CHAPTER – 7
Congruences on Pre A*-Algebra

In this chapter we study two types of fundamental congruences on a Pre A*-algebra $A$ and discuss various properties of these. It is proved that for any element $a$ of Pre A*-algebra $\theta_a = \beta_a$. The identities of the quotient algebras $A/\theta_a$ and $A/\beta_a$ were generated for both the operations $\wedge$ and $\vee$. We give sufficient conditions for two congruences on a Pre A*-algebra $A$ to be permutable. In particular we introduce the notion of an ideal congruence corresponding to a given ideal and prove several results on these. Also we characterise the factor congruences on a Pre A*-algebra $A$ and identify these with certain elements or set of elements of $A$.

7.1. Fundamental Congruences

In this section, we concentrate two types of fundamental congruences on a Pre A*-algebra and discuss various properties of these.

7.1.1. Definition: Let $A$ be a Pre A*-algebra and $\theta$ be a binary relation on $A$. Then $\theta$ is said to be an equivalence relation on $A$ if $\theta$ satisfies the following:

(i) Reflexive: $(x, x) \in \theta$, for all $x \in A$

(ii) Symmetric: $(x, y) \in \theta \Rightarrow (y, x) \in \theta$, for all $x, y \in A$

(iii) Transitive: $(x, y) \in \theta$ and $(y, z) \in \theta \Rightarrow (x, z) \in \theta$, for all $x, y, z \in A$.

We write $x \theta y$ to indicate $(x, y) \in \theta$.

7.1.2. Definition: [23] A relation $\theta$ on a Pre A*-algebra $(A, \wedge, \vee, (\cdot))$ is called congruence relation if

(i) $\theta$ is an equivalence relation
(ii) \( \theta \) is closed under \( \land, \lor, (-)^{-} \).

**7.1.3. Lemma:** [23] Let \((A, \land, \lor, (-)^{-})\) be a Pre A*-algebra and let \(a \in A\). Then the relation \(\theta_a = \{(x, y) \in A \times A/ a \land x = a \lor y\}\) is

(a) a congruence relation

(b) \(\theta_a \cap \theta_{a'} = \theta_{a \lor a'}\)

(c) \(\theta_a \cap \theta_b \subseteq \theta_{a \lor b}\)

(d) \(\theta_a \cap \theta_{a'} \subseteq \theta_{a \lor a'}\)

we will write \(x \theta_a y\) to indicate \((x, y) \in \theta_a\).

**7.1.4. Lemma:** [23] Let \(A\) be a Pre A*-algebra and \(a, b \in B(A)\), then \(\theta_a \cap \theta_b = \theta_{a \lor b}\)

**7.1.5. Theorem:** [23] Let \(A\) be a Pre A*-algebra, then \(A/\theta = \{\theta_a / a \in A\}\) is a Pre A*-algebra is called quotient Pre A*-algebra, whose operations are defined as

(i) \(\theta_a \land \theta_b = \theta_{a \land b}\)

(ii) \(\theta_a \lor \theta_b = \theta_{a \lor b}\)

(iii) \((\theta_a)^{-} = \theta_{a^{-}}\)

**7.1.6. Note:** \(A/\theta = \{a/\theta / a \in A\}\) we write \(\theta_a\) for the element of \(A/\theta\)

**7.1.7. Lemma:** [23] Let \((A, \land, \lor, (-)^{-})\) be a Pre A*-algebra and let \(a \in A\). Then the relation \(\beta_a = \{(x, y) \in A \times A/ a \lor x = a \lor y\}\) is

(a) congruence relation

(b) \(\beta_a \cap \beta_{a'} \subseteq \beta_{a \lor a'}\)

(c) \(\beta_a \cap \beta_a = \beta_{a \lor a'}\)

(d) \(\beta_a \cap \beta_b \subseteq \beta_{a \lor b}\)

We will write \(x \beta_a y\) to indicate \((x, y) \in \beta_a\).
7.1.8. **Lemma:** [23] Let $A$ be a Pre A*-algebra and $a, b \in B(A)$, then
\[ \beta_a \cap \beta_b = \beta_{a \wedge b} \]

7.1.9. **Theorem:** [23] Let $A$ be a Pre A*-algebra, then $A / \beta = \{ \beta_a / a \in A \}$ is a Pre A*-algebra is called quotient Pre A*-algebra, whose operations are defined as
(i) $\beta_a \cap \beta_b = \beta_{a \wedge b}$
(ii) $\beta_a \cup \beta_b = \beta_{a \vee b}$
(iii) $(\beta_a)^\ast = \beta_{a^\ast}$

7.1.10. **Definition:** Let $A$ be a Pre A*-algebra. Then the set of all congruences on $A$ is denoted by $\text{Con}(A)$.

7.1.11. **Definition:** Let $A$ be a Pre A*-algebra and $\alpha, \beta$ be binary relations on $A$. Then we define $\alpha \circ \beta = \{(x, y) \in A \times A / (x, z) \in \beta \text{ and } (z, y) \in \alpha \text{ for some } z \in A \}$.

In an Algebra if $\alpha, \beta$ are equivalence relations then $\alpha \circ \beta$ need not be an equivalence relation. However if $\alpha \circ \beta = \beta \circ \alpha$ then it is known that $\alpha \circ \beta$ is an equivalence relation. The same is the case with congruences on Pre A*-algebra.

7.1.12. **Definition:** Let $A$ be a Pre A*-algebra and $\alpha, \beta \in \text{Con}(A)$. Then $\alpha, \beta$ are said to be permutable if $\alpha \circ \beta = \beta \circ \alpha$. A subset $L$ of $\text{Con}(A)$ is called permutable if any two congruences in $L$ are permutable.

7.1.13. **Lemma:** The following are hold for any two elements $a$ and $b$ of Pre A*-algebra

1. $(a \wedge b, b) \in \theta_a$
2. $(a \wedge b, b \wedge a) \in \theta_a$

**Proof:** Let $a, b \in A$
(1) We know that \( a \land (a \land b) = a \land b \)
\[ \Rightarrow (a \land b, b) \in \theta_a \]

(2) We know that \( a \land (a \land b) = a \land (b \land a) \)
\[ \Rightarrow (a \land b, b \land a) \in \theta_a \]

7.1.14. Lemma: Let \( A \) be a Pre A*-algebra and \( a, b \in A \). Then the following hold:

(1) \( (b, a \lor b) \in \beta_a \)
(2) \( (a \lor b, b \lor a) \in \beta_a \)

Proof: Let \( a, b \in A \)

(1) We know that \( a \lor b = a \lor a \lor b \)
\[ \Rightarrow (b, a \lor b) \in \beta_a \]

(2) We know that \( a \lor (a \lor b) = a \lor (b \lor a) \)
\[ \Rightarrow (a \lor b, b \lor a) \in \beta_a \]

7.1.15. Lemma: For any element \( a \) in a Pre A*-algebra \( A \) the quotient algebra \( A/\theta_a \) has identities for both the operations \( \land \) and \( \lor \).

