CHAPTER-I

INTRODUCTION

1.1 Special Functions:

Most of the problems in Applied Mathematics lead to the solutions of differential equations or integral equations which satisfy certain prescribed conditions. Frequently, these solutions turn out to be new functions called special functions with interesting properties, the word special is used in the sense that they arise in the solutions of special problems.

The special functions occurring in Applied Mathematics such as Binomial, Exponential, Logarithmic and Trigonometric functions, Bessel and Whittaker functions, the Error functions and the Classical Orthogonal polynomials are all special cases of the generalized hypergeometric series:

\[ \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k x^k}{\prod_{j=1}^{q} (b_j)_k k!} \]

where

\[ (1.1.2) \ (a)_n = a(a+1)(a+2) \ldots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \ n \geq 1, \ (a)_0 = 1, \ a \neq 0 \]

The series on the right hand side of (1.1.1) is obviously convergent for all values of ‘x’ real or complex, when \( p \leq q \). Also when \( p = q + 1 \), the series is convergent if \( |x| < 1 \).

It converges when \( x = 1 \), if

\[ \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > 0 \]

and when \( x = -1 \), if

\[ \text{Re} \left( \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \right) > -1 \]
If \( p > q + 1 \), the series never converges except when \( x = 0 \), and the function is only defined when the series terminates. A comprehensive account of Confluent, Gauss’s and Generalized hypergeometric functions has been given in standard works by Slater [91], Exton [20] and Rainville [73].

In an attempt to give meaning to \( qF_p \) in the case when \( p > q + 1 \), T. M. Mac Robert and C. S. Meijer introduced and studied at full length the two special functions which are now well known in the literature as the E-function and the G-function, respectively. Apart from Meijer and Mac Robert, who have added a good account of research material on their respective functions, a number of other researchers have done a lot of useful work on these functions during the past years. A detailed account of the G-function is available in the work by Luke [44]. Again, the importance of the G-function in statistical theory of distributions, series expansion of a general G-function and its representations in computable forms when the poles of the integrand are not restricted to be simple and other uses of the G-function are available in the monograph by Mathai and Saxena [47].

In 1961 Charles Fox [21, 22, 23] defined a more general function called the H-function in terms of a Mellin-Barnes type integral, in an attempt to unify existing results on symmetrical Fourier kernels. The H-function of Fox defined and represented as follows:

\[
H[x] = H_{p, q}^{m, n}[x] = H_{p, q}^{m, n} \left[ x, \left( \begin{array}{c} a_1, A_1 \\ b_1, B_1 \end{array} \right), ..., \left( \begin{array}{c} a_p, A_p \\ b_p, B_p \end{array} \right) \right] 
\]

\[
= H_{p, q}^{m, n} \left[ x, \left( \begin{array}{c} a_1, A_1 \\ b_1, B_1 \end{array} \right), ..., \left( \begin{array}{c} a_p, A_p \\ b_p, B_p \end{array} \right) \right] = \frac{1}{2\pi i} \int_{L} F(s)x^s ds, \text{ where } i = \sqrt{-1}, x \neq 0,
\]

and

\[
F(s) = \prod_{j=1}^{m} \frac{\Gamma(b_j - B_j s)}{\Gamma(1 - a_j - A_j s)} \prod_{j=m+1}^{n} \frac{\Gamma(1 - b_j - B_j s)}{\Gamma(a_j - A_j s)}
\]

An empty product is interpreted as unity; \( m, n, p \) and \( q \) are integers satisfying \( 0 \leq m \leq q \), \( 0 \leq n \leq p \).
\[ 0 \leq n \leq p; A_j (j = 1, 2, \ldots, p), B_j (j = 1, 2, \ldots, q) \text{ are positive numbers and } a_j, j = 1, 2, \ldots, p, \]
\[ b_j, j = 1, 2, \ldots, q \text{ are complex numbers such that no poles of } \Gamma(b_j - B_j s), j = 1, 2, \ldots, m \]
\[ \text{coincide with any pole of } \Gamma(1 - a_j + A_j s), (j = 1, 2, \ldots, n), \text{ i.e. } A_j (b_k + N) \neq B_k (a_j - M - 1), \text{ where} \]
\[ k = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n; \quad M = 0, 1, 2, \ldots. \]

The contour L runs from \( \sigma - i \infty \) to \( \sigma + i \infty \), \( \sigma \) be a positive constant such that the points
\[ s = \frac{b_h + N}{B_h}, \quad h = 1, 2, \ldots, m; \quad N = 0, 1, 2, \ldots \]
which are the poles of \( \Gamma(b_j - B_j s), j = 1, 2, \ldots, m \) lie to the right and the points
\[ s = \frac{a_j - M - 1}{A_j}, \quad j = 1, 2, \ldots, n; \quad M = 0, 1, 2, \ldots \]
which are the poles of \( \Gamma(1 - a_j + A_j s), j = 1, 2, \ldots, n \) lie to the left of L.

In 1963 Braaksma \[6\] considered the asymptotic expansions and conditions of convergence of the integral (1.1.3).

The contour integral on the right hand side of (1.1.3) converges absolutely if
\[ |\text{arg } x| < \frac{1}{2} \Delta \pi, \quad \text{where} \]
\[ (1.1.4) \quad \Delta = \sum_{j=1}^{n} (A_j) - \sum_{j=n+1}^{p} (A_j) + \sum_{j=1}^{m} (B_j) - \sum_{j=m+1}^{q} (B_j) > 0, \]
when \( x \) is real and \( \Delta = 0 \), additional conditions are needed for the convergence of the integral (1.1.3).
again

\[ (1.1.5) \quad H_{p, q}^{m, n} = O(|x|^\mu), \text{ for small } x. \]

If
\[ (1.1.6) \quad \mu = \sum_{j=1}^{q} (B_j) - \sum_{j=1}^{p} (A_j) \geq 0, \]
\[ \alpha = \min \Re \left[ \frac{b_j}{B_j} \right], (j = 1, 2, \ldots, m), \text{ and} \]
(1.1.7) \( H_{p,q}^{m,n}[x] = O(x^{\beta}) \) for large \( x \).

