CHAPTER-VIII

SIMULTANEOUS CONFIDENCE BOUNDS FOR ALL RATIOS TO THE BEST : ASYMPTOTIC CASE

8.1 : INTRODUCTION

In chapters V and VI we have proposed classes of selection procedures to select a subset of the k populations, differing in the scale parameters, which includes the best population. The population associated with the smallest scale parameter was labelled as the best. These selection procedures are useful in agriculture and animal husbandry when the problem of interest is the identification of the best one or good ones of the k populations, differing only in scale parameters, where the best population is defined to be the one which has the smallest scale parameter and good ones are those within the small distances from the best. For example, in agriculture the small and marginal farmers always prefer the variety of a crop which is more consistent among those varieties having the same average yield. In animal husbandry, among different breeding methods with same average, one would prefer the most consistent breeding method.

Chen and Vanichbuncha (1989) used Hsu's (1981b) approach to derive the simultaneous confidence bounds for all distances from the largest location parameter of exponential distributions.
They pointed out that the proposed bounds are non-negative and hence can be used to assess the goodness of the populations from the best by ordering these simultaneous confidence bounds: a smaller interval corresponds to a good population. Thus, a reasonable rule to select a single best population is to select the one with the smallest confidence bound. Motivated with the fact that the goodness of populations from the best can be assessed by ordering the non-negative simultaneous confidence bounds, we propose simultaneous confidence bounds for all ratios to the best, using two-sample linear rank statistics, on the lines similar to Hsu's (1981b) approach. These simultaneous confidence bounds are non-negative and hence the ordering of these bounds reflects the ordering of the populations in terms of their scale parameters: smallest confidence interval corresponds to the population with smallest scale parameter.

Let \( \Pi_1, \ldots, \Pi_k \) be \( k \) independent populations differing in scale parameters such that the observations from \( \Pi_i \) have absolutely continuous cumulative distribution function (cdf)

\[
F_{\theta_i}(x) = F\left(-\frac{x}{\theta_i}\right) = H(x), \text{ say,}
\]

where \( H(.) \) is any continuous distribution function.

Let \( X = (X_{i1}, \ldots, X_{in}) \) be a random vector of observations from \( \Pi_i \) and suppose \( \Pi_{(i)} \) denotes the population associated with \( \theta_{[i]} \), the \( i \)th smallest of \( \theta_1 \)'s, \( i = 1, \ldots, k \). Throughout we assume that there is no prior knowledge about which of the \( \Pi_1, \ldots, \Pi_k \)
is \( \Pi(i) \), \( i=1,\ldots,k \) and \( \theta_1,\ldots,\theta_k \) are unknown. Call the population \( \Pi(1) \) as the best population. In case, more than one populations have \( \theta \)-values which are tied with the smallest, then exactly one of these tied populations is defined to be the best population according to some fixed rule.

In section 8.2 of this chapter we have formulated the problem and the proposed simultaneous confidence bounds are given in section 8.3. In section 8.4 we have shown that the proposed bounds can approximately be implemented with the help of existing tables and a numerical example is provided.

8.2 : CONFIDENCE BOUNDS FOR ALL RATIOS TO THE BEST

We shall find a set of simultaneous confidence intervals, 
\[ I_i = [D'_i, 1] \quad (0 \leq D'_i \leq 1) \]
for the ratios \( \frac{\theta_{[1]}}{\theta_i} \), \( i=1,\ldots,k \), such that the confidence interval \( I_i \) covers \( \frac{\theta_{[1]}}{\theta_i} \) for all \( i \) with probability at least \( P^* \), i.e.,
\[ P\left( \frac{\theta_{[1]}}{\theta_i} \in I_i, \ i=1,\ldots,k \right) \geq P^* \]
for some specified value of \( P^* (1 < P^* < 1) \).

Let \( R_{i/\beta}^{(j)}(\Delta) \) denote the rank of \( X_{\beta \Delta}^{(j)}/\Delta \) in the combined sample \( X_{i1}',\ldots,X_{in}',X_{j1}/\Delta,\ldots,X_{jn}/\Delta \), where \( \Delta = \theta_j/\theta_i > 0 \).