Proof: Consider the element \( a/\theta_a \) in \( A/\theta_a \)

For any \( b \in A \) we have
\[ a/\theta_a \land b/\theta_a = a \land b/\theta_a = b/\theta_a \]
and also
\[ b/\theta_a \land a/\theta_a = b \land a/\theta_a = b/\theta_a \]
(by 7.1.13. Lemma)

Hence \( a/\theta_a \) is identity for \( \land \) in \( A/\theta_a \).

Let \( b/\theta_a \) in \( A/\theta_a \)

We have \( a/\theta_a \lor b/\theta_a = a/\theta_a \lor b/\theta_a \)
\[ = a/\theta_a \lor b/\theta_a \]
\[ = (a \land b)/\theta_a \]
\[ = (b/\theta_a)^{\sim} \]
(since \( a/\theta_a \) is identity for \( \land \) in \( A/\theta_a \))
\[
\frac{b}{\theta_a} = \frac{b}{\theta_a}
\]
Similarly we can prove that \(\frac{b}{\theta_a} \lor a \sim/ \theta_a = \frac{b}{\theta_a}\).

\(a \sim/ \theta_a\) is identity for \(\lor\) in \(A/\theta_a\).

**7.1.16. Lemma:** Let \(A\) be a Pre A*-algebra and \(a \in A\). Then the quotient algebra \(A/\beta_a\) has identities for both the operations \(\land\) and \(\lor\).

**Proof:** Consider the element \(a/\beta_a\) in \(A/\beta_a\)

For any \(b \in A\) we have

\(a/\beta_a \lor b/\beta_a = a \lor b/\beta_a = b/\beta_a\) and also

\(b/\beta_a \lor a/\beta_a = b \lor a/\beta_a = b/\beta_a\) (by 7.1.14. Lemma)

Hence \(a/\beta_a\) is identity for \(\lor\) in \(A/\beta_a\).

Let \(b/\beta_a\) in \(A/\beta_a\)

We have \(a \sim/ \beta_a \land b/\beta_a = a \sim/ \beta_a \land b \sim/ \beta_a\)

\[= a \sim \land b \sim/ \beta_a\]

\[= (a \lor b/\beta_a)\sim\]

\[= (b/\beta_a)\sim\] (since \(a/\beta_a\) is identity for \(\lor\) in \(A/\beta_a\))

\[= b/\beta_a\]

Similarly we can prove that \(b/\beta_a \land a \sim/ \beta_a = b/\beta_a\).

\(a \sim/ \beta_a\) is identity for \(\land\) in \(A/\beta_a\).

**7.1.17. Lemma:** For any element \(a\) of Pre A*-algebra \(A\) we define

\(\theta_a = \{(x,y) \in A \times A | a \land x = a \land y\}\) then \(\theta_a \lor \beta_a = \{(x,y) \in A \times A | a \sim \lor x = a \sim \lor y\}\)

**Proof:** Let \(a, x, y \in A\).

Let \((x, y) \in \theta_a\) \(\Rightarrow a \land x = a \land y\)

\[\Rightarrow a \sim (a \land x) = a \sim (a \land y)\]
⇒ \( a \sim \lor x = a \sim \lor y \) (by 1.2.1 Definition (g))

⇒ \((x, y) \in \beta_a \)

Therefore \( \theta_a \subseteq \beta_a \)

Let \((x, y) \in \beta_a \) ⇒ \( a \sim \lor x = a \sim \lor y \)

⇒ \( a \land (a \sim \lor x) = a \land (a \sim \lor y) \)

⇒ \( a \land x = a \land y \) (by 1.2.1 Definition (g))

⇒ \((x, y) \in \theta_a \)

Therefore \( \beta_a \subseteq \theta_a \)

Hence \( \theta_a = \beta_a \)

7.1.18. Definition: Let \( A \) be a Pre A*-algebra and \( \theta \in \text{Con}(A) \). Then the map \( a \rightarrow \theta_a \) is an epimorphism of \( A \) onto \( A/\theta \) is called natural map or canonical map.

If \( A \) is a Pre A*-algebra then the congruences \( A \times A \) and \( \{(x, x) / x \in A\} \) are denoted by \( \nabla_A \) and \( \Delta_A \) respectively.

7.1.19. Lemma: Let \( A \) be a Pre A*-algebra and \( a, b \in A \) then \( \theta_a \circ \theta_b \subseteq \theta_{a \land b} \)

Proof: Let \((x, y) \in \theta_a \circ \theta_b \). Then there exist \( z \in A \) such that \((x, z) \in \theta_a \) and \((z, y) \in \theta_b \).

Thus \( b \land x = b \land z \) and \( a \land z = a \land y \).

Now \( a \land b \land x = a \land b \land z \)

\[ = a \land b \land a \land z \]

\[ = a \land b \land a \land y \]

\[ = a \land b \land y \]

Therefore \((x, y) \in \theta_{a \land b} \)

Hence \( \theta_a \circ \theta_b \subseteq \theta_{a \land b} \).
We remark that the converse of the above theorem is not true, that is 
\( \theta_{ab} \subseteq \theta_a \circ \theta_b \) is not true and which the congruence need not permute.

7.1.20. Example: Let \( G = \{a_1, a_2, a_3, a_4, a_5\} \) where \( a_1 = (1, 2), a_2 = (0, 2), a_3 = (2, 1), a_4 = (2, 0), a_5 = (2, 2) \). Then \( G \) is a Pre A*-algebra (a sub algebra of \( A \times A \)) under the point wise operations given in the 5.1.3. Example tables

This algebra \( (G, \wedge, \lor, (-)^{\sim}) \) is a Pre A*-algebra without 1.

Let \( A = \text{diagonal of } A \). Then we have the following.

\[
\theta_{a_1} = \{(x, y) / a_1 \wedge x = a_1 \wedge y\} \]  
\[
= \Delta_A \cup \{(a_3, a_4), (a_4, a_3), (a_4, a_5), (a_5, a_4), (a_5, a_3), (a_3, a_5)\}
\]

\[
\theta_{a_3} = \{(x, y) / a_3 \wedge x = a_3 \wedge y\} \]  
\[
= \Delta_A \cup \{(a_1, a_2), (a_2, a_1), (a_2, a_5), (a_5, a_2), (a_5, a_1), (a_1, a_5)\}
\]

Now \( \theta_{a_1} \circ \theta_{a_3} = \Delta_A \cup \theta_{a_1} \cup \theta_{a_3} \cup \{(a_2, a_4), (a_2, a_3), (a_1, a_4), (a_1, a_3)\} \)

\[
\theta_{a_3} \circ \theta_{a_1} = \Delta_A \cup \theta_{a_1} \cup \theta_{a_3} \cup \{(a_4, a_2), (a_3, a_2), (a_4, a_1), (a_3, a_1)\}
\]

Also \( \theta_{a_1} \wedge a_3 = \theta a_5 = A \times A \).

