INTRODUCTION

Lattice theory not only a necessary tool in the development of modern algebra in general and Universal algebra in particular. Lattice theory plays a major role in simplifying, unifying and generalizing many aspects of Mathematics and resembles Group theory, General topology and Functional analysis, because its central concept that of order, interwines through almost all mathematics. The beauty of Lattice theory is in its extreme simplicity of the basic concept, which is order or partial order, one gets interesting generalization of lattice concept by dropping one or more of the lattice identities.

In 1854, George Boole (1815–1864) introduced an important class of algebraic structures in his research work on mathematical logic. In his honor these structures have been called Boolean algebras. These are special type of lattices. In particular, congruence lattices play an important role.

It was E.Schroder, who about 1890, considered the lattice concept in today’s sense. At approximately the same time, R.Dedikind developed a similar concept in his work on groups and ideals. Dedikind defined modular and distributive lattices which are types of lattices of that are important in applications. The rapid development of lattice theory started around 1930. We could say that Boolean lattices or Boolean algebras are the simplest and at the same time the most important lattices for applications.

It was Garrett Birkhoff’s work in the mid thirties that started the general development of lattice theory. In a Brilliant series of papers he demonstrated the importance of the lattice theory and showed that it
provides a unifying framework for previously unrelated developments in any mathematical disciplines.

J.V. Numann, Oystein Ore, Orrin Frink, George Gratzer, E.T. Schmidt, G. Szasz, M.H. Stone and many others have worked on lattice theory. Stone also proved that Boolean algebras and Boolean rings are essentially the same in the sense that one can convert via terms from one to the other. Later many mathematicians developed different concepts and types of lattices.

In Mathematical order theory, a semilattice is a partially ordered set with in which either all binary sets have a supremum (join) or all binary sets have an infimum (meet). Consequently, one speaks of either a join semilattice or meet semilattice. Semilattices provide a generalization of the more prominent concept of a lattice and as such provide a natural way to introduce this concept as partial order which is both a meet and a join semilattice. As a natural consequence of the fact that semilattices are among the most basic “Lattice-like” structures, they can be characterized both in terms of order theory and Universal algebra.

In a draft paper [9], The Equational theory of Disjoint Alternatives, around 1989, E.G. Manes introduced the concept of Ada (Algebra of disjoint alternatives) \((A, \land, \lor, (\cdot), (\cdot), 0, 1, 2)\) which is however differ from the definition of the Ada of his later paper [10], Adas and the equational theory of if-then-else in 1993. While the Ada of the earlier draft seems to be based on extending the If-then-else concept more on the basis of Boolean algebras and the later concept is based on C-algebras \((A, \land, \lor, (\cdot))\) introduced by Fernando Guzman and Craig C. Squier [2]. In 1994, P. Koteswara Rao [7] first introduced the concept of A*-algebra \((A, \land, \lor, \ast, (\cdot)^{-1}, (\cdot)^{-1}, 0, 1, 2)\) not only studied the equivalence with Ada, C-algebra,
Ada’s connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and ideal of A*-algebra.

In 2000, J. Venkateswara Rao [29] introduced the variety generated by three element algebra associated with 3-valued conditional logic called Pre A*-algebra. Pre A*-algebra is the algebraic form of conditional logic. In 2009, K. Srinivasa Rao [23] introduced the concept of centre of Pre A*-algebra and obtained its useful characterizations. Further he studied on Pre A*-algebra as a poset, defined ternary operation on Pre A*-algebra obtained a Cayley theorem for centre of Pre A*-algebra and introduced the concept of Decomposition of Pre A*-algebra.

Boolean algebra is a two element conditional logic of two truth-values. C-algebra, Ada, A*-algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth-values, where the third truth-value stands for an undefined or unknown truth-value. The Pre A*-algebra structure is denoted by $(A, \land, \lor, (-)^-)$. Here A is non-empty set, $\land, \lor$ are binary operations and $(-)^-$ is a unary operation.

The Doctoral thesis ALGEBRAIC STUDY OF CERTAIN CLASSES OF PRE A*-ALGEBRAS AND C-ALGEBRAS consists of eight chapters.

The Chapter-1 originates with necessary and important definitions and results regarding partial orders, minimal and maximal elements, least and greatest elements, Hasse diagram of poset, lattices, Distributive lattices, Boolean algebra, A*-algebra and Pre A*-algebra etc., for our use of forthcoming chapters for ready reference.

The Chapter-2 depicts the concept of Semilattice structures on Pre A*-algebra, in this Chapter we define two binary operation * and $\oplus$ on Pre A*-
algebra $A$ and such that $<_{A,\star}>$ and $<_{A,\oplus}>$ become semilattices. We define $\preceq_\star$ and $\preceq_\oplus$ on a Pre $A^*$-algebra obtain the properties of Pre $A^*$-algebra as a poset. We obtain modular type results with respect to these operations. We observe that element 2 acts as the least element with respect to $\star$, where as the same element 2 acts as the greatest element with respect to $\oplus$ like the performance of a student in two different subjects, who obtains least mark in one subject and highest mark in the second subject. We give a number of equivalent conditions for Pre $A^*$-algebra $A$ to become a Boolean algebra in terms of the binary operations $\star$ and $\oplus$ and the partial orders $\preceq_\star$ and $\preceq_\oplus$. We derive the necessary conditions for $(A, \preceq_\star)$ and $(A, \preceq_\oplus)$ are to become a lattice. We also obtain $x \preceq_\star y$ if and only if $y \preceq_\oplus x$ and prove that the Pre $A^*$-algebra becomes trivial if the partial orders $\preceq_\star$ and $\preceq_\oplus$ are dual to each other. The main content of this chapter was published in Asian Journal of Scientific Research [15] and African Journal of Mathematics and Computer Science Research [14].