If

\[
\mu = \sum_{j=1}^{q} (B_j) - \sum_{j=1}^{p} (A_j) \geq 0, \Delta > 0, |\arg x| < \frac{1}{2} \Delta \pi, \text{ and}
\]

\[
\beta = \max \text{ Re} \left[ \frac{(a_j-1)}{A_j} \right], (j = 1, 2, \ldots, n), \Delta \text{ is given by (1.1.4)},
\]

Also from Braaksma’s same paper [6], we infer that when \( n = 0 \), the H-function vanishes exponentially for large \( x \).

\[
(1.1.9) \quad H_{p,q}^{m,n}[x] = O \left( \exp \left( -\mu x^{\frac{1}{\lambda}} \right) x^{\frac{1}{\lambda}} \delta + \frac{1}{2} \right)
\]

where

\[
(1.1.10) \quad \delta = \frac{1}{2} \left( \sum_{j=1}^{q} (b_j) - \sum_{j=1}^{p} (a_j) + \frac{1}{2} p - \frac{1}{2} q \right)
\]

and the following conditions are satisfied

\[
(1.1.11) \quad (i) \quad v = \frac{m}{2} \left( \sum_{j=1}^{m} (B_j) - \sum_{j=1}^{q} (B_j) - \sum_{j=1}^{p} (A_j) \right) > 0
\]

\[
(ii) \quad |\arg x| < \frac{1}{2} \pi \text{ and } \mu > 0
\]

where \( \mu \) is given by (1.1.6) and (1.1.12) \( \lambda = \sum_{j=1}^{p} (A_j) \prod_{j=1}^{q} (B_j)^{B_j} \)

Certain important properties of the H-function has been studied by Gupta and Jain [33]. Some of the properties, which are useful for later chapters, are listed below.

(i) The H-function is symmetric in the pairs \((a_1, A_1), \ldots, (a_n, A_n)\) likewise \((a_{n+1}, A_{n+1}), \ldots, (a_p, A_p)\) in \((b_1, B_1), \ldots, (b_m, B_m)\) and in \((b_{m+1}, B_{m+1}), \ldots, (b_q, B_q)\).
(ii) If one of the $(a_j, A_j), j = 1, 2, \ldots, n$ is equal to $(b_j, B_j), j = 1, 2, \ldots, m$ or one of the $(b_j, B_j), j = 1, 2, \ldots, m$ is equal to one of the $(a_j, A_j), j = n + 1, 2, \ldots, p$.

Then the H-function reduces to one of the lower order $p, q$ and $n$ (or $m$) decreased by unity.

\[
H^{m,n}_{p,q} \left[ \frac{(a_j, A_j)_h}{(b_j, B_j)_h} \right] = H^{n,m}_{q,p} \left[ \frac{(1-b_j, B_j)_h}{(1-a_j, A_j)_h} \right]
\]

\[
H^{m,n}_{p,q} \left[ \frac{(a_j, A_j)_h}{(b_j, B_j)_h} \right] = \alpha H^{n,m}_{q,p} \left[ \frac{(a_j, \alpha A_j)_h}{(b_j, \alpha B_j)_h} \right]; \sigma > 0
\]

\[
x^\sigma H^{m,n}_{p,q} \left[ \frac{(a_j, A_j)_h}{(b_j, B_j)_h} \right] = H^{n,m}_{q,p} \left[ \frac{(a_j + \sigma A_j, A_j)_h}{(b_j + \sigma B_j, B_j)_h} \right]
\]

The importance of the H-function lies in the fact that almost all the special functions occurring in the applied mathematics and statistics are its particular cases. The generalized Bessel function studied by Wright [110,111] and the generalized hypergeometric function defined by Fox [21] and Wright are all special cases of H-function, such a function cannot be obtained as special cases of Meijers G-function. Hence the study of H-function is more general in character.

A few interesting special cases of the H-function are:

(i) If all the $A_j = B_h = 1, (j = 1, 2, \ldots, p; h = 1, 2, \ldots, q)$ in (1.1.10) and using the definition of G-function,

\[
H^{m,n}_{p,q} \left[ \frac{(a_j, 1)_h}{(b_j, 1)_h} \right] = G^{m,n}_{p,q} \left[ \frac{(a)_p}{(b)_q} \right]; \text{ where } (a)_p = a_1, \ldots, a_p
\]

(ii) Also if $A_j = B_h = 1/k, (j = 1, 2, \ldots, p; h = 1, 2, \ldots, q)$ in (1.1.14) and then replacing $1/k$ by $k^1$,

\[
H^{m,n}_{p,q} \left[ \frac{(a_j, k^1)_h}{(b_j, k^1)_h} \right] = \frac{1}{k^l} G^{m,n}_{p,q} \left[ \frac{(a)_p}{(b)_q} \right]; k^1 > 0
\]
The great success and fruitful nature of the theorem of hypergeometric functions in one variable stimulated the study and the development of corresponding theory in two or more variables.

