Chapter 5

Longitudinal wave response of a chiral slab interposed between micropolar solid half-spaces

5.1 Introduction

Elphinstone and Lakhtakia (1994a, 94b) have investigated the response of plane waves incident on a chiral solid slab sandwiched between two elastic media. Elphinstone and Lakhtakia (1994a) have pointed out that when a linearly polarized plane wave propagating through an isotropic achiral (elastic) medium, encounters an interface with a chiral medium, then the refracted plane waves are either longitudinal or transverse circularly polarized waves. Recently, Hsia and Su (2008) have investigated the phenomena of reflection and transmission of incidence of a longitudinal plane wave at the elastic - microporous - elastic interfaces. In this chapter, we consider a chiral slab of uniform thickness, interposed between two different micropolar elastic solid half-spaces. A plane longitudinal displacement wave propagating through a micropolar elastic solid half-space is made to strike obliquely on the chiral slab. Reflection and transmission coefficients are obtained separately, using two distinct sets of boundary conditions. The effect of chirality parameter and the thickness of the chiral slab have been separately noticed on these amplitude ratios, in a specific model. Comparison of modulus values of the respective amplitude ratios corresponding to the two possible sets of boundary conditions, is also shown graphically. In a special case, the results of Hsia and Yang

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5.2 Formulation of the problem

(1999) have been recovered from the present problem.

5.2 Formulation of the problem

Consider a three layered model consisting of a chiral solid slab of finite thickness \( D \), interposed between two distinct semi-infinite micropolar elastic solids. Introducing the Cartesian co-ordinate system \((x, y, z)\), such that \(x-\) and \(y-\) axes are on horizontal plane and \(z-\) axes is pointing vertically downward. Let the intermediate layer occupying the region

\[ M = \{(x, z) : -\infty < x < \infty, 0 \leq z \leq D\} \]

be delineated by the planes \(z = 0\) and \(z = D\) and the two micropolar elastic solid half-spaces be occupying the regions

\[ M^{(1)} = \{(x, z) : -\infty < x < \infty, -\infty < z < 0\} \]

and

\[ M^{(2)} = \{(x, z) : -\infty < x < \infty, D < z < \infty\} \]

Denoting the elastic parameters and the density of the chiral slab \( M \) by \(\lambda, \mu, \alpha, \beta, \gamma, C_3, j, \rho\) and that of in the half-spaces \( M^{(l)} \) \((l = 1, 2)\) by the quantities \(\lambda^{(l)}, \mu^{(l)}, K^{(l)}, \alpha^{(l)}, \beta^{(l)}, \gamma^{(l)}, j^{(l)}, \rho^{(l)}\). The rough sketch of the problem considered is presented in Figure 5.1.

Following Eringen (1966), the force stress tensor \(t_{ij}^{(l)}\), the couple stress tensor \(m_{ij}^{(l)}\) and the equations of motion for micropolar elastic solid medium \( M^{(l)} \) \((l = 1, 2)\) in the absence of body force and body couple densities are given as

\[ t_{ij}^{(l)} = \lambda^{(l)} \delta_{ij} \delta_{ij} + \mu^{(l)} (u_{ij}^{(l)} + u_{ji}^{(l)}) + K^{(l)} (u_{ij}^{(l)} - c_{ijk} \phi_{k}^{(l)}), \quad (5.1) \]

\[ m_{ij}^{(l)} = \alpha^{(l)} \phi_{k}^{(l)} \delta_{ij} + \beta^{(l)} \phi_{ij}^{(l)} + \gamma^{(l)} \phi_{j}^{(l)} \phi_{j}^{(l)}, \quad (5.2) \]

\[ (c_1^{(l)2} + c_3^{(l)2}) \nabla \cdot \mathbf{u}^{(l)} - (c_2^{(l)2} + c_3^{(l)2}) \nabla \times \nabla \times \mathbf{u}^{(l)} + c_3^{(l)2} \nabla \times \phi^{(l)} = \ddot{u}^{(l)}, \quad (5.3) \]
\[ (c_4^{(i)})^2 + c_5^{(i)})^2 \nabla \cdot \phi^{(i)} - c_4^{(i)} \nabla \times \nabla \times \phi^{(i)} + \omega_0^{(i)} \nabla \times u^{(i)} - 2\omega_0^{(i)} \phi^{(i)} = \phi^{(i)}, \quad (5.4) \]

where

\[ c_1^{(i)} = \sqrt{(\lambda^{(i)} + 2\mu^{(i)})/\rho^{(i)}}, \quad c_2^{(i)} = \sqrt{\mu^{(i)}/\rho^{(i)}}, \quad c_3^{(i)} = \sqrt{K^{(i)}/\rho^{(i)}}, \]

\[ c_4^{(i)} = \sqrt{\gamma^{(i)}/\rho^{(i)}} j^{(i)}, \quad c_5^{(i)} = \sqrt{(\alpha^{(i)} + \beta^{(i)})/\rho^{(i)}} j^{(i)}, \quad \omega_0^{(i)} = \sqrt{K^{(i)}/\rho^{(i)}} j^{(i)} \]

and all other symbols have been defined earlier.

