences of $R \ni 0_R = \prod_{i \in I} \phi(\theta_i : i \in I)$. If the mapping $f : R \to \prod_{i \in I} R_i$ is defined by $f(x) = ([x]_{\theta_i} : i \in I) \ \forall x \in R$ where $R_i = R/\theta_i$, then $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$.

**Proof.** By lemma 1.2.4, $f(R)$ is a subdirect product of so-rings $R_i = R/\theta_i, i \in I$.

Let $(y_i : i \in I) \in f(R)^I \ni (y_i, y_j) \in f(\phi) \ \forall i, j \in I$.

Then $\exists x_i \in R \ \forall i \in I \ni f(x_i) = y_i \ \& (x_i, x_j) \in \phi \ \forall i, j \in I$.

$\Rightarrow \exists x \in R \ni (x, x_i) \in \theta_i \ \forall i \in I. \Rightarrow [x]_{\theta_i} = [x_i]_{\theta_i} \ \forall i \in I$.

$\Rightarrow f(x)(i) = f(y_i)(i) = y_i(i) \ \forall i \in I$ and hence $(y_i(i) : i \in I) = f(x) \in f(R)$.

Hence $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$. \qed

**Definition 1.3.11.** Let $R, R_i(i \in I)$ be so-rings, $\phi \in \text{Con}R$ and $f$ be an embedding of $R$ into $\prod_{i \in I} R_i$. If $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$, then the ordered pair $<(R_i : i \in I), f>$ is called as a $\phi$-representation of the so-ring $R$.

**Remark 1.3.12.** Let $R$ and $R_i(i \in I)$ be so-rings. Then

(i). $<(R_i : i \in I), f>$ is a $0_R$-representation of $R$ if and only if $f(R)$ is a subdirect product of $R_i, i \in I$

(ii). $<(R_i : i \in I), f>$ is a $1_R$-representation of $R$ if and only if $f(R)$ is the direct product of $R_i, i \in I$.

**Proof.**

(i). By remark 1.3.3(i), we have $R$ is a subdirect product of so-rings $R_t, t \in I$ if and only if $R$ is a $0_R$-product of so-rings $R_i, i \in I$.

Since $R \cong f(R)$, we have $f(R)$ is a subdirect product of so-rings $R_i, i \in I$ if and only if $f(R)$ is a $f(0_R)$-product of so-rings $R_i, i \in I$.

Hence $<(R_i : i \in I), f>$ is a $0_R$-representation of $R$ if and only if $f(R)$ is a subdirect product of $R_i, i \in I$. 

46
(ii). By remark 1.3.3(ii), we have $R$ is a $1_R$-product of $R_i$, $i \in I$ if and only if

$$R = \prod_{i \in I} R_i.$$  

Then $f(R)$ is a $f(1_R)$-product of $R_i$, $i \in I$ if and only if $f(R) = \prod_{i \in I} R_i$.

Hence $<(R_i : i \in I), f>$ is the direct product of $R_i$, $i \in I$.

\[ \square \]

**Theorem 1.3.13.** Let $R$ be a so-ring, $\phi$, $\theta_i (i \in I) \in \text{Con}R$. Define $f : R \to \prod_{i \in I} R_i$ by $x \mapsto (\theta_i : i \in I)$ where $R_i = R/\theta_i, i \in I$. Then $<(R_i : i \in I), f>$ is a $\phi$-representation of $R$ if and only if $0_R = \prod_\phi (\theta_i : i \in I)$.

**Proof.** Suppose $<(R_i : i \in I), f>$ is a $\phi$-representation of $R$.

Then $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$.

$$(f(x), f(y)) \in f(\theta_i) \iff (x, y) \in \theta_i \iff [x]_{\theta_i} = [y]_{\theta_i} \iff p_i | f(R) \iff (f(x), f(y)) \in \ker (p_i | f(R)).$$

Hence $f(\theta_i), i \in I$ is the kernel of the projection at $i$ restricted to $f(R)$.

Then by theorem 1.3.4, $0_{f(R)} = \bigcap (f(\theta_i) : i \in I)$.

Hence $0_{f(R)} = \prod_{f(\phi)} (f(\theta_i) : i \in I)$.

By lemma 1.3.9, $0_R = \prod_\phi (\theta_i : i \in I)$.

The converse part is trivial in view of theorem 1.3.10.

Hence the theorem.  

\[ \square \]

**Corollary 1.3.14.** (i). A system $(\theta_i : i \in I)$ of congruences of so-ring $R$ gives a subdirect representation if and only if $\bigcap_{i \in I} \theta_i = 0_R$.

(ii). A system $(\theta_i : i \in I)$ of congruences of so-ring $R$ constitutes a direct representation if and only if $0_R = \prod_{1_R} (\theta_i : i \in I)$. 

47