In 1880, Appell defined and studied systematically the four functions $F_1$, $F_2$, $F_3$ and $F_4$ which are generalization of the Gaussian hypergeometric functions in two variables, these functions are now properly known as Appell functions. Humbert [37] studied the confluent forms of these functions. Other hypergeometric functions of two variables were investigated by Horn in a long series of papers extending over a fifty year period [1883-1939]. A list of these functions is given by Erdelyi et al. [19] that is similar to the generalization of the single hypergeometric function, the function $F_1$ to $F_4$ and their confluent forms were further generalized by Kampe de Feriet who introduces the function defined by the following series:

\[
\sum_{\infty} \prod_{q=0} \Gamma(b_j + B jr) \prod_{j=1} \Gamma(B jr)^{-1} \frac{(-x)^r}{r!} = \prod_{p=0} \Gamma(a_j + A jr) \prod_{j=1} \Gamma(A jr)^{-1} \frac{(-y)^r}{r!}
\]

where $p^\nu_{q}$ be the Wright’s generalized hypergeometric function [105]

The notations on the left of (1.1.18) due essentially to Burchnall and Chaundy [8] are more compact than the one originally by Kampe de Feriet [2,p.150]

The above double series is absolutely convergent for all values of $x$ and $y$, if $u + v < w + t + 1$ and $u + p < w + q + 1$. Also when $u + v = w + t + 1$ and $u + p = w + q + 1$, the series becomes conditionally convergent.
The series is convergent if the following sets of conditions are holds

(i) \( u \leq w \) for \( \max \{|x|, |y|\} < 1 \),

(ii) \( u > w \) for \( |x|^{1/u-w} + |y|^{1/u-w} < 1 \)

The details regarding the convergence of the double series (1.1.18) can be found in the book by Exton [20,p.25]. A comprehensive account of all the earlier mentioned functions and their related functions have been given in the famous work by Appell and Kampe de Feriet [2].

The Kampe de Feriet function has been generalized by Srivastava and Daoust [97]. Their general function is defined and represented as follows:

\[
S[x, y] = \sum_{m, n = 0}^{\infty} \frac{P_1 \prod_{j=1}^{p_1} \Gamma(a_j + \alpha_j m + A_j n) P_2 \prod_{j=1}^{p_2} \Gamma(c_j + C_j m) P_3 \prod_{j=1}^{p_3} \Gamma(e_j + E_j n)}{\prod_{j=1}^{q_1} \Gamma(b_j + \beta_j m + B_j n) \prod_{j=1}^{q_2} \Gamma(d_j + D_j m) \prod_{j=1}^{q_3} (f_j + F_j n)} \frac{x^m y^n}{m! n!}
\]

where \((a_j: \alpha_j, A_j)_{1:p_1}\) abbreviates the array of \(p_1\) parameters \((a_1: \alpha_1, A_1), \ldots, (a_{p_1}: \alpha_{p_1}, A_{p_1})\) and so on. The series given by (1.1.19) converges absolutely, if

\[
1 + \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{q_2} D_j - \sum_{j=1}^{p_1} \alpha_j - \sum_{j=1}^{p_2} C_j \geq 0 , \text{ and}
\]

\[
1 + \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{q_3} F_j - \sum_{j=1}^{p_1} A_j - \sum_{j=1}^{p_3} E_j \geq 0 ,
\]

where each of the equalities holds when the variables are suitably constrained.

If in (1.1.19) all the Greek and capital letters to be equal to 1, it reduces to the Kampe de Feriet function (1.1.18).

When \(p_1 = q_1 = 0\) in (1.1.19), \(S[x, y]\) breaks into the product of two Wright’s generalized hypergeometric functions:
The study of the functions of two variables has greatly increased after the independent introduction of a generalization of the Kampe de Feriet function by Agarwal [1] and Sharma [83]. Agarwal calls his function Meijers G-function of two variables and Sharma on the other hand, calls his function the generalized function of two variables and represented it by the S-symbol.

Mittal and Gupta [52] have defined a more generalized function called the double H-function or the H-function of two variables, the double H-function is of very general value and it includes the H-function of Fox [23] and the product of two H-functions and most of the familiar functions of one and two variables such as the G-function of two variables introduced by Agarwal [1], S-function of two variables given by Sharma [8] and the Appel’s [2] functions F_1, F_2, F_3 and F_4.

Several definitions and notations of the double H-functions have appeared in the literature. The works of Chaturvedi and Goyal [10], Goyal [30], Mathur [48], Mourya [49], Munot and Kalla [54] and Pathak [58] may be mentioned in this condition.

The H-function of two variables given by Prasad and Gupta [66] is

\[
(1.1.20) \quad H_{x,y} = H_{P,Q}^{M,N;m,n;m,n} \left[ \left( a_j, \alpha_j, A_j \right)^{m,m} p_1^{(e_j, C_j}) \right] + \left( \beta_j, B_j \right)^{n,n} q_1^{(f_j, F_j)} \right]
\]

\[
\int_{L_1} \int_{L_2} \left[ \varphi(s) \psi(s) \right] x^y \psi(t) dt \right]
\]

where \( x, y > 0 \),
\[ \phi_1(s) = \frac{\prod_{j=1}^{m} \Gamma(d_j - D_j s) \prod_{j=1}^{n} \Gamma(1 - c_j + C_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - d_j + D_j s) \prod_{j=n+1}^{p} \Gamma(c_j - C_j s)} \]

\[ \phi_2(t) = \frac{\prod_{j=1}^{m} \Gamma(f_j - F_j t) \prod_{j=1}^{p} \Gamma(-e_j + E_j t)}{\prod_{j=g+1}^{q} \Gamma(-f_j + F_j t) \prod_{j=h+1}^{p} \Gamma(e_j - E_j t)} \]

\[ \psi(s,t) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s - B_j t) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s + B_j t) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s - A_j t)} \]