Therefore \( \theta_{a_1} \wedge a_3 \not\subseteq \theta_{a_1} \circ \theta_{a_3} \) and \( \theta_{a_1} \circ \theta_{a_3} \neq \theta_{a_3} \circ \theta_{a_1} \).

7.1.21. Example: Let \( H = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\} \) where \( b_1 = (1, 2), b_2 = (0, 2), b_3 = (2, 1), b_4 = (2, 0), b_5 = (2, 2), b_6 = (1, 1), b_7 = (0, 0) \). Then \( H \) is a Pre A*-algebra (a sub algebra of \( A \times A \)) under the point wise operations given in the 5.1.4. Example tables

This algebra \( (H, \wedge, \lor, (-)^{\sim}) \) is a Pre A*-algebra with 1 and \( 1 = b_6 \).

\[
\theta_{a_1} = \{(x, y) / a_1 \wedge x = a_1 \wedge y\} \]  
\[
= \Delta_A \cup \{(b_1, b_6), (b_6, b_1), (b_3, b_4), (b_4, b_3), (b_4, b_5), (b_5, b_4), (b_3, b_5),
\]
\[
(b_5, b_3), (b_2, b_7), (b_7, b_2)\}
\]

\[
\theta_{a_3} = \{(x, y) / a_3 \wedge x = a_3 \wedge y\} \]  
\[
= \Delta_A \cup \{(b_1, b_2), (b_2, b_1), (b_3, b_6), (b_6, b_3), (b_1, b_5), (b_5, b_1), (b_2, b_5),
\]
\[
(b_7, b_2), (b_2, b_7), (b_7, b_2)\}
\]

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Now $\theta_a \circ \theta_a = \Delta_4 \cup \theta_a \cup \theta_a \cup \{(b_1, b_7), (b_2, b_6), (b_3, b_1), (b_6, b_4), (b_6, b_5), (b_5, b_7), (b_4, b_2), (b_7, b_5), (b_7, b_3)\}$

$(b_6, b_3), (b_1, b_4), (b_5, b_6), (b_2, b_4), (b_2, b_3), (b_5, b_7), (b_4, b_2), (b_7, b_5), (b_7, b_3)\}$

$\theta_a \circ \theta_a = \Delta_4 \cup \theta_a \cup \theta_a \cup \{(b_1, b_3), (b_6, b_2), (b_6, b_5), (b_3, b_7), (b_4, b_6), (b_4, b_1), (b_4, b_2), (b_5, b_7), (b_5, b_6), (b_3, b_1), (b_3, b_2), (b_7, b_5), (b_2, b_4), (b_7, b_1)\}$

Also $\theta_a \land a_3 = \theta a_5 = A \times A$.

Therefore $\theta_a \land a_3 \not\subseteq \theta_a \circ \theta_a$ and $\theta_a \circ \theta_a \neq \theta_3 \circ \theta_a$.

**7.1.22. Theorem:** Let $A$ be a Pre $A^*$-algebra with $1$ and $a, b \in B(A)$ then $\theta_a, \theta_b$ are permute and $\theta_a \circ \theta_b = \theta_a \circ \theta_b$

**Proof:** Let $A$ be a Pre $A^*$-algebra with $1$ and $a, b \in B(A)$.

By lemma 7.19 we have $\theta_a \circ \theta_b \subseteq \theta_a \circ \theta_b$

Let $(p, q) \in \theta_a \circ \theta_b$ $\Rightarrow \ a \land b \land p = a \land b \land q$ -------------- ( i )

Consider $r = (b \land p) \lor (b \sim q)$

Now $b \land r = b \land ((b \land p) \lor (b \sim q))$

$= (b \land b \land p) \lor (b \land b \sim q)$

$= (b \land p) \lor (0 \land q)$ \ (since $b \in B(A)$)

$= (b \land p) \lor 0$ \ (provided $q \neq 2$)

$= b \land p$

If $q = 2$ then $r = (b \land p) \lor (b \sim q) = 2$ and also $b \land r = 2$. For $b \land p = 2$, $p$ should be $2$.

Therefore $b \land r = b \land p$

$\Rightarrow (p, r) \in \theta_b$

Now $a \land r = a \land ((b \land p) \lor (b \sim q))$

$= (a \land b \land p) \lor (a \land b \sim q)$
\[(a \land b \land q) \lor (a \land b \lor q) \quad \text{(by (i))}\]

\[= a \land ((b \lor b) \land q)\]

\[= a \land (1 \land q) \quad \text{(since } b \in B(A))\]

\[= a \land q\]

Therefore \(a \land r = a \land q\) so that \((r, q) \in \theta_a\)

We have \((p, r) \in \theta_a\) and \((r, q) \in \theta_a\) hence \((p, q) \in \theta_a \circ \theta_a\), which implies that \(\theta_{a \land b} \subseteq \theta_a \circ \theta_b\).

Hence \(\theta_a \circ \theta_b = \theta_{a \land b}\).

We have \(\theta_a \circ \theta_b = \theta_{a \land b} = \theta_{b \land a} = \theta_b \circ \theta_a\).

Hence \(\theta_a, \theta_b\) are permute congruences.

7.1.23. Theorem: Let \(A\) be a Pre A*-algebra with 1 and \(a, b \in A\) then \(\theta_a = \theta_b\) if and only if \(a = b\).

Proof: Suppose that \(\theta_a = \theta_b\)

We have \(b \land a = b \land (b \land a)\) then \((a, b \land a) \in \theta_b\)

\[\Rightarrow (a, b \land a) \in \theta_a\]

\[\Rightarrow a \land a = a \land (b \land a)\]

\[\Rightarrow a = b \land a\]

Also we have \(a \land b = a \land (a \land b)\) then \((b, a \land b) \in \theta_a\)

\[\Rightarrow (b, a \land b) \in \theta_b\]

\[\Rightarrow b \land b = b \land (a \land b)\]

\[\Rightarrow b = b \land a\]

We have \(a \land b = b \land a\) implies that \(a = b\). Converse is trivial.

7.1.24. Theorem: Let \(A\) be a Pre A*-algebra with 1 then

(1) \(\theta_1 = \Delta_A\)

(2) \(\theta_0 = A \times A\), if \(A = B(A)\)
Proof: (1) \[ \theta_i = \{ (x, y) / 1 \land x = 1 \land y \} \]
\[ = \{ (x, y) / x = y \} \]
\[ = \{ (x, x) / x \in A \} \]
\[ = \Delta_A \]

(2) Let \( x, y \in A = B(A) \)
Since \(0 \land x = 0 = 0 \land y \) (since \( x, y \in B(A) \))
\[ \Rightarrow (x, y) \in \theta_0 \]
Therefore \( A \times A \subseteq \theta_0 \)
Thus \( \theta_0 = A \times A. \)

7.1.25. Theorem: Let \((A, \land, \lor, (-))\) be a Pre A*-algebra with 1 if A be a Boolean algebra then \( \theta_i \cap \theta_{x} = \Delta_A \) and hence \( \theta_{x \lor x'} = \Delta_A \) for all \( x \in A \)

Proof: Suppose \( A \) be a Boolean algebra and \( x \in A \)
Let \((p, q) \in \theta_i \cap \theta_{x'} \)
Then \( x \land p = x \land q \) and \( x' \land p = x' \land q. \)
Now \( p = 1 \land p = (x \lor x') \land p \)
\[ = (x \land p) \lor (x' \land p) \]
\[ = (x \land q) \lor (x' \land q) \]
\[ = (x \lor x') \land q \]
\[ = 1 \land q = q \]
Therefore \( p = q \) so that \((p, q) \in \Delta_A. \) Hence \( \theta_i \cap \theta_{x'} \subseteq \Delta_A. \)
Thus \( \theta_i \cap \theta_{x'} = \Delta_A. \)
Also \( \theta_{x \lor x'} = \theta_i \cap \theta_{x'}. \) (by 7.1.4.Lemma)

7.1.26. Theorem: The following are equivalent for any element \( x \) in a Pre A*-algebra A.
(1) \( x \) is the identity element for \( \lor \)
(2) $\beta_x=\Delta_A$, the diagonal on $A$.