For \( 1 \leq i,j \leq k, i \neq j \), let
\[ S_{i}^{(j)}(\Delta) = \frac{(2/n)}{\sum_{\beta=1}^{n} a_{2n}(R_{i}^{(j)}(\Delta))} - n \cdot a_{2n}, \]

where \( a_{m}(\beta) \) are some given scores satisfying the following assumptions:

(i) For any positive integer \( m \), the scores \( a_{m}(1), \ldots, a_{m}(m) \) are generated by a non-constant square integrable function \( J(u)(0<u<1) \), which is nonincreasing (nondecreasing) for \( 0<u \leq 1/2 \) and nondecreasing (nonincreasing) for \( 1/2 \leq u<1 \), in either of the following two ways:

\[ a_{m}(\beta) = J(\beta/(m+1)) \quad \text{or} \quad a_{m}(\beta) = E[J(U_{m}^{(1)})], \]

where \( \beta=1, \ldots, m \) and \( U_{m}^{(1)} \leq \ldots \leq U_{m}^{(m)} \) denote the order statistics corresponding to a random sample of size \( m \) from the uniform distribution on the interval \((0,1)\).

(ii) \( J(u)(0<u<1) \) called the limiting score function is such that

\[ \sigma_{m}^{2} = \frac{1}{m} \sum_{\beta=1}^{m} (a_{m}(\beta)-\bar{a}_{m})^{2}/(m-1) \rightarrow \sigma^{2} = \int_{0}^{1} (J(u)-\bar{J})^{2} du < \infty, \]

where \( \bar{a}_{m} = (1/m) \sum_{\beta=1}^{m} a_{m}(\beta) \) and \( \bar{J} = \int_{0}^{1} J(u)du. \)

It may be noted that \( J(u)=(u-1/2)^{2} \) and \( J(u)=|u-1/2|(0<u<1) \) correspond to the Mood(1954) scores, i.e., \( a_{m}(\beta)=(\beta/(m+1)-1/2)^{2} \) and the Ansari-Bradley(1960) scores, i.e., \( a_{m}(\beta)=|\beta/(m+1)-1/2| \) respectively.

Furthermore for the asymptotic results we assume in the underlying scale model that \( F \) has a finite variance, the quantity \( I(f) = \int_{-\infty}^{\infty} [-1-x f'(x)/f(x)]^{2} dx \) is finite and the
limiting score function $J$ has a bounded second derivative.

Let $S_i^j = S_i^j(1)$ and for a given probability $P^*(k^{-1} < P^* < 1)$, the nonnegative constants $d_i^j(n, P^*)$ be such that

$$P_0[S_i^j \leq d_i^j(n, P^*) \forall i \neq j, i = 1, \ldots, k] \geq P^*,$$

where $P_0$ indicates that the probability is computed under parametric configuration $\theta_1 = \ldots = \theta_k$ and $n=(n,...,n)^t$.

### 8.3: Proposed Confidence Intervals for All Ratios to the Best

For $1 \leq i, j \leq k$, $i \neq j$, let

$$D_i^j = \sup\{\Delta: S_i^j(\Delta) > d_i^j(n, P^*)\},$$

and for $i=1, \ldots, k$, let

$$D_i' = \min\{\min_{j \neq i} D_i^j, 1\}.$$

Theorem below gives simultaneous confidence bounds for all ratios $\frac{\theta[1]}{\theta_i}$, $i=1, \ldots, k$.

**Theorem 8.1:** A set of $100P^\%$ simultaneous confidence intervals for $\frac{\theta[1]}{\theta_1}, \ldots, \frac{\theta[1]}{\theta_k}$ is given by

$$[D'_1, 1], \ldots, [D'_k, 1]$$

**Proof:**

$$P^* \leq P\left\{S_i^{(1)}\left(\frac{\theta[1]}{\theta_i}\right) \leq d_i^{(1)}(n, P^*), i=1, \ldots, k, i \neq (1)\right\}$$

$$\leq P\left\{D_i^{(1)} \leq \frac{\theta[1]}{\theta_i}, i=1, \ldots, k, i \neq (1)\right\}.$$
Thus the Theorem follows since $0 < \frac{\theta [1]}{\theta_i} \leq 1$ for $i = 1, \ldots, k$.