Introducing the scalar potentials \( q^{(i)} \) and \( \zeta^{(i)} \), the vector potentials \( U^{(i)} \) and \( \Pi^{(i)} \), through the Helmholtz decomposition of vector, as follows

\[ u^{(i)} = \nabla q^{(i)} + \nabla \times U^{(i)}, \quad \phi^{(i)} = \nabla \zeta^{(i)} + \nabla \times \Pi^{(i)}, \quad \nabla \cdot U^{(i)} = \nabla \cdot \Pi^{(i)} = 0, \]

and using these relations into equations (5.3) and (5.4), it has been shown by Parfitt and Eringen (1969) that there exist following four basic waves propagating with distinct
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(i) a longitudinal displacement wave propagating with phase speed $V^{(l)}_1$ given by

$$V^{(l)}_1 = \left[ \frac{a^2 + c^2_3}{c_1^2 + c_3^2} \right]^{1/2}.$$  

(ii) a longitudinal microrotational wave traveling with speed $V^{(l)}_2$ given by

$$V^{(l)}_2 = \left[ \frac{c_4^2 + c_5^2 + \frac{2\omega_0 c_2}{k(0)^2}}{c_1^2 + c_3^2} \right]^{1/2}.$$  

(iii) two sets of coupled transverse waves propagating with phase speeds $V^{(l)}_3$ and $V^{(l)}_4$, given by

$$V^{(l)}_3 = \left[ \frac{1}{2A} \left( B + \sqrt{B^2 - 4AC} \right) \right]^{1/2}$$

and

$$V^{(l)}_4 = \left[ \frac{1}{2A} \left( B - \sqrt{B^2 - 4AC} \right) \right]^{1/2},$$

where

$$A = 1 - \Omega^0, \quad B = c_4^2 + c_5^2 \left( 1 - \Omega^0 \right) + c_3^2 \left( 1 - \frac{\Omega^0}{2} \right),$$

$$C = c_4^2 \left( c_2^2 + c_3^2 \right) \quad \text{and} \quad \Omega^0 = \frac{2\omega_0}{\omega(0)^2}.$$  

The force stress tensor $t_{ij}$, the couple stress tensor $m_{ij}$ and the elastodynamic equations with vanishing body force and body couple densities, for a chiral solid $M$ mentioned already in the previous chapter and are reproduced as [Lakes and Benedict (1982), Nowacki (1986)]

$$t_{ij} = \lambda u_{i,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) + C_3 \phi_{j,i}, \quad (5.5)$$

$$m_{ij} = \alpha \phi_{k,k} \delta_{ij} + \beta \phi_{i,j} + \gamma \phi_{j,i} + C_3 u_{j,i} - C_4 e_{ijk} \phi_k, \quad (5.6)$$

$$c_1^2 \nabla \nabla \cdot \mathbf{u} - c_2^2 \nabla \times \nabla \times \mathbf{u} + c_3^2 \nabla \nabla \cdot \mathbf{\phi} - c_4^2 \nabla \nabla \times \mathbf{\phi} = -c_5^2 \nabla \nabla \times \mathbf{\phi} = \mathbf{\ddot{u}}, \quad (5.7)$$

$$\left( c_2^2 + c_3^2 \right) \nabla \nabla \cdot \mathbf{\phi} - \frac{c_4^2}{c_2} \nabla \times \nabla \times \mathbf{\phi} + 2c_5^2 \nabla \times \mathbf{\phi} + c_6^2 \nabla \nabla \cdot \mathbf{u} - c_2^2 \nabla \times \nabla \times \mathbf{u} = \mathbf{\ddot{\phi}}, \quad (5.8)$$
where
\[ c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad c_4^2 = \frac{\gamma}{\rho j}, \]
\[ c_3^2 = \frac{\alpha + \beta}{\rho j}, \quad c_6^2 = \frac{C_3}{\rho}, \quad c_7^2 = \frac{C_3}{j}, \quad c_8^2 = \frac{C_3}{\rho j}. \]
As earlier, decomposing the vectors \( \mathbf{u} \) and \( \phi \) by using the scalar potentials \( q \) and \( \xi \), the vector potentials \( \mathbf{U} \) and \( \Pi \), respectively, and inserting the resulting relations into equations (5.7) and (5.8), one can obtain the phonon dispersion relations [see (4.9) and (4.10) with unprimed symbols]. In an infinite chiral medium, Lakhtakia et al. (1988) have shown that there exist following six waves traveling with different phase speeds

(i) two sets of coupled longitudinal waves traveling with speeds \( V_1 \) and \( V_2 \) given by [see (4.11) with unprimed symbols]
\[ V_{1,2}^2 = \Gamma \pm \sqrt{\Gamma^2 - c_1^2(c_4^2 + c_7^2) + c_2^2c_3^2}, \quad \Gamma = \frac{1}{2}(c_1^2 + c_4^2 + c_7^2), \] (5.9)

(ii) four sets of coupled transverse waves traveling with speeds \( V_i \) (\( i = 3, 4, 5, 6 \)), where \( V_i^2 \) are the roots of the dispersion equation (4.10) (without prime). Out of these four distinct sets of coupled transverse waves, the two sets are coupled right circularly polarized (RCP) waves and the remaining two sets are the coupled left circularly polarized (LCP) waves.

5.3 Reflection and transmission of waves

Let a plane longitudinal displacement wave of amplitude \( A_0^{(1)} \) traveling through the micropolar elastic solid medium \( M^{(1)} \) be incident at the interface \( z = 0 \) making an angle \( \theta_0^{(1)} \) with the \( z \)-axis. A part of this incident energy will be reflected back (in the form of reflected waves) into the medium \( M^{(1)} \) and rest will be transmitted into the medium \( M \). Now, the waves associated with transmitted energy will proceed through the medium \( M \) to interact with the boundary \( z = D \), where again some part of this energy will be reflected and rest will be transmitted into the medium \( M^{(2)} \). The reflected energy further proceeds back to interact with the boundary \( z = 0 \) and the process will repeat.

To satisfy the boundary conditions at both the interfaces, i.e., at \( z = 0 \) and \( z = D \), we shall take the following reflected and refracted waves into consideration.
5.3. Reflection and transmission of waves

Reflected waves:
In medium \( M^{(1)} \): (i) a longitudinal displacement wave traveling with speed \( V_{1}^{(1)} \) and making an angle \( \theta_{1}^{(1)} \) with the normal,
(ii) two sets of coupled transverse waves propagating with speeds \( V_{3,4}^{(1)} \) and making angles \( \theta_{3,4}^{(1)} \) with the normal.
In medium \( M \): (i) two sets of coupled longitudinal waves traveling with speeds \( V_{1,2} \) and making angles \( \theta_{1,2} \) with the normal,
(ii) four sets of coupled transverse waves propagating with speeds \( V_{i} \) and making angles \( \theta_{i} \) \( (i = 3, 4, 5, 6) \) with the normal.