(3) $x$ is identity for $\wedge$.

**Proof:** (1) $\Rightarrow$ (2) Let $x$ is the identity element for $\vee$. Then $x \vee a = a$, for all $a \in A$

Let $(p, q) \in \beta_x \Rightarrow x \vee p = x \vee q \Rightarrow p = q$.

Then $\beta_x = \Delta_A$.

(2) $\Rightarrow$ (1) Suppose $\beta_x = \Delta_A$.

Clearly $(a, x \vee a)$ and $(a, a \vee x) \in \beta_x = \Delta_A$.

Therefore $x \vee a = a = a \vee x$ for any $a \in A$ and hence $x$ is the identity element for $\vee$.

Thus (1) and (2) are equivalent.

The equivalence of (1) and (3) follows from the De Morgan law $(a \wedge b)^\sim = a^\sim \vee b^\sim$ and the axiom $a^\sim \sim = a$.

**7.1.27. Theorem:** The following are equivalent for any element $x$ in a Pre $A^*$-algebra $A$.

(1) $x$ is the identity element for $\wedge$.

(2) $\theta_x = \Delta_A$, the diagonal on $A$.

(3) $x^\sim$ is identity for $\vee$.

(4) $\beta_x = \Delta_A$.

**Proof:** (1) $\Rightarrow$ (2) Let $x$ is the identity element for $\wedge$. Then $x \wedge a = a$, for all $a \in A$

Let $(p, q) \in \theta_x \Rightarrow x \wedge p = x \wedge q \Rightarrow p = q$.

Then $\theta_x = \Delta_A$.

(2) $\Rightarrow$ (1) Suppose $\theta_x = \Delta_A$.

Clearly $(x \wedge a, a)$ and $(a, a \wedge x) \in \theta_x = \Delta_A$.
Therefore \( x \land a = a = a \land x \) for any \( a \in A \) and hence \( x \) is the identity element for \( \land \).

Thus (1) and (2) are equivalent.

The equivalence of (1) and (3) follows from the De Morgan law \((a \lor b) \sim = a\sim \land b\sim\) and the axiom \( a \sim \sim = a \).

(3) \( \Rightarrow \) (4) Let \( x \sim \) is identity for \( \lor \). Then \( x \sim \lor a = a \), for all \( a \in A \).

Let \( (p, q) \in \beta_{x} \Rightarrow x \sim \lor p = x \sim \lor q \Rightarrow p = q \).

Then \( \beta_{x} = \Delta_{A} \).

(4) \( \Rightarrow \) (3) Suppose that \( \beta_{x} = \Delta_{A} \).

Clearly \((a, x \sim \lor a)\) and \((a, a \lor x \sim) \in \beta_{x} = \Delta_{A} \).

Therefore \( x \sim \lor a = a = a \lor x \sim \) for any \( a \in A \) and hence \( x \sim \) is identity for \( \lor \).

Thus (3) and (4) are equivalent.

**7.1.28. Lemma:** Let \( A \) be a Pre \( A^* \)-algebra and \( x, y \in A \). Then \( \theta_{x} \subseteq \theta_{y} \) if and only if \( y = x \land y \)

**Proof:** Suppose that \( \theta_{x} \subseteq \theta_{y} \).

Since \( x \land y = x \land x \land y \), we have \((y, x \land y) \in \theta_{x} \) and therefore \((y, x \land y) \in \theta_{y} \)

\[
\Rightarrow y \land y = y \land x \land y
\]

\[
\Rightarrow y = x \land y
\]

Conversely suppose that \( y = x \land y \)

Let \((p, q) \in \theta_{x} \Rightarrow x \land p = x \land q \)

Now \( y \land p = x \land y \land p \) (by supposition)

\[
= y \land x \land p
\]

\[
= y \land x \land q
\]

\[
= y \land q \text{ (by supposition)}
\]

Therefore \((p, q) \in \theta_{x} \) and hence \( \theta_{x} \subseteq \theta_{y} \).
7.1.29. **Lemma:** Let $A$ be a Pre $A^*$-algebra and $x, y \in A$. Then $\theta_x \subseteq \theta_{x \land y}$ and $\theta_y \subseteq \theta_{x \land y}$.

**Proof:** We know that $x \land (x \land y) = x \land y$, by above Lemma we have $\theta_x \subseteq \theta_{x \land y}$.

Also we know that $y \land (x \land y) = x \land y$, by above Lemma we have $\theta_y \subseteq \theta_{x \land y}$.

7.1.30. **Lemma:** Let $A$ be a Pre $A^*$-algebra and $x, y \in B(A)$. Then $\theta_{x \lor y} \subseteq \theta_x$.

**Proof:** Let $(p, q) \in \theta_{x \lor y}$. Then $(x \lor y) \land p = (x \lor y) \land q$.

Now $x \land p = x \land (x \lor y) \land p$ (by 1.2.10. Lemma)

$= x \land (x \lor y) \land q$

$= x \land q$

Therefore $(p, q) \in \theta_x$ and hence $\theta_{x \lor y} \subseteq \theta_x$.

Note that it is similar way $\theta_{x \lor y} \subseteq \theta_y$.

7.1.31. **Theorem:** Let $A$ be a Pre $A^*$-algebra with $1$ and $x \in A$. Then $\theta_x$ is the smallest congruence on $A$ containing $(1, x)$.

**Proof:** We know that $\theta_x$ is a congruence of $A$ and clearly $(1, x) \in \theta_x$.

Let $\theta$ be a congruence on $A$ and $(1, x) \in \theta$.

Suppose that $(p, q) \in \theta_x \Rightarrow x \land p = x \land q$.

Since $(1, x) \in \theta$ we have $(1 \land p, x \land p)$ and $(1 \land q, x \land q) \in \theta$; that is $(p, x \land p)$ and $(q, x \land q) \in \theta$. Therefore $(p, q) \in \theta$ and hence $\theta_x \subseteq \theta$.

Therefore $\theta_x$ is the smallest congruence on $A$ containing $(1, x)$.