8.4 : DETERMINATION OF THE CONSTANTS $d_{ij}^{d}(D, P^{*})$

Let $S = (S_1, \ldots, S_k, \ldots, S_{k-1})^t$ be a vector of $k(k-1)$ components. Gill and Mehta (1989) have shown that under the configuration $\theta_1 = \ldots = \theta_k$ and as $n \to \infty$ the random vector

$$(N^{1/2}/\sigma_N) S = (N^{1/2}/\sigma_N) (S_2, \ldots, S_k, \ldots, S_{k-1})^t$$

is asymptotically normally distributed with mean vector $0$ and covariances as

$$(N/c_N^2) E[S_i^j S_h^e] = \begin{cases} 2/p & \text{for } i = h, j = e \\ 1/p & \text{for } i = h, j \neq e (i \neq h, j = e) \\ -1/p & \text{for } i = e, j \neq h (i \neq e, j = h) \\ 0 & \text{otherwise,} \end{cases} ...(8.4.1)$$

where $N = nk$, $p = 1/k$, $\sigma_m^2 = \sum_{\beta=1}^{m} (a_m(\beta) - \bar{a}_m)^2 / (m-1) \to \sigma^2 = \int 1(J(u)J)du$ and 

$$\bar{a}_m = (1/m) \sum_{\beta=1}^{m} a_m(\beta) \to J.$$ 

Let

$$Z_i^j = (n/2\sigma_N^2)^{1/2} S_i^j, i \neq j, i, j = 1, \ldots, k.$$ 

Then it follows from (8.4.1) that the asymptotic distribution of the random vector $Z_j = (Z_1^j, \ldots, Z_{j-1}^j, Z_{j+1}^j, \ldots, Z_k^j)$ is multivariate
normal of equally correlated standard normal variables (the limiting value of this correlation is \( r = 1/2 \)). Now the constants \( d^j_i(n, P^*) \), \( i \neq j \), \( i, j = 1, \ldots, k \) are determined such that
\[
P^* = P_0\left[ z^j_i \leq e' \ \forall \ i \neq j, \ i = 1, \ldots, k \right]
\]
\[
= P_0\left[ \max_{i \neq j} z^j_i \leq e' \ \forall \ i \neq j \right].
\]
Here
\[
e' = \left( \frac{n}{2 \sigma^2_N} \right)^{1/2} d^j_i(n, P^*). \tag{8.4.2}
\]

Now we can use Table-1 of Gupta, Nagel, and Panchapakesan (1973) to read the constant \( e' \) (reading \( N, \rho \) and \( 1 - \alpha \)) in those tables as \( k - 1, r \) and \( P^* \) respectively and thereby get the values of the constants \( d^j_i(n, P^*) \).

Example 8.4.1: Let us now determine confidence bounds corresponding to Ansari-Bradley (1960) statistics.

Let \( W_{2n}(X_i, X_j) \) be Ansari-Bradley statistic for testing \( \theta_i = \theta_j \) against \( \theta_j < \theta_i \). Bhattacharya (1977) has defined an equivalent version \( h_{2n}(X_i, X_j) \) of \( W_{2n}(X_i, X_j) \) and proposed the estimator \( T^{(2n)}_{ji} \) of \( \theta_j / \theta_i = \Delta \), on the lines similar to those of Hodges and Lehmann (1963). The estimator \( T^{(2n)}_{ji} \) of \( \theta_j / \theta_i = \Delta \) is obtained as follows:

Subtract the combined sample median of the \( i \)th and \( j \)th sample observations from the observations \( x_{i1}, \ldots, x_{in} \), \( x_{j1}, \ldots, x_{jn} \); \( i \neq j ; i = 1, \ldots, k ; j = 1, \ldots, k \). Let \( X'_{ji} (\beta = 1, \ldots, n) \) and \( X'_{i\alpha} (\alpha = 1, \ldots, n) \) be the observations from the \( j \)th and \( i \)th population samples, after subtracting the combined sample
median. For the adjusted observations \( X'_{j\beta} \) \((\beta=1,\ldots,n)\) and \( X'_{i\alpha} \) \((\alpha=1,\ldots,n)\) Bhattacharya(1977) has defined the concept of relevant pairs to obtain the estimator \( T_{ji}^{(2n)} \) of \( \theta_j/\theta_i \) given below:

**Definition 8.4.1:** A relevant pair is a pair \((X'_{j\beta}, X'_{i\alpha})\), where \( X'_{j\beta} \) and \( X'_{i\alpha} \) are both negative or both positive.