Refracted waves:
In medium \( M \): (i) two sets of coupled longitudinal waves traveling with speeds \( V_{1,2} \) and making angles \( \theta_{1,2} \) with the normal,
(ii) four sets of coupled transverse waves propagating with speeds \( V_{i} \) and making angles \( \theta_{i} \) \( (i = 3, 4, 5, 6) \) with the normal.
In medium \( M^{(2)} \): (i) a longitudinal displacement wave traveling with speed \( V_{1}^{(2)} \) and making an angle \( \theta_{1}^{(2)} \) with the normal,
(ii) two sets of coupled transverse waves propagating with speeds \( V_{3,4}^{(2)} \) and making angles \( \theta_{3,4}^{(2)} \) with the normal.

Since we have considered a two-dimensional problem in \( x - z \) plane, therefore, we take the following form of potentials

In the medium \( M^{(1)} \)

\[
q^{(1)} = A_{0}^{(1)} P_{0}^{+ (1)} + A_{1}^{(1)} P_{1}^{(-1)},
\]

\[
U^{(1)} = \sum_{p=3}^{4} A_{pp}^{(1)} \hat{e}_{p} P_{p}^{(-1)},
\]

\[
\Pi^{(1)} = \sum_{p=3}^{4} (B_{ps}^{(1)} \hat{e}_{s} + B_{ps}^{(1)} \hat{e}_{s}) P_{p}^{(-1)},
\]
In the sandwiched medium $M$

$$q = \sum_{p=1}^{2} (A_p P_p^+ + B_p P_p^-), \quad (5.13)$$

$$\xi = \sum_{p=1}^{2} D_p (A_p P_p^+ + B_p P_p^-), \quad (5.14)$$

$$U = \sum_{p=3}^{6} (A_p P_p^+ + B_p P_p^-)(\Delta_1^1 \hat{e}_x + \Delta_2^2 \hat{e}_y + \Delta_3^3 \hat{e}_z), \quad (5.15)$$

$$\Omega = \sum_{p=3}^{6} D_p (A_p P_p^+ + B_p P_p^-)(\Delta_1^1 \hat{e}_x + \Delta_2^2 \hat{e}_y + \Delta_3^3 \hat{e}_z), \quad (5.16)$$

In the medium $M^{(2)}$

$$q^{(2)} = A_1^{(2)} P_1^{+ (2)}, \quad (5.17)$$

$$U^{(2)} = \sum_{p=3}^{4} A_p^{(2)} \hat{e}_p P_p^{+ (2)}, \quad (5.18)$$

$$\Omega^{(2)} = \sum_{p=3}^{4} (B_p^{(2)} \hat{e}_p + B_p^{(2)} \hat{e}_s) P_p^{+ (2)}, \quad (5.19)$$

where

$$P_0^{+ (1)} = \exp[i k_0^{(1)} (\sin \theta_0 x + \cos \theta_0 z) - \omega_1^{(1)} t],$$

$$P_0^{-(1)} = \exp[i k_0^{(1)} (\sin \theta_0 x - \cos \theta_0 z) - \omega_1^{(1)} t],$$

$$P_p^{(1)} = \exp[i k_p (\sin \theta_p x \pm \cos \theta_p z) - \omega_p^{(1)} t],$$

$$P_p^{(2)} = \exp[i k_p^{(2)} (\sin \theta_p x + \cos \theta_p z) - \omega_p^{(2)} t],$$

$$\omega_r^{(l)} = k_r^{(l)} v_r^{(l)} (r = 1, 3, 4, \ l = 1, 2), \quad \omega_s = k_s V_s (s = 1, 2, ..., 6),$$
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and $k_p^{(2)}$ and $k_s$ being the respective wavenumbers. $A_0^{(1)}$, $A_1^{(1)}$, $A_3^{(1)}$ and $A_4^{(1)}$ denote the amplitudes of the incident longitudinal displacement wave, the reflected longitudinal displacement wave, the reflected set of coupled transverse waves propagating with speed $V_3^{(1)}$ and the reflected set of coupled transverse waves propagating with speed $V_4^{(1)}$, respectively, in medium $M^{(1)}$. $A_5$, and $B_s$ are the amplitudes of the respective refracted and reflected waves traveling with phase speed $V_s$; $A_1^{(2)}$, $A_3^{(2)}$, and $A_4^{(2)}$ denote the amplitudes of the refracted longitudinal displacement wave, the refracted set of coupled transverse waves propagating with speed $V_3^{(2)}$ and the refracted set of coupled transverse waves propagating with speed $V_4^{(2)}$, respectively, in medium $M^{(2)}$.

Parfitt and Eringen (1969) have shown that the coefficients $A_p$ and $B_p$ ($p = 3, 4, l = 1, 2$) are connected to each other through the relation given by

$$B_p^{(0)} = \frac{\omega_0^{(l)} A_p^{(l)} \left( \cos \theta_p^{(l)} \mathbf{e}_x + \sin \theta_p^{(l)} \mathbf{e}_z \right)}{k_p^{(l)} \left( c_s^{(l)2} + 2 \omega_0^{(l)} k_p^{(l)2} - V_s^{(l)2} \right)}.$$  \hspace{1cm} (5.20)

The expressions of the quantities $D_s$ ($s = 1, 2, ..., 6$) are [Yang and Hsia (1995)]

$$D_s = \begin{cases} \rho \omega_s^2 - k_p^2 (\lambda + 2\mu), & \text{when } s = 1, 2 \\ \frac{k_p^2 C_3}{k_s^2 C_3}, & \text{when } s = 3, 4, 5, 6. \end{cases}$$  \hspace{1cm} (5.21)

When a RCP or a LCP plane wave propagates in the $x-z$ plane, the presentation of the quantities $\Delta_1$, $\Delta_2$ and $\Delta_3$ can be specified as

$$\Delta_1^*: \Delta_2^*: \Delta_3^* = \pm \epsilon \cos \theta_s : 1 : \mp \epsilon \sin \theta_s,$$

where the upper signs ‘+’ in $\Delta_1^*$ and ‘+’ in $\Delta_3^*$ refer to the RCP plane waves and the lower signs in $\Delta_1^*$ and $\Delta_3^*$ are used to express the LCP plane waves.