7.1.32. **Theorem:** Let $A$ be a Pre $A^*$-algebra with $1$ and $x, y \in A$ then the following are equivalent.

1. $x, y \in B(A)$
2. $\theta_{x \lor y} \subseteq \theta_y$
3. $\theta_{x \lor y} \subseteq \theta_x \cap \theta_y$
(4) $\theta_{xy} = \theta_x \cap \theta_y$

**Proof:** (1) $\Rightarrow$ (2) Suppose that $x, y \in B(A)$
Let $(p, q) \in \theta_{xy}$.

Then $(x \lor y) \land p = (x \lor y) \land q$
Now $y \land p = y \land (y \lor x) \land p$ (by 1.2.10. Lemma)
\[= y \land (x \lor y) \land q \]
\[= y \land q \]
Therefore $(p, q) \in \theta_y$ and hence $\theta_{xy} \subseteq \theta_y$.

(2) $\Rightarrow$ (3) Suppose that $\theta_{xy} \subseteq \theta_y$.
By symmetry we have $\theta_{xy} \subseteq \theta_x$.

Therefore $\theta_{xy} \subseteq \theta_x \cap \theta_y$.

(3) $\Rightarrow$ (4) Suppose that $\theta_{xy} \subseteq \theta_x \cap \theta_y$.

We know that $\theta_x \cap \theta_y \subseteq \theta_{xy}$ (by 7.1.3. Lemma)
Therefore $\theta_{xy} = \theta_x \cap \theta_y$

(4) $\Rightarrow$ (1) Suppose that $\theta_{xy} = \theta_x \cap \theta_y$

Then $(1, x \lor y) \in \theta_{xy} = \theta_x \cap \theta_y$ (by supposition)

Therefore $(1, x \lor y) \in \theta_x$ and $(1, x \lor y) \in \theta_y$

$\Rightarrow x \land 1 = x \land (x \lor y)$ and $y \land 1 = y \land (x \lor y)$

$\Rightarrow x = x \land (x \lor y)$ and $y = y \land (x \lor y)$

$\Rightarrow x, y \in B(A)$ (by 1.2.10. Lemma)

**7.2 Ideal congruences**

Now we introduce the notion of the ideal congruence on a Pre $A^*$-algebra $A$ corresponding to an ideal $I$ of $A$. 
7.2.1. **Definition:** For any ideal I of a Pre A*-algebra A we define 
\[ \beta_1 = \{(x, y) / a \lor x = a \lor y, \text{ for some } a \in I\}. \] That is \[ \beta_1 = \bigcup_{a \in I} \beta_a = \bigcup_{a \in I} \theta_a. \]

7.2.2. **Theorem:** \( \beta_1 \) is a congruence on a Pre A*-algebra A for any ideal I of A.

**Proof:** We know that the union of a class of congruences on A is again a congruence on A if the given class is directed above, in the sense that, for any two members \( \beta_1 \) and \( \beta_2 \) in that class there exist a member \( \beta \) in the class containing both \( \beta_1 \) and \( \beta_2 \).

Now consider \( C = \{ \beta_a / a \in I\} \)

Since each \( \beta_a \) is congruence on A, C is a class of congruence on A. Also for any \( a, b \in I \) we have \( a \lor b \in I \) and \( \beta_a \lor \beta_b = \beta_{a \lor b} \in C \)

Therefore C is a directed above class of congruences and \( \bigcup_{a \in I} \beta_a (= \beta_1) \) is a congruence on A.

7.2.3. **Remark:** If \( < x > = \{a \land x / a \in A\} \) is the principal ideal generated by an element x in a Pre A*-algebra A, then clearly \( \beta_x \subseteq \beta_{<x>} \). However equality does not hold as in the case of distributive lattices. For, consider the three element Pre A*-algebra \( A = \{0, 1, 2\} \), \( < 0 > = \{0, 2\} \) and \( \beta_0 = \Delta_A \) and \( \beta_{<0>} = A \times A \). Hence \( \beta_{<0>} \not\subseteq \beta_0 \).

7.2.4. **Theorem:** Let I be an ideal a Pre A*-algebra A. Then \( \beta_1 \) is the smallest congruence on A containing \( I \times I \).

**Proof:** We know that \( \beta_1 \) is a congruence on a Pre A*-algebra A.

Also for any \( x, y \in I \) we have \( x \lor y \in I \)

Now \( (x \lor y) \lor x = x \lor y = (x \lor y) \lor y \) hence \( (x, y) \in \beta_1 \) (since \( x \lor y \in I \))

Therefore \( I \times I \subseteq \beta_1 \).
Now \( \beta \) is any congruence on \( A \) such that \( I \times I \subseteq \beta \).

Then \((x, y) \in \beta_1 \Rightarrow a \lor x = a \lor y \) for some \( a \in I \)

We have \((x \land x \sim, a) \in \beta \) (since \( x \land x \sim \in I \) and \( a \in I \))

\[ \Rightarrow ((x \land x \sim) \lor x, a \lor x) \in \beta \] (since \( \beta \) is congruence)

\[ \Rightarrow (x, a \lor x) \in \beta \] and for similar reason \((y, a \lor y) \in \beta \)

\[ \Rightarrow (x, y) \in \beta \] (since \( \beta \) is transitive and \( a \lor x = a \lor y \))

Therefore \( \beta_1 \subseteq \beta \).

Then \( \beta_1 \) is the smallest congruence on \( A \) containing \( I \times I \).

**7.2.5. Theorem:** For any ideals \( I \) and \( J \) of a Pre A*-algebra \( A \) the following hold.

1. \( I \subseteq J \Rightarrow \beta_I \subseteq \beta_J \)
2. \( \beta_I \cap \beta_J = \beta_{I \cap J} \)
3. \( \beta_I \lor \beta_J = \beta_{I \lor J} \)

**Proof:** Let \( I \) and \( J \) are ideals of a Pre A*-algebra \( A \).

(1) Suppose that \( I \subseteq J \).

Let \( a \in I \Rightarrow a \in J \)

Let \((x, y) \in \beta_I \Rightarrow a \lor x = a \lor y \) for some \( a \in I \)

\[ \Rightarrow a \lor x = a \lor y \] for some \( a \in J \)

\[ \Rightarrow (x, y) \in \beta_J \]

Therefore \( \beta_I \subseteq \beta_J \).