H[x,y] defined by (1.1.21) is an analytic function of x and y, if \( V_1 < 0, V_2 < 0 \) where \( V_1 \) and \( V_2 \) are defined by

\[ \text{(1.1.22)} \quad V_1 = \sum_{j=1}^{P} \alpha_j + \sum_{j=1}^{P} C_j - \sum_{j=1}^{Q} \beta_j - \sum_{j=1}^{Q} D_j \]

and

\[ \text{(1.1.23)} \quad V_2 = \sum_{j=1}^{P} A_j + \sum_{j=1}^{P} E_j - \sum_{j=1}^{Q} B_j - \sum_{j=1}^{Q} F_j \]

The double integral in (1.1.21) converges absolutely; if \(|\arg x| < (1/2) \pi \Delta_1\) and

\[ |\arg y| < \pi \Delta_2 \] where

\[ \text{(1.1.23)} \quad \Delta_1 = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{P} \alpha_j + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j + \sum_{j=1}^{M} D_j - \sum_{j=M+1}^{Q} D_j + \sum_{j=1}^{P} C_j - \sum_{j=n_t+1}^{P} C_j \]
and

\[(1.1.24) \quad \Delta_2 = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{P} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j + \sum_{j=1}^{m_2} F_j - \sum_{j=m_1+1}^{q_2} F_j + \sum_{j=1}^{n_2} E_j - \sum_{j=n_1+1}^{p_2} E_j \]

A modified H-function of two variables is introduced by Prasad and Prasad [68], as a generalization of H[x,y] is given by:

\[(1.1.25) \quad H[x,y] = H^{M,N}:M,N:m,n;r,l \quad \left[ x \Gamma(a_j,A_j) , p : (c_j,C_j)_1, p : (e_j,E_j)_1, p : (g,H)_1, u \right] \]

\[\left( y \Gamma(b_j,B_j) , q : (d_j,D_j)_1, q : (f_j,F_j)_1, q : (h,H)_1, v \right) \]

\[= \frac{1}{(2\pi)^2} \int \int \psi(s,t)\theta_1(s)\theta_2(t) x^y y^t ds dt, \quad i = \sqrt{-1} \]

where

\[(1.1.26) \quad \psi(s,t) = \prod_{j=1}^{l} \Gamma(b_j - \beta_j s - B_j t) \prod_{j=1}^{l} \Gamma(l - a_j + \alpha_j s + A_j t) \]

\[\prod_{j=1}^{l} \Gamma(l - a_j + \alpha_j s + A_j t) \prod_{j=1}^{l} \Gamma(b_j - \beta_j s - B_j t) \]

\[\times \prod_{j=1}^{l} \Gamma(c_j - \gamma_j s - C_j t) \prod_{j=1}^{l} \Gamma(1 - c_j + \gamma_j s - C_j t) \]

and \(\theta_1(s)\) and \(\theta_2(t)\) have similar definitions as \(\phi_1(s)\) and \(\phi_2(t)\), as defined in (1.1.21) with changed parameters \(M^1,N^1,P^1,Q^1,M,N,P,Q,m,n,p,q,r,l,u,v\) are all non negative integers such that \(0 \leq M^1 \leq Q^1, 0 \leq N^1 \leq P^1, 0 \leq M \leq Q, 0 \leq N \leq P, 0 \leq m \leq q, 0 \leq n \leq p, 0 \leq r \leq v, 0 \leq l \leq u\) and \(\alpha_j, \beta_j, \gamma_j, \delta_j, A_j, B_j, C_j, D_j, E_j, F_j, G_j, H_j\) are all positive with \(x, y \neq 0\).

The function \(H[x,y]\) defined by (1.1.25) is an analytic function of \(x\) and \(y\) if
\begin{equation}
(1.1.27) \quad V_1 = \sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{p} \gamma_j + \sum_{j=1}^{p} E_j - \sum_{j=1}^{Q} \beta_j - \sum_{j=1}^{Q} \delta_j - \sum_{j=1}^{Q} F_j < 0
\end{equation}

and

\begin{equation}
(1.1.28) \quad V_2 = \sum_{j=1}^{p} A_j + \sum_{j=1}^{Q} D_j + \sum_{j=1}^{u} G_j - \sum_{j=1}^{Q} B_j - \sum_{j=1}^{Q} C_j - \sum_{j=1}^{Q} H_j < 0
\end{equation}

exist. The double integral defined in (1.1.25) converges absolutely; if

\begin{equation}
|\arg x| < (1/2)\pi \Delta_1, |\arg y| < (1/2)\pi \Delta_2,
\end{equation}

\begin{equation}
(1.1.29) \quad \Delta_1 = \sum_{j=1}^{N} \alpha_j - \sum_{j=N+1}^{M} \alpha_j + \sum_{j=1}^{M} \beta_j - \sum_{j=M+1}^{Q} \beta_j + \sum_{j=N+1}^{Q} \gamma_j - \sum_{j=1}^{P} D_j + \sum_{j=1}^{P} \delta_j
\end{equation}

- \sum_{j=M+1}^{Q} \delta_j + \sum_{j=1}^{n} E_j - \sum_{j=n+1}^{P} E_j + \sum_{j=1}^{m} F_j - \sum_{j=m+1}^{Q} F_j > 0

and

\begin{equation}
(1.1.30) \quad \Delta_1 = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{M} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j + \sum_{j=1}^{Q} C_j - \sum_{j=1}^{Q} C_j + \sum_{j=1}^{Q} D_j
\end{equation}

- \sum_{j=M+1}^{Q} D_j + \sum_{j=1}^{l} G_j - \sum_{j=l+1}^{Q} G_j + \sum_{j=1}^{r} H_j - \sum_{j=r+1}^{Q} F_j > 0

and other conditions have similarity with H[x, y] defined in (1.1.21).