The estimator \( T_{ji}^{(2n)} \) of \( \theta_j/\theta_i = \Delta \) is then

\[
T_{ji}^{(2n)} = \hat{\Delta} = \text{median} (X'_{j\beta}/X'_{i\alpha}),
\]

where \( X'_{j\beta} \) and \( X'_{i\alpha} \) in \((X'_{j\beta}/X'_{i\alpha})\) form a relevant pair, that is, \( \hat{\Delta} \) is the median of the ratios \((X'_{j\beta}/X'_{i\alpha})\).

Let \( \mu_{2n} \) be the mean of \( h_{2n}(X'_{i},X'_{j}) \) under \( \theta_i = \theta_j \); \( X_{i\alpha} (\alpha=1,\ldots,n) \) have distribution function \( H(x) \); and \( X_{jt} (t=1,\ldots,n) \) have distribution function \( H(x(1+c_j N^{-1/2})) \).

Bhattacharya(1977) has shown that the random variables

\[
(2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1) - c_j N^{-1/2}]
\]

and

\[
v_j^i = (2/n)[h_{2n}(X'_{i},X'_{j}) - \mu_{2n}]
\]

are asymptotically equal in probability under \( \theta_i = \theta_j \). Let

\[
w_j^i = (2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1)]
\]

and \( S_j^i \) be the statistics obtained from \( v_j^i \) by replacing \( h_{2n}(X'_{i},X'_{j}) \) and \( \mu_{2n} \) by their respective scores when \( \theta_i = \theta_j \). In this case \( h_{2n}(X'_{i},X'_{j}) \) is an equivalent version of Ansari-Bradley statistic, thus the scores corresponding to the statistic \( h_{2n}(X'_{i},X'_{j}) \) will be \(|\beta/(2n+1)-1/2|\) and the limiting score

\[
(2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1) - c_j N^{-1/2}]
\]

and

\[
v_j^i = (2/n)[h_{2n}(X'_{i},X'_{j}) - \mu_{2n}]
\]

are asymptotically equal in probability under \( \theta_i = \theta_j \). Let

\[
w_j^i = (2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1)]
\]

and \( S_j^i \) be the statistics obtained from \( v_j^i \) by replacing \( h_{2n}(X'_{i},X'_{j}) \) and \( \mu_{2n} \) by their respective scores when \( \theta_i = \theta_j \). In this case \( h_{2n}(X'_{i},X'_{j}) \) is an equivalent version of Ansari-Bradley statistic, thus the scores corresponding to the statistic \( h_{2n}(X'_{i},X'_{j}) \) will be \(|\beta/(2n+1)-1/2|\) and the limiting score

\[
(2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1) - c_j N^{-1/2}]
\]

and

\[
v_j^i = (2/n)[h_{2n}(X'_{i},X'_{j}) - \mu_{2n}]
\]

are asymptotically equal in probability under \( \theta_i = \theta_j \). Let

\[
w_j^i = (2/n)[(2n)^{1/2}(T_{ji}^{(2n)}-1)]
\]

and \( S_j^i \) be the statistics obtained from \( v_j^i \) by replacing \( h_{2n}(X'_{i},X'_{j}) \) and \( \mu_{2n} \) by their respective scores when \( \theta_i = \theta_j \). In this case \( h_{2n}(X'_{i},X'_{j}) \) is an equivalent version of Ansari-Bradley statistic, thus the scores corresponding to the statistic \( h_{2n}(X'_{i},X'_{j}) \) will be \(|\beta/(2n+1)-1/2|\) and the limiting score
function as $J(u) = |u-1/2|$. Thus it follows that under $\theta_i = \theta_j$,

$$P[ w_i^j \leq N^{-1/2} d_i^j(n, P^*) ] \approx P[ S_i^j \leq d_i^j(n, P^*) ],$$

where $d_i^j(n, P^*) = (2nc_j + n^{-1/2} d_i^j(n, P^*) \sqrt{n}) / n^2$

$$\approx N^{-1/2} d_i^j(n, P^*). \quad \ldots(8.4.3)$$

For the relevant pairs of the type $(\chi_{j\beta}', \chi_{i\alpha}')$ suppose there are $m'$ ratios of the form $(\chi_{j\beta}' / \chi_{i\alpha}')$, $1 \leq i, j \leq k$, $i \neq j$ and let $Q_i[1] \leq \ldots \leq Q_i[m']$ denote the ordered values of these ratios. Let