(2) Since \( I \cap J \subseteq I \) and \( I \cap J \subseteq J \) we get \( \beta_{I \cap J} \subseteq \beta_I \cap \beta_J \) (by (1))

Let \((x, y) \in \beta_I \cap \beta_J \Rightarrow (x, y) \in \beta_I \) and \((x, y) \in \beta_J \)

\[ \Rightarrow a \lor x = a \lor y \text{ and } b \lor x = b \lor y, \text{ where } a \in I, b \in J \]

Now \( a \land b \in I \cap J \) and also \((a \land b) \lor x = (a \lor x) \land (b \lor x)\)

\[ = (a \lor y) \land (b \lor y) \]
\[(a \land b) \lor y\]

Therefore \((x, y) \in \beta_{I \cap J}\)

Hence \(\beta_1 \cap \beta_1 \subseteq \beta_{I \cap J}\)

Therefore \(\beta_1 \cap \beta_1 = \beta_{I \cap J}\)

(3) Since \(I \subseteq I \lor J\) and \(J \subseteq I \lor J\) we have \(\beta_1 \subseteq \beta_{I \lor J}\), \(\beta_1 \subseteq \beta_{I \lor J}\) and hence

\[\beta_1 \lor \beta_1 \subseteq \beta_{I \lor J}\]

Let \((x, y) \in \beta_z\) where \(z \in I \lor J\)

\[\Rightarrow z = \bigvee_{i=1}^{n} x_i\] for some \(x_i \in I \lor J\) and \((x, y) \in \bigvee_{i=1}^{n} \beta_{x_i}\) (since \(
\beta_{a \lor b} = \beta_a \lor \beta_b\))

\[\Rightarrow (x, y) \in \bigvee_{i=1}^{n} \beta_{x_i} \subseteq \beta_1 \lor \beta_1\] (since each \(x_i \in I\) or \(J\))

\[\Rightarrow (x, y) \in \beta_1 \lor \beta_1\]

\[\Rightarrow \beta_{I \lor J} \subseteq \beta_1 \lor \beta_1\]

Therefore \(\beta_1 \lor \beta_1 = \beta_{I \lor J}\).

Let us recall that the set \(\text{Con}(A)\) of all congruences on any algebra \(A\) is an algebraic lattice under the inclusion ordering in which the g.l.b and l.u.b of any subset \(\bar{C}\) of \(\text{Con}(A)\) are given by g.l.b \(\bar{C} = \bigcap_{\theta \in \bar{C}} \theta\) and l.u.b \(\bar{C} = \bigcup \{ \theta_1 \circ \theta_2 \circ \ldots \circ \theta_n \mid \theta_i \in \bar{C} \}\). Also it is known that the set \(\text{F}(A)\) of all ideals of a Pre \(A^*\)-algebra \(A\) forms an algebraic lattice under the inclusion of ordering. Now we have the following.

**7.2.6. Theorem:** Let \(\text{F}(A)\) be the lattice of all ideals of Pre \(A^*\)-algebra \(A\). Then \(I \rightarrow \beta_1\) is homomorphism of the lattice \(\text{F}(A)\) into the lattice \(\text{Con}(A)\) of all congruences on \(A\).
Proof: From 7.2.5. Theorem (2 and 3) it follows that $I \to \beta_1$ is lattice homomorphism of $F(A)$ into the lattice $\text{Con}(A)$.

The above map $I \to \beta_1$ need not be an injection, in general. However, we have the following.

7.2.7. Theorem: For any Pre $A^*$-algebra $A$, the map $I \to \beta_1$ of $F(A)$ into $\text{Con}(A)$ is an injective if $A$ is a Boolean algebra.

Proof: Suppose that $A$ is a Boolean algebra and $I, J$ are ideals of $A$ such that $\beta_1 = \beta_J$.

Then for any $a \in I$ and $b \in J$, we have $a \lor (a \lor b) = a \lor b$

$$\Rightarrow (a \lor b, b) \in \beta_a \subseteq \beta_1 = \beta_J$$

and hence $x \lor (a \lor b) = x \lor b$, for some $x \in J$ which implies that

$$a = a \land (x \lor (a \lor b)) \quad \text{(Since $A$ is a Boolean algebra)}$$

$$= a \land (x \lor b)$$

$$\in J \quad \text{(Since $x \lor b \in J$, $J$ is an ideal)}$$

Therefore $I \subseteq J$ and similarly $J \subseteq I$ and hence $I = J$

Thus $I \to \beta_1$ is an injective.

7.3. Factor congruences

It is well known that an algebra $A$ can be factored as $A \cong B \times C$ (as a product of two algebras of the same type as $A$) if and only if there exist congruences $\alpha$ and $\beta$ on $A$ such that $\alpha \cap \beta = \Delta_A$ and $\alpha \circ \beta = A \times A$

$$A / \alpha \cong B$$

and

$$A / \beta \cong C.$$}

For this reason, a pair $(\alpha, \beta)$ of congruences is called a pair of factor congruences if $\alpha \cap \beta = \Delta_A$ and $\alpha \circ \beta = A \times A$.

In this section we shall discuss various properties of factor congruences on a Pre $A^*$-algebra.
7.3.1. Definition: Let $A$ be a Pre $A^*$-algebra and $\alpha \in \text{Con}(A)$. Then $\alpha$ is called factor congruence if there exist $\beta \in \text{Con}(A)$ such that $\alpha \cap \beta = \Delta_A$ and $\alpha \circ \beta = A \times A$. In this case $\beta$ is called direct complement of $\alpha$.

Now we specialise factor congruence on a Pre $A^*$-algebra with $1$, where $1$ is the identity for operator $\wedge$ in $A$ or equivalently $1^\sim = 0$ is the identity for the operator $\vee$.

7.3.2. Note: In this section we consider Pre $A^*$-algebra $A$ induced by a Boolean algebra to the following theorems.

7.3.3. Theorem: Let $A$ be a Pre $A^*$-algebra with $1$, $\theta$ is a factor congruence on $A$ and $\beta$ a direct complement of $\theta$. Then there exist unique $a \in A$ such that $\theta = \theta_a$ and $\beta = \theta_a \cdot (= \beta_a)$.

Proof: Let $1^\sim = 0$. Then $1$ and $0$ are identities for operators $\wedge$ and $\vee$ respectively in $A$.

We have $\theta \cap \beta = \Delta_A$ and $\theta \circ \beta = A \times A$.

Then clearly $\theta \circ \beta = \beta \circ \theta = A \times A$.

Since $(0, 1) \in A \times A = \theta \circ \beta$, there exist $a \in A$ such that $(0, a) \in \beta$ and $(a, 1) \in \theta$.

First we observe that $a$ is a unique element with the above property. If $b \in A$ also is such that $(0, b) \in \beta$ and $(b, 1) \in \theta$ then by the transitive and symmetry of $\beta$ and $\theta$ we get $(a, b) \in \theta \cap \beta = \Delta_A$, the diagonal of $A$, and hence $a = b$.

Thus $a$ is unique such that $(0, a) \in \beta$ and $(a, 1) \in \theta$.

Now we prove that $\theta = \theta_a$ and $\beta = \theta_a$.

For any $x, y \in A$ we have

$(0, a \land x) = (0 \land x, a \land x) \in \beta$ (since $(0, a) \in \beta$) and hence $(a \land x, a \land y) \in \beta$.

Now $(x, y) \in \theta \Rightarrow (a \land x, a \land y) \in \theta \cap \beta = \Delta_A$.
\[ a \land x = a \land y \]
\[ (x, y) \in \theta_a \]

Therefore \( \theta \subseteq \theta_a \).