We note that there are eighteen unknown, namely, $A_1^{(l)}$, $A_3^{(l)}$, $A_4^{(l)}$, $A_s$, and $B_s$ ($l = 1, 2$, $s = 1, 2, ..., 6$) occurring in the expressions of potentials given in (5.10)-(5.19). Hence, eighteen linearly independent boundary conditions are required to solve the problem. The classical theory of elasticity provides us the following twelve boundary conditions at the two interfaces (i.e., at $z = 0$ and at $z = D$), corresponding to the
continuity of displacement and surface traction, given as [see Achenbach (1973)]

\[ U^{(\text{inc})} - U^{(\text{ref})} = U^{(\text{fr})} + u^{(\text{ref})} \] at \( z = 0 \), (5.22)

\[ \hat{\epsilon}_z \cdot (T^{(\text{inc})} + T^{(\text{ref})}) = \hat{\epsilon}_z \cdot (T^{(\text{ref})} + T^{(\text{fr})}) \] at \( z = 0 \), (5.23)

\[ u^{(\text{fr})} + u^{(\text{ref})} = u^{(0)} \] at \( z = D \), (5.24)

\[ \hat{\epsilon}_z \cdot (T^{(\text{ref})} + T^{(\text{fr})}) = \hat{\epsilon}_z \cdot T^{(0)} \] at \( z = D \), (5.25)

where \( T \) represents the stress vector.

In micropolar continuum, in addition to the above mentioned boundary conditions, the continuity of microrotation or the continuity of couple stress is required. These conditions are given by [see Chapter- 4]

\[ \phi^{(\text{fr})} = \phi^{(\text{ref})} \] at \( z = 0 \), (5.26)

\[ \phi^{(\text{fr})} + \phi^{(\text{ref})} = \phi^{(0)} \] at \( z = D \), (5.27)

(corresponding to the continuity of microrotation),

\[ \hat{\epsilon}_z \cdot (M^{(\text{inc})} + M^{(\text{ref})}) = \hat{\epsilon}_z \cdot (M^{(\text{ref})} + M^{(\text{fr})}) \] at \( z = 0 \), (5.28)

\[ \hat{\epsilon}_z \cdot (M^{(\text{ref})} + M^{(\text{fr})}) = \hat{\epsilon}_z \cdot M^{(0)} \] at \( z = D \), (5.29)

(corresponding to the continuity of couple stress \( M \)).

Thus, we have two possible sets of boundary conditions at the two bi-material interfaces. The Set-I contains the conditions given in equations (5.26) and (5.27), while the Set-II contains the conditions given in equations (5.28) and (5.29). Note that the conditions given in equations (5.22)-(5.25) are common in both the sets of boundary conditions. These sets of boundary conditions are suffice to solve the boundary value problem. In the present problem, we have obtained the solution for the unknown by
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using separately both the possible sets (Set-I and Set-II) of boundary conditions.

Inserting the expressions of the potentials given in (5.10)-(5.19) into the boundary conditions of Set-I and assuming

\[ k_1^{(1)} \sin \theta_0^{(1)} = k_r^{(i)} \sin \theta_r^{(i)} = k_s \sin \theta_s, \]

\[ \omega_r^{(i)} = \omega_s = \omega \text{ (say)} \quad (r = 1, 3, 4, \ l = 1, 2, \ s = 1, 2, \ldots, 6) \]

at the interfaces, we obtain

\[ \sin \theta_0^{(1)} k_1^{(1)} A_0^{(1)} + \sin \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \cos \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \sum_{p=1}^{2} \sin \theta_p k_p A_p \]

\[ - \sum_{p=3}^{6} \cos \theta_p k_p A_p + \sum_{p=3}^{2} \sin \theta_p k_p B_p + \sum_{p=3}^{6} \cos \theta_p k_p B_p, \]  

(5.30)

\[ \sum_{p=3}^{4} k_p A_p - \sum_{p=5}^{6} k_p A_p - \sum_{p=3}^{4} \cos 2\theta_p k_p B_p + \sum_{p=5}^{6} \cos 2\theta_p k_p B_p = 0, \]  

(5.31)

\[ \cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \sin \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \sum_{p=1}^{2} \cos \theta_p k_p A_p \]

\[ + \sum_{p=3}^{6} \sin \theta_p k_p A_p - \sum_{p=1}^{2} \cos \theta_p k_p B_p + \sum_{p=3}^{6} \sin \theta_p k_p B_p, \]  

(5.32)

\[ (2\mu^{(1)} + \kappa^{(1)}) \sin \theta_0^{(1)} \cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - (2\mu^{(1)} + \kappa^{(1)}) \sin \theta_1^{(1)} \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} \]

\[ - \sum_{p=3}^{4} \left[ \mu^{(1)} \cos 2\theta_p^{(1)} + \kappa^{(1)} \cos^2 \theta_p^{(1)} - \frac{K^{(1)} \omega_0^{(1)} k_p^{(1)} A_p^{(1)}}{c_4^{(1)} k_p^{(1)} A_p^{(1)} + 2\omega_0^{(1)} - \omega_p^{(1)} \omega_0^{(1)}} \right] \]

\[ = \sum_{p=1}^{2} (2\mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 A_p - \sum_{p=3}^{6} (\mu \cos 2\theta_p + C_3 \cos^2 \theta_p D_p) k_p^2 A_p \]

\[ - \sum_{p=1}^{2} (2\mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 B_p - \sum_{p=3}^{6} (\mu \cos 2\theta_p + C_3 \cos^2 \theta_p D_p) k_p^2 B_p, \]  