$$t_i^j(n, P^*) = [ d_i^j(n, P^*) + 1 ]^*,$$

where $[x]^*$ denotes the smallest integer greater than or equal to $x$. By the definition of $D_i^j$, given in section 8.3, it is easy to see that

$$D_i^j = Q_i[t_i^j(n, P^*)].$$

Thus $D_i^j = \min(\min_{j \neq i} Q_i[t_i^j(n, P^*)], 1)$, $i = 1, \ldots, k \quad \ldots(8.4.4)$

are the confidence bounds of the simultaneous confidence intervals for the ratios $\frac{\theta_i[1]}{\theta_i}$, $i = 1, \ldots, k$, based on Ansary-Bradley statistic.

As an illustration let us have an example.

**Example 8.4.2**: In order to determine confidence bounds for all ratios to the best we consider the following data:
Population Observations (arranged in ascending order)

<table>
<thead>
<tr>
<th>Population</th>
<th>-0.80</th>
<th>-0.25</th>
<th>0.19</th>
<th>0.30</th>
<th>0.42</th>
<th>0.63</th>
<th>0.86</th>
<th>0.94</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi_1 )</td>
<td>-3.40</td>
<td>-2.94</td>
<td>0.38</td>
<td>0.96</td>
<td>1.14</td>
<td>1.46</td>
<td>1.68</td>
<td>1.80</td>
</tr>
<tr>
<td>( \Pi_2 )</td>
<td>-4.20</td>
<td>-1.80</td>
<td>-1.17</td>
<td>-0.93</td>
<td>-0.42</td>
<td>1.77</td>
<td>3.24</td>
<td>3.51</td>
</tr>
</tbody>
</table>

Suppose \( P^* = 0.95 \). In this example \( n=8 \) and \( k=3 \). From table-1 of Gupta, Nagel and Panchapakesan (1973), it follows that \( e' = 1.9163 \) (here \( N=k-1=2 \), \( \rho=0.5 \), \( \alpha=1-P^* = 0.05 \)). Thus from (8.4.2) and (8.4.3) we have

\[
d_j^*(n,P^*) = N^{1/2} d_j^*(n,P^*)
\]

\[
= N^{1/2} [1.9163 \sigma_N (2/n)^{1/2}]
\]

\[
\approx (24)^{1/2} [1.9163 (1/48)^{1/2} (2/n)^{1/2}]
\]

\[
= 0.6774 \forall i \neq j, i,j=1,2,3.
\]

Therefore, \( t_j^*(n,P^*) = 2 \forall i \neq j, i,j=1,2,3 \).

The combined ordered sample from populations \( \Pi_1 \) and \( \Pi_2 \) becomes

\[
X_2 \quad X_2 \quad X_1 \quad X_1 \quad X_1 \quad X_2 \quad X_1 \quad X_2
\]

\[-3.40 \quad -2.94 \quad -0.80 \quad -0.25 \quad 0.19 \quad 0.30 \quad 0.38 \quad 0.42\]

\[
X_1 \quad X_1 \quad X_1 \quad X_2 \quad X_2 \quad X_2 \quad X_2 \quad X_2
\]

\[0.63 \quad 0.86 \quad 0.94 \quad 0.96 \quad 1.14 \quad 1.46 \quad 1.68 \quad 1.80\]

After subtracting the median \( M_1 \) of combined samples, \( M_1 = \frac{0.42 + 0.63}{2} = 0.525 \), the combined ordered sample becomes

\[
X_2 \quad X_2 \quad X_1 \quad X_1 \quad X_1 \quad X_2 \quad X_1 \quad X_2
\]

\[-3.925 \quad -3.465 \quad -1.325 \quad -0.775 \quad -0.335 \quad -0.225 \quad -0.145 \quad -0.105\]

\[
X_1 \quad X_1 \quad X_1 \quad X_2 \quad X_2 \quad X_2 \quad X_2 \quad X_2
\]

\[0.105 \quad 0.335 \quad 0.415 \quad 0.435 \quad 0.615 \quad 0.935 \quad 1.155 \quad 1.275\]