When M^1 = M = N = P = Q = 0 in (1.1.25), the modified H-function of two variables reduces to the H-function of two variables defined by Mittal and Gupta [52].

The H-function of two variables is defined by Mittal and Gupta [52] has recently been generalized to the H-function of several complex variables (or the multivariable H-function)
x_1, x_2, \ldots, x_r by Srivastava and Panda [101] in terms of the multiple contours integral as

\[(1.1.31) \quad H[x_1, \ldots, x_r] = H[P, Q : p_1, q_1, \ldots, p_r, q_r] = \frac{1}{(2\pi i)^r} \prod_{k=1}^r \left[ \theta(s_k) \phi_k(s_k) x_k^{s_k} \right] ds_1 \ldots ds_r,
\]

where \(i = \sqrt{-1}\)

\[(1.1.32) \quad \theta(s_1, \ldots, s_r) = \prod_{j=1}^n \frac{\Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}{\prod_{j=1}^{n+1} \Gamma(a_j - \sum_{k=1}^r \alpha_j^{(k)} s_k)} \prod_{j=1}^{n+1} \frac{\Gamma(1 - b_j + \sum_{k=1}^r \beta_j^{(k)} s_k)}{\Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k)}
\]

\[(1.1.33) \quad \phi_k(s_k) = \frac{\prod_{j=1}^m \Gamma(d_j^{(k)} - D_j^{(k)} s_k)}{\prod_{j=m+1}^{m_k} \Gamma(1 - d_j^{(k)} + D_j^{(k)} s_k)} \frac{\prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + C_j^{(k)} s_k)}{\prod_{j=n_k+1}^{n} \Gamma(c_j^{(k)} - C_j^{(k)} s_k)}, \quad k = 1, \ldots, r
\]

An empty product is interpreted as unity. \(n, p, q, m_k, n_k, p_k, k = 1, 2, \ldots, r\) are non-negative integers such that: \(0 \leq n \leq p, q \geq 0, 0 \leq m_k \leq q_k \quad 0 \leq n_k \leq p_k, k = 1, 2, \ldots, r\) and \(\alpha_j^{(k)}, \beta_j^{(k)}, C_j^{(k)}, D_j^{(k)}\) are all positive.

The contour \(L_k\) in the complex plane \(s_k\) is of the Mellin-Barnes type which runs from \(-i\infty\) to \(i\infty\) with indentations, if necessary to ensure that all the poles of \(\Gamma(d_j^{(k)} - D_j^{(k)} s_k), j = 1, 2, \ldots, m_k\) are to the right path, and those of \(\Gamma(1 - c_j^{(k)} + C_j^{(k)} s_k), j = 1, 2, \ldots, n_k\) and

\[\Gamma\left(1 - a_j + \sum_{j=1}^r \alpha_j^{(k)} s_k\right), \quad (j = 1, 2, \ldots, n)\]

are to the left of \(L_k\).

The various parameters are so restricted that the poles of all gamma functions involved in (1.1.31) are simple and none of them coincide.
By applying the method of Braaksma [6], it can be proved that the H-function of several variables (or the multivariable H-function) is an analytic function, if

\[
A_k = \sum_{j=1}^{p} \alpha_j^{(k)} - \sum_{j=1}^{q} \beta_j^{(k)} + \sum_{j=1}^{p_k} C_j^{(k)} - \sum_{j=1}^{q_k} D_j^{(k)} \leq 0
\]

The integral (1.1.31) converges absolutely, if \(|\arg x| < (1/2) \pi \Delta_k; \ k = 1, 2, \ldots, r\) where

\[
\Delta_k = -\sum_{j=n+1}^{p} \alpha_j^{(k)} + \sum_{j=1}^{n_k} C_j^{(k)} - \sum_{j=1}^{p_k} C_j^{(k)} - \sum_{j=1}^{q_k} D_j^{(k)} - \sum_{j=1}^{m_k} D_j^{(k)} > 0,
\]

\(k = 1, 2, \ldots, r\).

The H-function of ‘r’ complex variables (or multivariable H-function) is abbreviated in the form \(H[x_1, \ldots, x_r]\) and in the form \(H_1[x_1, \ldots, x_r]\), when \(n = 0\). also

\[
H[x_1, \ldots, x_r] = \begin{cases}
O\left[\left|x_1^{\alpha_1}, \ldots, x_r^{\alpha_r}\right|, \max\left\{x_1, \ldots, x_r\right\} \to 0\right] \\
O\left[\left|x_1^{\beta_1}, \ldots, x_r^{\beta_r}\right|, n = 0, \max\left\{x_1, \ldots, x_r\right\} \to \infty\right]
\end{cases}
\]

where

\(\alpha_k = (d_j^{(k)}) / (D_j^{(k)}), j = 1, 2, \ldots, n_k; \ \beta_k = (1-c_j^{(k)}) / (C_j^{(k)}), j = 1, 2, \ldots, n_k\)

When \(n = p = q = 0\), (1.1.31) degenerate into the product of ‘r’ mutually independent H-function of Fox: \(H[x_1, \ldots, x_r] = H[x_1] \cdots H[x_r]\), where \(H[x_i]\) denotes the H-function of Fox given by (1.1.3) \((i = 1, 2, \ldots, r)\).

If all the \(\alpha\)’s, \(\beta\)’s, \(C\)’s and \(D\)’s are chosen to unity, the H-function is defined by (1.1.31) reduces to the corresponding G-function of ‘r’ variables.