(5.33)
\[ \sum_{p=3}^{4}(\mu + C_3D_p) \cos \theta_p k_p^2 A_p - \sum_{p=5}^{6}(\mu + C_3D_p) \cos \theta_p k_p^2 A_p \]
\[ = \sum_{p=3}^{4}(\mu + C_3D_p) \cos 2\theta_p \cos \theta_p k_p^2 B_p - \sum_{p=5}^{6}(\mu + C_3D_p) \cos 2\theta_p \cos \theta_p k_p^2 B_p = 0, \quad (5.34) \]
\[ |\lambda| + (2\mu^{(1)} + K^{(1)}) \cos^2 \theta_0^{(1)} k_0^{(1)} A_0^{(1)} + |\lambda| + (2\mu^{(1)} + K^{(1)}) \cos^2 \theta_1^{(1)} k_1^{(1)} A_1^{(1)} \]
\[ - (2\mu^{(1)} + K^{(1)}) \sum_{p=3}^{4} \sin \theta_p^{(1)} \cos \theta_p^{(1)} k_p^{(1)} A_p^{(1)} - (2\mu^{(1)} + C_3D_p) \cos^2 \theta_p k_p^2 A_p \]
\[ + \sum_{p=3}^{6}(2\mu + C_3D_p) \sin \theta_p \cos \theta_p k_p^2 A_p + \sum_{p=1}^{2}[\lambda + (2\mu + C_3D_p) \cos^2 \theta_p] k_p^2 B_p \]
\[ - \sum_{p=3}^{6}(2\mu + C_3D_p) \sin \theta_p \cos \theta_p k_p^2 B_p, \quad (5.35) \]
\[ \sum_{p=1}^{2} \sin \theta_p D_p k_p A_p - \sum_{p=1}^{6} \cos \theta_p D_p k_p A_p + \sum_{p=1}^{2} \sin \theta_p D_p k_p B_p + \sum_{p=1}^{6} \cos \theta_p D_p k_p B_p = 0, \quad (5.36) \]
\[ \sum_{p=3}^{4} \omega_p^{(1)^2} k_p^{(1)^2} A_p^{(1)} = \sum_{p=3}^{4} D_p k_p A_p - \sum_{p=5}^{6} D_p k_p A_p \]
\[ - \sum_{p=3}^{4} \cos 2\theta_p D_p k_p B_p + \sum_{p=3}^{6} \cos 2\theta_p D_p k_p B_p. \quad (5.37) \]
\[ \sum_{p=1}^{2} \cos \theta_p D_p k_p A_p + \sum_{p=3}^{6} \sin \theta_p D_p k_p A_p - \sum_{p=1}^{2} \cos \theta_p D_p k_p B_p + \sum_{p=3}^{6} \sin \theta_p D_p k_p B_p = 0, \quad (5.38) \]
\[ \sum_{p=1}^{2} \sin \theta_p k_p \exp[i k_p \cos \theta_p D] A_p - \sum_{p=3}^{6} \cos \theta_p k_p \exp[i k_p \cos \theta_p D] A_p \]
\[ + \sum_{p=1}^{2} \sin \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=3}^{6} \cos \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p \]
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\[
\begin{align*}
\sin \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A^{(2)}_1 & \quad - \sum_{p=3}^{4} \cos \theta_p^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A^{(2)}_{p-1} \\
= & \quad \sum_{p=3}^{4} k_p \exp[i k_p \cos \theta_p D] A_p - \sum_{p=5}^{6} k_p \exp[i k_p \cos \theta_p D] A_p
\end{align*}
\]

\[
\begin{align*}
& \quad - \sum_{p=3}^{4} \cos 2 \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=5}^{6} \cos 2 \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p = 0
\end{align*}
\]

\[
\begin{align*}
& \quad \sum_{p=1}^{2} \cos \theta_p k_p \exp[i k_p \cos \theta_p D] A_p + \sum_{p=3}^{6} \sin \theta_p k_p \exp[i k_p \cos \theta_p D] A_p
\end{align*}
\]

\[
\begin{align*}
& \quad - \sum_{p=1}^{2} \cos \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=3}^{6} \sin \theta_p k_p \exp[-i k_p \cos \theta_p D] B_p
\end{align*}
\]

\[
= \cos \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} + \sum_{p=3}^{4} \sin \theta_p^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p
\]

\[
= \sum_{p=1}^{2} (2 \mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 \exp[i k_p \cos \theta_p D] A_p
\]

\[
= \sum_{p=1}^{2} (2 \mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 \exp[-i k_p \cos \theta_p D] B_p - \sum_{p=3}^{6} (\mu \cos 2 \theta_p + C_3 \cos^2 \theta_p D_p) k_p^2 \\
& \quad \exp[-i k_p \cos \theta_p D] B_p = (2 \mu^{(2)} + K^{(2)}) \sin \theta_1^{(2)} \cos \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)}
\]

\[
- \sum_{p=3}^{4} \left[ \mu^{(2)} \cos 2 \theta_p^{(2)} + K^{(2)} \cos^2 \theta_p^{(2)} + \frac{K^{(2)} \omega_0^{(2)} \cos 2 \theta_p^{(2)}}{k_p^{(2)^2}} + 2 \omega_0^{(2)^2} - \omega_p^{(2)^2} \right] \\
\times k_p^{(2)^2} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}
\]

\[
= \sum_{p=1}^{2} (\mu + C_3 D_p) \cos \theta_p k_p^2 \exp[i k_p \cos \theta_p D] A_p - \sum_{p=5}^{6} (\mu + C_3 D_p) \cos \theta_p k_p^2 \exp[i k_p \cos \theta_p D] A_p
\]