There will be 30 ratios \( (x_2/x_1) \) of relevant pairs as follows:
\((-3.925/-1.325)\) \((-3.925/-0.775)\) \((-3.925/-0.335)\) \((-3.925/-0.225)\) \\
\((-3.925/0.105)\) \((3.405/-1.325)\) \((3.465/0.775)\) \((3.465/-0.335)\) \((-3.465/-0.225)\) \((-3.465/-0.105)\) \((-0.145/-1.325)\) \((-0.145/-0.775)\) \((-0.145/-0.335)\) \((-0.145/-0.225)\) \((-0.145/-0.105)\) \((0.435/0.105)\) \\
\((0.435/0.335)\) \((0.435/0.415)\) \((0.615/0.105)\) \((0.615/0.335)\) \((0.615/0.415)\) \((0.935/0.105)\) \((0.935/0.335)\) \((0.935/0.415)\) \\
\((1.155/0.105)\) \((1.155/0.335)\) \((1.155/0.415)\) \((1.275/0.105)\) \((1.275/0.335)\) \((1.275/0.415)\).

These ratios when arranged in the increasing order will be

0.109, 0.187, 0.432, 0.644 and so on.

Similarly the ratios \(x_1/x_2\) of relevant pairs are

\((-1.325/-3.925)\) \((-1.325/-3.465)\) \((-1.325/-0.145)\) \\
\((-0.755/-3.925)\) \((-0.755/-3.465)\) \((-0.755/-0.145)\) \\
\((-0.335/-3.925)\) \((-0.335/-3.465)\) \((-0.335/-0.145)\) \\
\((-0.225/-3.925)\) \((-0.225/-3.465)\) \((-0.225/-0.145)\) \\
\((-0.105/-3.925)\) \((-0.105/-3.465)\) \((-0.105/-0.145)\) \\
\((0.105/0.435)\) \((0.105/0.615)\) \((0.105/0.935)\) \((0.105/1.155)\) \((0.105/1.275)\)

These ratios when arranged in increasing order will be

0.026, 0.030 and so on.

Similarly the ratios \(x_3/x_1\) of relevant pairs in increasing order will be

0.636, 0.956 and so on (Total number = 44).

Similarly the ratios \(x_1/x_3\) of relevant pairs in increasing
order will be 0.012, 0.010 and so on (Total number = 44).

Similarly the ratios \(\frac{x_3}{x_2}\) of relevant pairs in increasing order will be 0.267, 0.302 and so on (Total number = 30).

Similarly the ratios \(\frac{x_2}{x_3}\) of relevant pairs in increasing order will be 0.059, 0.102 and so on (Total number = 30).

Thus for the above data

\[
D'_1 = \min\left(\min Q^j_{1[2]}, 1\right) = \min\left(\min \left(Q^2_{1[2]}, Q^3_{1[2]}\right), 1\right) = \min\left(\min \left(0.187, 0.956\right), 1\right) = 0.187,
\]

\[
D'_2 = \min\left(\min Q^j_{2[2]}, 1\right) = \min\left(\min \left(Q^1_{2[2]}, Q^3_{2[2]}\right), 1\right) = \min\left(\min \left(0.030, 0.302\right), 1\right) = 0.030,
\]

\[
D'_3 = \min\left(\min Q^j_{3[2]}, 1\right) = \min\left(\min \left(Q^1_{3[2]}, Q^2_{3[2]}\right), 1\right) = \min\left(\min \left(0.016, 0.102\right), 1\right) = 0.016,
\]
and hence a set of 95% simultaneous confidence intervals for \( \frac{\theta_1}{\theta_2} \) and \( \frac{\theta_1}{\theta_3} \) is \([0.187, 1]\), \([0.030, 1]\) and \([0.016, 1]\).

Remark 8.4.1: In the above data random samples of size 8 were generated through computer from each of the three normal populations \( \pi_1, \pi_2 \) and \( \pi_3 \) with means zero and variances 1, 4 and 9 respectively. We see that the smallest confidence interval, \([0.187, 1]\), corresponds to the best population, \( \pi_1 \), second smallest confidence interval, \([0.030, 1]\), corresponds to the second best population, \( \pi_2 \), and the largest confidence interval, \([0.016, 1]\), corresponds to the population \( \pi_3 \), with the largest variance.