Lauricella’s hypergeometric function of several variables [47, p.162] is a special case of (1.1.31).
When \( r = 2 \), (1.1.31) reduces to the H-function of two variables defined by Mittal and Gupta [54].

A slightly modified form of the definition of the multivariable H-function given by Saxena [82].

Prasad and Singh [69] have discussed a modified H-function of several complex variables as an extension of the H-function of several complex variables defined by Sriastava and Panda [101].

The A-function of several variables defined by Gautam and Goyal [26] as follows:

\[
A[z_1, \ldots, z_r] = A_{P, Q}^{M, N} : \Gamma(p) : (p_1, q_1) : \ldots : (p_r, q_r)
\]

\[
= \frac{1}{(2\pi)^r} \int \ldots \int \varphi(s_1, \ldots, s_r) z_1^{s_1} \ldots z_r^{s_r} ds_1 \ldots ds_r,
\]

where

\[
\varphi(i)_{s_i} = \prod_{j=1}^{m_i} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} s_i\right) \prod_{j=1}^{q_i} \Gamma\left(1-c_j^{(i)} + \gamma_j^{(i)} s_i\right), \quad i = 1, \ldots, r
\]

\[
\theta(s_1, \ldots, s_r) = \prod_{j=N+1}^{P} \Gamma\left(a_j - \sum_{i=1}^{\ell} \alpha_j^{(i)} s_i\right) \prod_{j=M+1}^{Q} \Gamma\left(b_j + \sum_{i=1}^{\ell} \beta_j^{(i)} s_i\right)
\]

\( M, N, P, Q, m_j, n_j, p_j, q_j \) are positive real numbers, \( a_j, b_j, d_j^{(i)}, c_j^{(i)}, \delta_j^{(i)}, \alpha_j^{(i)}, \beta_j^{(i)} \) are real numbers, \( \xi_i^* = 0, \eta_i > 0 \) and \( |\arg(\xi_i z_k)| < (\pi/2)\eta_i \).
where

\[ \xi_i = \prod_{k_j} \beta_j^{(i)} \, \gamma_j^{(i)} , \quad i = 1, 2, \ldots, r \]

(1.1.40)

\[ \xi_i^* = \text{Im} \left\{ \frac{p}{\sum_{j=1}^{P} \alpha_j^{(i)}} - \frac{q_i}{\sum_{j=1}^{Q} \delta_j^{(i)}} + \frac{p_j}{\sum_{j=1}^{P} \gamma_j^{(i)}} \right\} \]

(1.1.41)

\[ \eta_i = \text{Re} \left\{ \frac{N}{\sum_{j=1}^{N} \alpha_j^{(i)}} - \frac{p}{\sum_{j=1}^{P} \alpha_j^{(i)}} + \frac{M}{\sum_{j=1}^{M} \beta_j^{(i)}} - \frac{Q}{\sum_{j=1}^{Q} \delta_j^{(i)}} + \frac{n_i}{\sum_{j=1}^{N} \gamma_j^{(i)}} - \frac{p_i}{\sum_{j=1}^{P} \gamma_j^{(i)}} \right\} \]

(1.1.42)

\[ \alpha_j, \beta_j, \gamma_j, \delta_j \text{ are positive numbers.} \]

When \( M = 0 \), A-function of ‘r’ variables reduces to the H-function of ‘r’ variables given by Srivastava and Panda [104].

1.2 Class of polynomials:

According to Srivastava [94, p.1, eq. (1)] is defined as,

\[ S_n^{m} [x] = \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_n, r x^r , \quad n = 0, 1, 2, \ldots \]

(1.2.1)

where \( m \) is an arbitrary positive integer and the coefficients \( A_n, r \) (\( n, r \geq 0 \)) are arbitrary constants, real or complex. On suitably specializing the coefficients \( A_n, r \), \( S_n^{m} [x] \) yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and others [104, pp.158-161].

According to Srivastava and Garg [98, p.686, eq. (1.4)] the first class of multivariable polynomials is defined as follows

\[
S^m_{\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_l}[x_1, \ldots, x_l] = \frac{U_{1}k_1 + \ldots + U_{l}k_l}{k_1 \ldots k_l}, \quad k_1, \ldots, k_l = 0
\]

(1.2.2)
The second class of multivariable polynomials give by Srivastava [95, p.183] is defined and represented in the following slightly modified form

\[
S_{V_1, \ldots, V_t}^{U_1, \ldots, U_t}[x_1, \ldots, x_t] = \sum_{k_1=0}^{V_1} \cdots \sum_{k_t=0}^{V_t} (-V_1)_{k_1}^{U_1} \cdots (-V_t)_{k_t}^{U_t} A(V_1, k_1; \ldots; V_t, k_t) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_t^{k_t}}{k_t!} 
\]

\[V = 0, 1, 2, \ldots.\]

On taking \( A(V_1, k_1; \ldots; V_t, k_t) = A_1(V_1, k_1) \ldots A_t(V_t, k_t) \) in (6.3.1) the multivariable polynomial \( S_{V_1, \ldots, V_t}^{U_1, \ldots, U_t}[x_1, \ldots, x_t] \) reduced to the product of \( t \) well known general class of polynomials \( s_t^U[x] \).