On the other hand for any \( x \in A \), \( (a \land x, x) = (a \land x, 1 \land x) \in \theta \) (since \( (a, 1) \in \theta \))

Now \( (x, y) \in \theta_a \Rightarrow a \land x = a \land y \)

We have \( (a \land x, x) \in \theta, (a \land y, y) \in \theta \) and \( a \land x = a \land y \Rightarrow (x, y) \in \theta \)

Therefore \( \theta_a \subseteq \theta \).

Thus \( \theta = \theta_a \).

Also from \( (0, a) \in \beta \) and \( (a, 1) \in \theta \) we have that \( (0, a) \in \theta \) and \( (a, 1) \in \beta \) and by interchanging \( \theta \) and \( \beta \) in the above argument we get that

\[ \beta = \theta_a^- = \beta_a = \{ (x, y) \in A \times A / a \lor x = a \lor y \} \]

We have already proved that \( a \) is unique with this property.

**7.3.4. Theorem:** Let \( A \) be a Pre A*-algebra with 1 and \( a \in A \) then \( \theta_a \) is a factor congruence on \( A \) and \( \theta_a \) is a unique direct complement.

**Proof:** It is enough to prove that \( \theta_a \cap \theta_a^\perp = \Delta_A \) and \( \theta_a \circ \theta_a^\perp = A \times A \).

For any \( x, y \in A \)

Let \( (x, y) \in \theta_a \cap \theta_a^\perp \Rightarrow a \land x = a \land y \) and \( a \land x = a \land y \)

Now \( (a \lor a^\perp) \land x = (a \land x) \lor (a \land x) \)

\[ = (a \land y) \lor (a \land y) \]

\[ = (a \lor a^\perp) \land y \]

\[ \Rightarrow x = y \quad \text{(since } a \lor a^\perp = 1) \]

Therefore \( \theta_a \cap \theta_a^\perp = \Delta_A \).

Also for any \( x, y \in A \) such that \( z = (a \land x) \lor (a \land y) \)

Now \( a \land z = a \land ((a \land x) \lor (a \land y)) \)
\[= (a \land a \land x) \lor (a \land a \neg \land y)\]
\[= (a \land x) \lor (0 \land y)\]
\[= (a \land x) \lor 0\]
\[= a \land x\]

and similarly \(a \neg \land z = a \neg \land y\) and hence \((x, z) \in \theta_a\) and \((z, y) \in \theta_a\) so that
\((x, y) \in \theta_a \circ \theta_a\).

Thus \(\theta_a \circ \theta_a = A \times A\) and hence \(\theta_a \circ \theta_a = A \times A\).

Therefore \(\theta_a\) is a factor congruence on \(A\).

**7.3.5. Definition:** A congruence \(\beta\) on Pre A*-algebra \(A\) is called balanced if
\[(\beta \lor \theta) \cap (\beta \lor \theta^\sim) = \beta\] for any direct factor congruences \(\theta\) and any of its
direct comoplement \(\theta^\sim\) on \(A\).

**7.3.6. Theorem:** Let \(A\) be a Pre A*-algebra with 1 and \(\theta\) be a congruence on \(A\). Then \(\theta_x\) is a factor congruence on \(A\) if and only if \(\theta = \theta_x\) for some
\(x \in B(A)\).

**Proof:** Suppose that \(\theta = \theta_x\) for some \(x \in B(A)\).

Then \(x^\sim \in B(A)\) and \(\theta_x \cap \theta_x^\sim = \theta_{x \land x^\sim} = \theta_1 = \Delta_A\)

and \(\theta_x \circ \theta_x = \theta_{x \land x^\sim} = \theta_0 = A \times A\)

Thus \(\theta_x\) is a factor congruence on \(A\).

Conversely suppose that \(\theta\) is a factor congruence on \(A\).

Then there exist a congruence \(\beta\) on \(A\) such that \(\theta \cap \beta = \Delta_A\) and
\(\theta \circ \beta = A \times A\).

Now we show that \(\theta = \theta_x\)

Suppose that \((p, q) \in \theta\), then \(x \land p = x \land q\).

Since \((x, 1) \in \theta\) we have \((x \land p, 1 \land p), (x \land q, 1 \land q) \in \theta\) that is \((x \land p, p), (x \land q, q) \in \theta\) which imply that \((p, q) \in \theta\).
Hence $\theta_x \subseteq \theta$.

Suppose $(p, q) \in \theta$. Then $(x \land p, x \land q) \in \theta$.

Since $(0, x) \in \beta$ we have $(0 \land p, x \land p), (0 \land q, x \land q) \in \beta$ which is $(0, x \land p), (0, x \land q) \in \beta$ which implies that $(x \land p, x \land q) \in \beta$

Therefore $(x \land p, x \land q) \in \beta \cap \theta = \Delta_A$ and hence $x \land p = x \land q \Rightarrow (p, q) \in \theta$

Hence $\theta \subseteq \theta_x$

Thus $\theta = \theta_x$.

Hence $\theta_x$ is a factor congruence on $A$.

**7.3.7. Theorem:** Let $A$ be a Pre A*-algebra with 1. Then any factor congruence on $A$ is balanced.

**Proof:** Let $\beta$ is a factor congruences on Pre A*-algebra $A$ and $\theta$ another factor congruence on $A$ and $\theta^\sim$ a direct complement of $\theta$.

Then by 7.3.3. Theorem there exist $a, b \in A$ such that $\beta = \beta_a$ and $\theta = \theta_b = \beta_b$ and $\theta^\sim = \theta_b^\sim$.

Now $(\beta \lor \theta) \cap (\beta \lor \theta^\sim) = (\beta_a \lor \beta_b) \cap (\beta_a \lor \beta_b^\sim)$

$$= \beta_{a \lor b} \cap \beta_{a \lor b}$$

$$= \beta_{(a \lor b) \land (a \lor b)}$$

$$= \beta_{a \lor (b \land b)}$$

$$= \beta_{a \lor 0} = \beta_a = \beta$$

Thus $\beta$ is balanced.

Therefore the set of balanced congruence which admit a balanced complement is precisely the set $\mathbb{B}(A) = \{ \theta_x / x \in B(A) \}$ and hence $\mathbb{B}(A)$ is the Boolean centre of $A$. 

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7.3.8. Theorem: Let A be a Pre $A^*$-algebra and $a \in A$. Let $A_a = \{x \in A / a \land x = x\}$. Then $A_a$ is closed under the operations $\land$ and $\lor$. Also for any $x \in A_a$ define, $x^a = a \land x^\sim$. Then $(A_a, \land, \lor, a)$ is a Pre $A^*$-algebra with 1 (here $a$ is itself is the identity for $\land$ in $A_a$; that is 1 in $A_a$).

Proof: Let $x, y \in A_a$. Then $a \land x = x$ and $a \land y = y$.

Now $a \land (x \land y) = (a \land x) \land y = x \land y \Rightarrow x \land y \in A_a$

Also $a \land (x \lor y) = (a \land x) \lor (a \land y) = x \lor y \Rightarrow x \lor y \in A_a$

Therefore $A_a$ is closed under the operation $\land$ and $\lor$.