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\[ + \sum_{p=3}^{4} (\mu + C_3 D_p) \cos 2\theta_p \cos \theta_p k_p^2 \exp[-i k_p \cos \theta_p D] B_p \]

\[ - \sum_{p=3}^{6} (\mu + C_3 D_p) \cos 2\theta_p \cos \theta_p k_p^2 \exp[-i k_p \cos \theta_p D] B_p = 0, \tag{5.43} \]

\[ + \sum_{p=1}^{2} (\lambda + (2\mu + C_3 D_p) \cos^2 \theta_p) k_p^2 \exp[i k_p \cos \theta_p D] A_p \]

\[ + \sum_{p=3}^{6} (2\mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 \exp[i k_p \cos \theta_p D] A_p \]

\[ + \sum_{p=1}^{2} [\lambda + (2\mu + C_3 D_p) \cos^2 \theta_p] k_p^2 \exp[-i k_p \cos \theta_p D] B_p - \sum_{p=3}^{6} (2\mu + C_3 D_p) \sin \theta_p \cos \theta_p k_p^2 \exp[-i k_p \cos \theta_p D] B_p = \lambda^{(2)} + (2\mu^{(2)} + K^{(2)}) \cos^2 \theta_p^{(2)} k_p^{(2)^2} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)} \]

\[ + (2\mu^{(2)} + K^{(2)}) \sum_{p=3}^{4} \sin \theta_p^{(2)} \cos \theta_p^{(2)} k_p^{(2)^2} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}, \tag{5.44} \]

\[ \sum_{p=1}^{2} \sin \theta_p D_p k_p \exp[i k_p \cos \theta_p D] A_p - \sum_{p=3}^{6} \cos \theta_p D_p k_p \exp[i k_p \cos \theta_p D] A_p \]

\[ + \sum_{p=1}^{2} \sin \theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=3}^{6} \cos \theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p = 0, \tag{5.45} \]

\[ - \sum_{p=3}^{4} D_p k_p \exp[i k_p \cos \theta_p D] A_p - \sum_{p=3}^{6} D_p k_p \exp[i k_p \cos \theta_p D] A_p \]

\[ - \sum_{p=3}^{4} \cos 2\theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=5}^{6} \cos 2\theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p \]

\[ = - \sum_{p=3}^{4} \frac{\omega_0^{(2)^2} \cos 2\theta_p^{(2)} k_p^{(2)^2}}{c_4^{(2)^2} k_p^{(2)^2} + 2\omega_0^{(2)^2} - \omega_p^{(2)^2}} \frac{\exp[i k_p^{(2)} \cos \theta_p^{(2)} D]}{A_p^{(2)}}, \tag{5.46} \]
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\[
\sum_{p=1}^{2} \cos \theta_p D_p k_p \exp[\pm i k_p \cos \theta_p D] A_p + \sum_{p=3}^{6} \sin \theta_p D_p k_p \exp[\pm i k_p \cos \theta_p D] A_p
\]

\[
- \sum_{p=1}^{2} \cos \theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p + \sum_{p=3}^{6} \sin \theta_p D_p k_p \exp[-i k_p \cos \theta_p D] B_p = 0. \quad (5.47)
\]

These equations can be written in matrix form as

\[
[a_{i,j}] [X] = [N], \quad (5.48)
\]

where \([a_{i,j}]\) is a 18 \times 18 matrix, \([N]\) is a 18 \times 1 matrix, \([X]\) = \([Z_1, Z_2, Z_3, \ldots, Z_{18}]\). \(Z_1 = A_1^{(1)}/A_0^{(1)}\) and \(Z_{r-1} = A_r^{(1)}/A_0^{(1)}\) \((r = 3, 4)\) are the reflection coefficients in medium \(M^{(1)}\). \(Z_{r+3} = A_r/A_0^{(1)}\) and \(Z_{s+6} = B_s/A_0^{(1)}\) \((s = 1, 2, \ldots, 6)\) are, respectively, the transmission and reflection coefficients in the medium \(M\), \(Z_{16} = A_1^{(2)}/A_0^{(1)}\) and \(Z_{r+14} = A_r^{(2)}/A_0^{(1)}\) \((r = 3, 4)\) are the transmission coefficients in medium \(M^{(2)}\).

The non-zero coefficients of the matrix \([a_{i,j}]\) of the matrix equation (5.48) are given as

\[
a_{1,1} = \sin \theta_0^{(1)},
\]

\[
a_{1,2} = \sqrt{1 - v_{31}^{(1)} \sin^2 \theta_0^{(1)}/v_{31}^{(1)}},
\]

\[
a_{1,3} = \sqrt{1 - v_{41}^{(1)} \sin^2 \theta_0^{(1)}/v_{41}^{(1)}},
\]

\[
a_{1,4} = a_{1,4+6} = -\sin \theta_0^{(1)},
\]

\[
a_{1,5} = a_{1,5+6} = \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}/V_{p1}},
\]

\[
a_{1,6} = a_{1,6+6} = \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}/V_{s1}},
\]

\[
a_{2,5} = 1/V_{p1},
\]

\[
a_{2,6} = -1/V_{s1},
\]

\[
a_{2,5+6} = -(1 - 2V_{p1}^2 \sin^2 \theta_0^{(1)})/V_{p1},
\]