If we take \( A(V_1, k_1; \ldots; V_t, k_t) = \frac{(\beta_1)_{k_1} \phi_1 + \cdots + k_t \phi_t}{(V_1)_{k_1} \phi_1 + \cdots + k_t \phi_t} \) in (1.2.2), the polynomial \( S_{V_1, \ldots, V_t}^{U_1, \ldots, U_t}[x_1, \ldots, x_t] \) reduces to the first class of multivariable hypergeometric polynomial defined by Carlitz and Srivastava [100, p.462, eq.9.4.4]. Also, if we take same in (1.2.3) \( S_{V_1, \ldots, V_t}^{U_1, \ldots, U_t}[x_1, \ldots, x_t] \) reduces to the second class of multivariable hypergeometric polynomial defined by Carlitz and Srivastava [100, p.463, eq.9.4.8].

The generalized Struve’s function is defined by Kanth [38] as:

\[
H_{\nu, y; u}^{\lambda, k}[z] = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+u)}
\]

\[\text{Re } (k) > 0, \text{ Re } (\lambda) > 0, \text{ Re } (y) > 0, \text{ Re } (v+u) > 0\]

When \( k = 1 \), (1.2.4) reduces to the Struve’s function defined by Prasad and Chaudhary as:

\[
H_{\nu, y; u}^{\lambda}[z] = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma(m+y)\Gamma(v+\lambda m+u)}
\]

When \( \mu = 3/2 \), (1.2.4) reduces to Struve’s function defined by Singh [88] as:
(1.2.6) \[ H_{\lambda,k,v,y}[z] = \sum_{m=0}^{\infty} \frac{(-1)^m(z/2)^{y+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\frac{3}{2})} \]

where \( \text{Re}(k) > 0, \text{Re}(\lambda) > 0, \text{Re}(y) > 0, \text{Re}(v+3/2) > 0 \)

Put \( k = 1, y = 3/2 = \mu \) in (1.2.4) to get generalized Struve’s function defined by Bhoumick[5] as:

(1.2.7) \[ H_{\lambda,v}[z] = \sum_{m=0}^{\infty} \frac{(-1)^m(z/2)^{v+2m+1}}{\Gamma(m+3/2)\Gamma(v+m+3/2)} \],

where \( \text{Re}(\lambda) > 0, \text{Re}(v+3/2) > 0. \)

When \( \lambda = k = 1, y = 3/2 = \mu \) in (1.2.4) reduces to generalized Struve’s function defined by Watson [112] as:

(1.2.8) \[ H_{\lambda,v}[z] = \sum_{m=0}^{\infty} \frac{(-1)^m(z/2)^{v+2m+1}}{\Gamma(m+3/2)\Gamma(v+m+3/2)} \],

where \( \text{Re}(v+3/2) > 0. \)

According to Sinha [90] Struve’s function is defined by

(1.2.9) \[ H_{\lambda,k,v,y,u}[z] = \frac{(z/2)^{v+1}}{\Gamma(y)\Gamma(v+u)} \binom{1}{k, \ldots, 1} \binom{v+k-1}{k} \binom{v+u-1}{\lambda} \binom{z^2/4}{\lambda^2} \]

On putting \( k = 1 \) and replacing \( y \) by \( \alpha \) and \( u \) by \( \beta \) in (1.2.8) to get

(1.2.10) \[ H_{\lambda,\alpha,\beta}[z] = \frac{(z/2)^{v+1}}{\Gamma(\alpha)\Gamma(v+\beta)} \binom{v+\beta-1}{\lambda} \binom{-z^2/4}{\lambda^2} \]

which is the result of Prasad and Chaudhary.

On putting \( y = 3/2 = u, k = 1 \) in (1.2.8) we get

(1.2.11) \[ H_{\lambda,v}[z] = \frac{2(z/2)^{v+1}}{\sqrt{\pi}\Gamma(v+3/2)} \binom{3/2}{2} \binom{3v/2}{\lambda} \binom{2v+2\lambda+1}{4\lambda^3} \]

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which is result for Struve’s function of Bhoumick [5].

On putting \( y = 3/2 = u \) and \( \lambda = k = 1 \) in (1.2.8), we get

\[
H_v[z] = \frac{2(z/2)^{v+1}}{\sqrt{\pi} \Gamma(v + \frac{3}{2})} {}_1F_2 \left[ \begin{array}{c}
\frac{3}{2}, \frac{2v+3}{4} \\
-\frac{z^2}{4}\lambda
\end{array} \right],
\]

which is result for Struve’s function of Watson [113].

1.3 Integral Transforms:

The study of integral transforms involving special functions also play an important role in finding the solutions of differential and integral equations of Applied Mathematics.

If \( K(x, s) \) denotes a definite function of \( x \) on the interval \((a, b)\), then the transformation of the function \( F(x) \) with respect to \( K(x, s) \), called the kernel, is defined by

\[
(1.3.1) \quad f(x) = \int_a^b K(x, s)F(x)dx
\]

Also if \( F(x) \) can be expressed interms of \( f(s) \) by an integral of the form

\[
(1.3.2) \quad F(x) = \int_c^d f(s)\theta(x, s)ds
\]

is called inverse transform of (1.3.1).