$a \land x^a = a \land (a \land x\sim) = a \land x = x^a \Rightarrow x^a \in A_a$

Thus $A_a$ is closed under $a$.

Now for any $x, y, z \in A_a$

(1) $x^{2a} = (a \land x\sim)^a = a \land (a \land x\sim) = a \land (a \lor x) = a \land x = x$

(2) $x \land x = (a \land x) \land (a \land x) = a \land x = x$

(3) $x \land y = (a \land x) \land (a \land y) = (a \land x) \land (a \land y) = y \land x$

(4) $(x \land y)^a = a \land (x \land y)^\sim = a \land (x\sim \lor y\sim) = (a \land x\sim) \lor (a \land y\sim) = x^a \lor y^a$

(5) $x \land (y \land z) = (a \land x) \land ((a \land y) \land (a \land z))$

$\Rightarrow a \land \{x \land (y \land z)\}$

$\Rightarrow a \land \{(x \land y) \land z\} \text{ (since } x, y, z \in A)$

$\Rightarrow (x \land y) \land z$

(6) $x \land (y \lor z) = (a \land x) \land ((a \land y) \lor (a \land z))$

$= \{(a \land x) \land (a \land y)\} \lor \{(a \land x) \land (a \land z)\}$

$= \{a \land (x \land y)\} \lor \{(a \land (x \land z))\}$

$= (x \land y) \lor (x \land z)$
(7) $x \land (x^a \lor y) = x \land \{ (a \land x^-) \lor y \}$

$$= \{ x \land (a \land x^-) \} \lor (x \land y)$$

$$= (x \land x^-) \lor (x \land y) \quad \text{(since } a \land x = x \text{)}$$

$$= x \land (x^- \lor y)$$

$$= x \land y$$

Finally, $x \in A_a$ implies that $a \land x = x = x \land a$. Thus $(A_a, \land, \lor, ^a)$ is a Pre $A^*$-algebra with $a$ as identity for $\land$.

**7.3.9. Theorem:** Let $\theta$ be a congruence on $A$. Then $\theta \cap (A_a \times A_a)$ is a congruence on $A_a$, for each $a \in A$.

**Proof:** Suppose $\theta$ be a congruence on $A$, and $a \in A$.

Let $x \in A_a$ we have that $(x, x) \in A_a \times A_a$

Since $\theta$ be congruence we have $(x, x) \in \theta \cap (A_a \times A_a)$

Therefore the result is reflexive.

Let $(x, y) \in \theta \cap (A_a \times A_a)$

Then $(x, y) \in \theta$ and $(x, y) \in A_a \times A_a$

$\Rightarrow (y, x) \in \theta$ and $(y, x) \in A_a \times A_a$

$\Rightarrow (y, x) \in \theta \cap (A_a \times A_a)$

The result is Symmetric.

Let $(x, y), (y, z) \in \theta \cap (A_a \times A_a)$

$\Rightarrow (x, y), (y, z) \in \theta$ and $(x, y), (y, z) \in A_a \times A_a$

$\Rightarrow (x, z) \in \theta$ and $(x, z) \in A_a \times A_a$

$\Rightarrow (x, z) \in \theta \cap (A_a \times A_a)$

The result is Transitive.

Hence the relation is an equivalence relation.

Let $(x, y), (z, t) \in \theta \cap (A_a \times A_a)$
Since $x, y, z, t \in A_a$, we have
\[ a \land x \land z = x \land z \Rightarrow x \land z \in A_a \]
\[ a \land y \land t = y \land t \Rightarrow y \land t \in A_a \]
\[ \Rightarrow (x \land z, y \land t) \in A_a \times A_a \]

Since $\theta$ be a congruence we have $(x \land z, y \land t) \in \theta \cap (A_a \times A_a)$

Now $(x, y) \in \theta \Rightarrow (x, y) \in \theta$
\[ \Rightarrow (a \land x, a \land y) \in \theta \cap (A_a \times A_a) \]
\[ \Rightarrow (x^a, y^a) \in \theta \cap (A_a \times A_a) \]

Therefore $\theta \cap (A_a \times A_a)$ is compatible (closed) with the binary operation $\land$ and unary operation $^a$ on $A_a$.

Let $(x, y), (z, t) \in \theta \cap (A_a \times A_a)$

Since $x, y, z, t \in A_a$, we have
\[ a \land (x \lor z) = (a \land x) \lor (a \land y) = x \lor z \Rightarrow x \lor z \in A_a \]
\[ a \land (y \lor t) = (a \land y) \lor (a \land t) = y \lor t \Rightarrow y \lor t \in A_a \]
\[ \Rightarrow (x \lor z, y \lor t) \in A_a \times A_a \]

Therefore $\theta \cap (A_a \times A_a)$ is compatible with $\lor$ also.

Thus $\theta \cap (A_a \times A_a)$ is a congruence relation on $A_a$.

7.3.10. Theorem: Let $\theta$ be a factor congruence on a Pre $A^*$-algebra $A$. Then $\theta \cap (A_a \times A_a)$ is a factor congruence on $A_a$.

Proof: Since $\theta$ be a factor congruence on $A$ there is a congruence $\theta^\sim$ on $A$ such that $\theta \cap \theta^\sim = \Delta_A$ and $\theta \circ \theta^\sim = A \times A$.

Consider $[ \theta \cap (A_a \times A_a) ] \cap [ \theta^\sim \cap (A_a \times A_a) ] = (\theta \cap \theta^\sim) \cap (A_a \times A_a)$

\[ = \Delta_A \cap (A_a \times A_a) \]
\[ \Delta_{A_a}, \text{ the diagonal on } A_a \]

Observe that every element in \( A_a \) is the form \( a \wedge x \) for some \( x \in A \).

Now, let \( (a \wedge x, a \wedge y) \in A_a \times A_a \). Then \( (a \wedge x, a \wedge y) \in A \times A = \theta \circ \theta^\sim \) which implies that there exist \( z \in A \) such that \( (a \wedge x, z) \in \theta \) and \( (z, a \wedge y) \in \theta^\sim \).

Now \( (a \wedge x, a \wedge z) \in \theta \) and \( (a \wedge z, a \wedge y) \in \theta^\sim \) and \( a \wedge z \in A_a \).

Therefore \( (a \wedge x, a \wedge z) \in \theta \cap (A_a \times A_a) \) and \( (a \wedge z, a \wedge y) \in \theta^\sim \cap (A_a \times A_a) \) and hence \( (a \wedge x, a \wedge y) \in [\theta^\sim \cap (A_a \times A_a)] \circ [\theta \cap (A_a \times A_a)] \)

Therefore \( [\theta \cap (A_a \times A_a)] \circ [\theta^\sim \cap (A_a \times A_a)] = A_a \times A_a \)

Therefore \( \theta \cap (A_a \times A_a) \) is a factor congruence on \( A_a \) and \( \theta^\sim \cap (A_a \times A_a) \) is a direct complement of \( \theta \cap (A_a \times A_a) \).