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\[
\begin{align*}
    a_{2,8+6} &= (1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)})/V_{s1}, \\
    a_{3,1} &= \cos \theta_0^{(1)}, \\
    a_{3,2} &= a_{3,3} = -\sin \theta_0^{(1)} = -a_{3,p} = -a_{3,s} = -a_{3,p+6} = -a_{3,s+6}, \\
    a_{3,t} &= -a_{3,t+6} = \sqrt{1 - V_{t1}^2 \sin^2 \theta_0^{(1)}}/V_{t1}, \\
    a_{4,4} &= a_{4,10} = \sin \theta_0^{(1)}, \\
    a_{4,5} &= a_{4,11} = D_2 \sin \theta_0^{(1)}/D_1, \\
    a_{4,p} &= -a_{4,p+6} = -D_{p-3}\sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}}/(D_1 V_{p1}), \\
    a_{4,s} &= -a_{4,s+6} = -D_{s-3}\sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}/(D_1 V_{s1}), \\
    a_{5,2} &= S_1^{(1)} k_2^{(1)}/(D_1 v_{21}^{(1)}), \\
    a_{5,3} &= S_2^{(1)} k_3^{(1)}/(D_1 v_{31}^{(1)}), \\
    a_{5,p} &= -D_{p-3}/(D_1 V_{p1}), \\
    a_{5,s} &= D_{s-3}/(D_1 V_{s1}), \\
    a_{5,p+6} &= D_{p-3}(1 - 2V_{p1}^2 \sin^2 \theta_0^{(1)})/(D_1 V_{p1}), \\
    a_{5,s+6} &= -D_{s-3}(1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)})/(D_1 V_{s1}), \\
    a_{6,4} &= -a_{6,10} = \sqrt{1 - v_{11}^2 \sin^2 \theta_0^{(1)}}/v_{11}, \\
    a_{6,5} &= -a_{6,11} = D_2 \sqrt{1 - v_{21}^2 \sin^2 \theta_0^{(1)}}/(D_1 v_{21}),
\end{align*}
\]
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\[ a_{6,p} = a_{6,s+6} = D_{p-3} \sin \theta_0^{(1)} / D_1, \]
\[ a_{6,s} = a_{6,s+6} = D_{s-3} \sin \theta_0^{(1)} / D_1, \]
\[ a_{7,1} = \sin \theta_0^{(1)} \cos \theta_0^{(1)}, \]
\[ a_{7,2} = \frac{(\mu^{(1)} + K^{(1)} - K^{(1)} S_1^{(1)})/(B_1^{(1)} v_{31}^{(1)^2}) - \sin^2 \theta_0^{(1)}}, \]
\[ a_{7,3} = \frac{(\mu^{(1)} + K^{(1)} - K^{(1)} S_2^{(1)})/(B_1^{(1)} v_{41}^{(1)^2}) - \sin^2 \theta_0^{(1)}}, \]
\[ a_{7,t+6} = -a_{7,t+6} = \frac{(2\mu + C_3 D_{t-3}) \sin \theta_0^{(1)} \sqrt{1 - V^2_{t+1} \sin^2 \theta_0^{(1)}}/(B_1^{(1)} V_{t+1})}, \]
\[ a_{7,p} = a_{7,p+6} = -[\mu + C_3 D_{p-3} - (2\mu + C_3 D_{p-3}) V^2_{p+1} \sin^2 \theta_0^{(1)}]/(B_1^{(1)} V_{p+1}), \]
\[ a_{7,s} = a_{7,s+6} = -[\mu + C_3 D_{s-3} - (2\mu + C_3 D_{s-3}) V^2_{s+1} \sin^2 \theta_0^{(1)}]/(B_1^{(1)} V_{s+1})}, \]
\[ a_{8,p} = (\mu + C_3 D_{p-3}) \sqrt{1 - V^2_{p+1} \sin^2 \theta_0^{(1)}}/(\mu^{(1)} V_{p+1}^2), \]
\[ a_{8,s} = -(\mu + C_3 D_{s-3}) \sqrt{1 - V^2_{s+1} \sin^2 \theta_0^{(1)}}/(\mu^{(1)} V_{s+1}^2), \]
\[ a_{8,p+6} = (\mu + C_3 D_{p-3})(1 - V^2_{p+1} \sin^2 \theta_0^{(1)} \sqrt{1 - V^2_{p+1} \sin^2 \theta_0^{(1)}})/(\mu^{(1)} V_{p+1}^2), \]
\[ a_{8,s+6} = -[\mu + C_3 D_{s-3})(1 - V^2_{s+1} \sin^2 \theta_0^{(1)} \sqrt{1 - V^2_{s+1} \sin^2 \theta_0^{(1)}}/(\mu^{(1)} V_{s+1}^2), \]
\[ a_{9,1} = 1, \]
\[ a_{9,2} = -B_1^{(1)} \sin \theta_0^{(1)} \sqrt{1 - v_{31}^{(1)^2} \sin^2 \theta_0^{(1)}}/(B_1^{(1)} v_{31}^{(1)}), \]
\[ a_{9,3} = -B_1^{(1)} \sin \theta_0^{(1)} \sqrt{1 - v_{41}^{(1)^2} \sin^2 \theta_0^{(1)}}/(B_1^{(1)} v_{41}^{(1)}), \]
\[ a_{9,t} = a_{9,t+6} = -[\lambda + (2\mu + C_3 D_{t-3})(1 - V^2_{t+1} \sin^2 \theta_0^{(1)})]/(B_2^{(1)} V_{t+1}^2), \]
\[ a_{9,p} = a_{9,p+6} = -(2\mu + C_3 D_{p-3}) \sin \theta_0^{(1)} \sqrt{1 - V^2_{p+1} \sin^2 \theta_0^{(1)}}/(B_2^{(1)} V_{p+1}), \]
\[ a_{9,s} = -a_{9,s+6} = -(2\mu + C_4 D_{s-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} / (B_2^{(1)} V_{a1}), \]

\[ a_{10,t} = \sin \theta_0^{(1)} \exp[i k_{t-3} \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} D], \]

\[ a_{10,p} = -\sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D]/V_{p1}, \]

\[ a_{10,s} = -\sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/V_{s1}, \]

\[ a_{10,p+6} = \sin \theta_0^{(1)} \exp[-i k_{t-3} \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} D], \]

\[ a_{10,p+6} = \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \exp[-i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D]/V_{p1}, \]

\[ a_{10,s+6} = \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp[-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/V_{s1}, \]

\[ a_{10,16} = -\sin \theta_0^{(1)} \exp[i k_{16} \sqrt{1 - \tau_{11}^2 \sin^2 \theta_0^{(1)}} D], \]

\[ a_{10,17} = \sqrt{1 - \tau_{31}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{17} \sqrt{1 - \tau_{31}^2 \sin^2 \theta_0^{(1)}} D]/\tau_{31}, \]

\[ a_{10,18} = \sqrt{1 - \tau_{41}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{18} \sqrt{1 - \tau_{41}^2 \sin^2 \theta_0^{(1)}} D]/\tau_{41}, \]