The Chapter-3 elucidates the concept of Pre $A^*$-algebra with order relation. Here we define a relation $\preceq_\wedge$ on $A$ by $x \preceq_\wedge y$ iff $x \wedge y = x$ and $x \vee y = y$. We derive some important properties of $(A, \preceq_\wedge)$ which leads to number of equivalent conditions $A$ to become a Boolean algebra in terms of this partial ordering. We establish necessary conditions for a poset to become a lattice. The main content of this chapter was published in International Journal of contemporary Mathematical Sciences [19].

The Chapter-4 explores the concept of Pre $A^*$-modules and If-then-else algebras over Pre $A^*$-algebra. We define the If-then-else operation $if_\wedge(p, q) = (x \wedge p) \vee (x^\sim \wedge q)$ on Pre $A^*$-algebra $A$ ($if_\wedge(p, q)$ should be viewed as conditional “if $x$, then $p$, else $q$”) and derive the most important properties
of the operation $if_x(-,-)$ ($x \in A$). We observe that either $x$ or $p$ or $q$ is 2 then $if_x(p,q)$ is 2. We establish the diagonal property on the If-then-else with a constraint, further we note that if the If-then-else operation defined on $B(A)$ then diagonal property holds good. Also we obtain the result that the variety $A$-ITE of all If-then-else algebras over Pre $A^*$-algebras $A$ and the variety $A$-Mod of Pre $A^*$-modules over $A$ are equivalent. The main content of this chapter was published in International Journal of Systemics, Cybernetics and Informatics [13].

The Chapter-5 *Prime and Maximal Ideals of Pre $A^*$-algebra* deals with the notation of an ideal, prime ideal, maximal ideal and minimal prime ideal of Pre $A^*$-algebra $A$ and discuss certain examples. We prove several fundamental properties of these, in particular we extend to prove that every ideal $I$ of a Pre $A^*$-algebra $A$ is the intersection of all prime ideals of $A$ containing $I$. Also we prove that for any ideals $I$ and $J$ of Pre $A^*$-algebra $A$, $(I:J)$ is precisely equal to the intersection of all prime ideals containing $J$ and not containing $I$. We also show that every maximal ideal is necessarily prime, while the converse is true for special cases only. The main content of this chapter was published in International Journal of Computational Cognition [17] and Trends in Applied Sciences Research [16].

In Chapter-6 *The category of Pre $A^*$-algebra* we cram the categorical aspects of Pre $A^*$-algebras. Here we establish the categorical equivalence by showing that the category of Pre $A^*$ of Pre $A^*$-algebras is equivalent to the category $B$ of Boolean algebras, to the category $A$ of Adas and to the category $3$-ring of 3-rings. Also this chapter concerns with the study of products and co-products in the category of Pre $A^*$-algebras. We obtain the results that the product (co-product) of the family of Pre $A^*$-algebras...
induces a Pre A*-epimorphism (Pre A*-monomorphism). The main content of this chapter was published in International Journal of Computational and Applied Mathematics [18] and Journal of Mathematical Sciences [21].

The Chapter-7 endows Congruences on Pre A*-algebra. We study two types of fundamental congruences on a Pre A*-algebra A and discuss various properties of these. It is proved that for any element a of Pre A*-algebra \( \theta_a = \beta_a \). The identities of the quotient algebras \( A/\theta_a \) and \( A/\beta_a \) were generated for both the operations \( \wedge \) and \( \vee \). We give sufficient conditions for two congruences on a Pre A*-algebra A to be permutable. In particular we introduce the notion of an ideal congruence corresponding to a given ideal and prove several results on these. Also we characterise the factor congruences on a Pre A*-algebra A and identify these with certain elements or set of elements of A. The main content of this chapter was published in International Journal of Applied Mathematics [22].

The Chapter-8 enlightens the concept of Partial ordering on C-algebra. In this chapter we give the definition of a C-algebra A and different properties. We define a partial ordering \( \leq \) on a C-algebra A by \( x \leq y \) if and only if \( x \vee y = y \) and give some examples. In general, in a C-algebra A the infimum of \( \{ x, y \} \) need not exist under this partial ordering. In fact, we prove that if \( \text{Inf} \{ x, y \} \) exist for all \( x, y \) in the poset \( (A, \leq) \) then \( <A, \wedge, \vee, \text{'}> \) is a Boolean algebra. We also give a particular cases when the supremum and infimum of \( \{ x, y \} \) exist in a C-algebra. We give a number of equivalent conditions for a C-algebra to become a Boolean algebra, in terms of \( \leq \). The main content of this chapter was published in International Journal of Computational Cognition [20].