The most powerful mathematical technique used in solving the differential equations and general system analysis is the Laplace transform which is defined by

\[
(1.3.3) \quad L(F(t)) = f(p) = \int_0^\infty e^{-pt}F(t)dt, \quad \text{Re} \ (p) > 0
\]

Its inversion formula is given by

\[
(1.3.4) \quad L^{-1}(f(p)) = F(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt}f(p)dp
\]

where \( i = \sqrt{-1} \) and \( c \) is a positive constant. Symbolically the Laplace transform is denoted by
The Mellin transform of $F(t)$ is denoted by $M[F(t)]$ and it is defined as

\[(1.3.6) \quad M[F(t)] = f(s) = \int_0^\infty t^{s-1}F(t)dt\]

and its inversion formula is given by

\[(1.3.7) \quad M^{-1}[f(s)] = F(t) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} f(s)t^{-s}ds\]

The Hankel transform, which is used in solving various boundary value problems defined by

\[(1.3.8) \quad H_n[F(t)] = f(p) = \int_0^\infty J_n(pt)F(t)dt\]

and its inversion formula is given by

\[(1.3.9) \quad H_n^{-1}[f(p)] = F(t) = \int_0^\infty pJ_n(pt)f(p)dp\]

The Fourier transform of $f(t)$ is defined as,

\[(1.3.10) \quad F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}f(t)dt\]

and its inversion formula is given by,

\[(1.3.11) \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist}F(s)ds, \text{ where } F(s) \text{ is Fourier transform of } f(t)\]

Other integral transforms worth mentioning are Kontorvich-Lebedev transforms, Mehler-Fock transforms, Stieltjes transforms, Y and K transforms and the H-function transforms defined by Gupta and Mittal [34].

The Laplace transform is generalized by different authors like Bhise [4], Kurushreshtha [42], Mainra [45], Meijer [51], Sharma [84], Varma [106] and Kapoor and Masood [39].
Apart from the infinite integral transforms discussed above, finite integral transforms are also defined and studied by different authors. Doetsch [18] defined the first finite integral transform with trigonometric kernels. The finite Fourier Sine transform of a function \( F(x) \) is given by

\[
\int_{0}^{\pi} F(x) \sin(px) dx
\]

and the inversion formula is given by

\[
\sum_{n=1}^{\infty} f(n) \sin(nx) dx
\]

Further generalizations on this are developed by Ko Schmieder [40] and Roettinger [77]. This has been applied to the classical boundary value problems by Brown [7]. The use of finite Hankel transforms was suggested by Sneddon [92] and its applications were studied by Marchi and Zgrablich [46] and Cinelli[14] for the solution of boundary value problems with more complicated boundary conditions. Gupta [35] generalized the finite Hankel transforms to solve the boundary value problems relating to hollow and confocal elliptical cylinders, Spheroids and Spheroidal shells.

Finite Gegenbauer transform was introduced by Lakshman Rao [43] and Conte [15]. Other useful finite transforms are studied by Churchill [11], Debnath [16], Mc Culley [50], Naylor [56] and Tranter [105].

Double integral transforms are also introduced by different authors like Goals [27], Kurushreshta[41], Pandey and Pandey [57], Rattan Singh [76], Varma [106], Vasishtha and Goyal [107] and Verma [108,109].

The Double Laplace transform is defined by

\[
F(p,q) = \int_{0}^{\infty} \int_{0}^{\infty} e^{-ps-qt} f(s,t) dsdt, \text{ Re}(p,q) > 0
\]

The triple integral transforms are studied by Sharma [87] and Singhal and Bhati [89]. The triple infinite Spheroidal transforms defined by Sharma [87] is also extended to n-variables.
Studies on multi dimensional integral transforms involving a product of two multi variable H-functions as kernel were introduced and discussed by Prasad and Nath [67] and Prasad and Dubey [65]. An inversion formula for the said transform is also given.

The multi dimensional H-function transform is defined as

\[ \mathcal{H}\{f(x_1,\ldots,x_r); p_1,\ldots,p_r\} \]

\[
= \int_0^\infty \cdots \int_0^\infty \mathcal{H}^{N_0}\{m_1,\ldots,m_n,\ldots,m_r,\ldots,n_r; p_1,\ldots,p_r,\ldots,q_r\}
\begin{bmatrix}
\alpha_1 \ldots \alpha_j \\
\vdots \\
\beta_1 \ldots \beta_j \\
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \ldots \gamma_j \\
\vdots \\
\delta_1 \ldots \delta_j \\
\end{bmatrix}
\frac{1}{p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot q_1^{\beta_1} \cdots q_r^{\beta_r}}
\left\{f(x_1,\ldots,x_r)\right\} dx_1 \cdots dx_r
\]

Provided the multiple integral occurring on the right hand side of (1.3.13) is absolutely convergent.

The multi dimensional Laplace transform is defined as

\[ \mathcal{L}\{f(x_1,\ldots,x_r); p_1,\ldots,p_r\} = \int_0^\infty \cdots \int_0^\infty \mathcal{L}^{N_0}\{m_1,\ldots,m_n,\ldots,m_r,\ldots,n_r; p_1,\ldots,p_r,\ldots,q_r\}
\begin{bmatrix}
\alpha \ldots \alpha_j \\
\vdots \\
\beta \ldots \beta_j \\
\end{bmatrix}
\begin{bmatrix}
\gamma \ldots \gamma_j \\
\vdots \\
\delta \ldots \delta_j \\
\end{bmatrix}
\frac{1}{p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot q_1^{\beta_1} \cdots q_r^{\beta_r}}
\left\{f(x_1,\ldots,x_r)\right\} e^{-(p_1^{\alpha_1} \cdots p_r^{\alpha_r} + q_1^{\beta_1} \cdots q_r^{\beta_r})} dx_1 \cdots dx_r
\]

Provided the multiple integral occurring on the right hand side of (1.3.14) is absolutely convergent.

The transforms (1.3.15) and (1.3.16) are due to Srivastava and Panda [101].

From the Table of Integrals [24], we have

\[ \int_{-1}^{1} (1-x)^p (1+x)^q dx = 2^{p+q+1} B(p+1, q+1) \]

\[ \int_0^\infty \left( ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx = \frac{\sqrt{\pi} \Gamma(p+\frac{1}{2})}{2a(4ab+c)^{p+\frac{1}{2}} \Gamma(p+1)} \]

\( \text{Re} (p) + \frac{1}{2} > 0 \)