\[ a_{11,p} = \exp[i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D]/V_{p1}, \]

\[ a_{11,s} = -\exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/V_{s1}, \]

\[ a_{11,p+6} = -(1 - 2V_{p1}^2 \sin^2 \theta_0^{(1)}) \exp[-i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D]/V_{p1}, \]

\[ a_{11,s+6} = (1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)}) \exp[-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/V_{s1}, \]

\[ a_{12,t} = \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{t-3} \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} D]/V_{a1}, \]

\[ a_{12,p} = \sin \theta_0^{(1)} \exp[i k_{p-3} \sqrt{1 - V_{a1}^2 \sin^2 \theta_0^{(1)}} D], \]
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\[ a_{12,s} = \sin \theta_0^{(1)} \exp \left[ i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{12,t+6} = -\sqrt{1 - V_{t1}^2 \sin^2 \theta_0^{(1)}} \exp \left[ -i k_{t-3} \sqrt{1 - V_{t1}^2 \sin^2 \theta_0^{(1)}} D \right] / V_{t1}, \]

\[ a_{12,p+6} = \sin \theta_0^{(1)} \exp \left[ -i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{12,s+6} = \sin \theta_0^{(1)} \exp \left[ -i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{12,16} = -\sqrt{1 - v_{11}^{(2)2} \sin^2 \theta_0^{(1)}} \exp \left[ i k_{1}^{(2)} \sqrt{1 - v_{11}^{(2)2} \sin^2 \theta_0^{(1)}} D \right] / v_{11}^{(2)}, \]

\[ a_{12,17} = -\sin \theta_0^{(1)} \exp \left[ i k_{3}^{(2)} \sqrt{1 - v_{31}^{(2)2} \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{12,18} = -\sin \theta_0^{(1)} \exp \left[ i k_{4}^{(2)} \sqrt{1 - v_{41}^{(2)2} \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{13,4} = \sin \theta_0^{(1)} \exp \left[ i k_{1} \sqrt{1 - v_{11}^{2} \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{13,5} = D_{2} \sin \theta_0^{(1)} \exp \left[ i k_{2} \sqrt{1 - v_{21}^{2} \sin^2 \theta_0^{(1)}} D \right] / D_{1}, \]

\[ a_{13,p} = -D_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \exp \left[ i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right] / (D_{1} V_{p1}), \]

\[ a_{13,s} = -D_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp \left[ i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right] / (D_{1} V_{s1}), \]

\[ a_{13,10} = \sin \theta_0^{(1)} \exp \left[ -i k_{1} \sqrt{1 - v_{11}^{2} \sin^2 \theta_0^{(1)}} D \right], \]

\[ a_{13,11} = D_{2} \sin \theta_0^{(1)} \exp \left[ -i k_{2} \sqrt{1 - v_{21}^{2} \sin^2 \theta_0^{(1)}} D \right] / D_{1}, \]

\[ a_{13,p+6} = D_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \exp \left[ -i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right] / (D_{1} V_{p1}), \]

\[ a_{13,s+6} = D_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp \left[ -i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right] / (D_{1} V_{s1}), \]

\[ a_{14,p} = D_{p-3} \exp \left[ i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right] / (D_{1} V_{p1}). \]
\[ a_{14,s} = -D_{s-3} \exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}] / (D_1 V_{s1}), \]

\[ a_{14,p+6} = -D_{p-3} (1 - 2v_{p1}^2 \sin^2 \theta_0^{(1)}) \exp[-i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}}] / (D_1 V_{p1}), \]

\[ a_{14,s+6} = D_{s-3} (1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)}) \exp[-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}] / (D_1 V_{s1}), \]

\[ a_{14,17} = S_1^{(2)} k_3^{(2)} (1 - 2v_{3}^{(2)} \sin^2 \theta_0^{(1)}) \exp[i k_3^{(2)} \sqrt{1 - v_{31}^{(2)} \sin^2 \theta_0^{(1)}}] / (D_1 v_{31}^{(2)}), \]

\[ a_{14,18} = S_2^{(2)} k_4^{(2)} (1 - 2v_{41}^{(2)} \sin^2 \theta_0^{(1)}) \exp[i k_4^{(2)} \sqrt{1 - v_{41}^{(2)} \sin^2 \theta_0^{(1)}}] / (D_1 v_{41}^{(2)}), \]

\[ a_{15,4} = \sqrt{1 - v_{11}^2 \sin^2 \theta_0^{(1)}} \exp[i k_1 \sqrt{1 - v_{11}^2 \sin^2 \theta_0^{(1)}}] / v_{11}, \]

\[ a_{15,5} = D_2 \sqrt{1 - v_{21}^2 \sin^2 \theta_0^{(1)}} \exp[i k_2 \sqrt{1 - v_{21}^2 \sin^2 \theta_0^{(1)}}] / (D_1 v_{21}), \]

\[ a_{15,q} = D_{p-3} \sin \theta_0^{(1)} \exp[i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}}] / D_1, \]

\[ a_{15,s} = D_{s-3} \sin \theta_0^{(1)} \exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}] / D_1, \]

\[ a_{15,10} = -\sqrt{1 - v_{11}^2 \sin^2 \theta_0^{(1)}} \exp[-i k_1 \sqrt{1 - v_{11}^2 \sin^2 \theta_0^{(1)}}] / v_{11}, \]

\[ a_{15,11} = -D_2 \sqrt{1 - v_{21}^2 \sin^2 \theta_0^{(1)}} \exp[-i k_2 \sqrt{1 - v_{21}^2 \sin^2 \theta_0^{(1)}}] / (D_1 v_{21}), \]

\[ a_{15,p+6} = D_{p-3} \sin \theta_0^{(1)} \exp[-i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}}] / D_1, \]

\[ a_{15,s+6} = D_{s-3} \sin \theta_0^{(1)} \exp[-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}] / D_1, \]

\[ a_{16,4} = (2\mu + C_3 D_{l-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp[i k_{l-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}}] / (\mu v_{11}), \]

\[ a_{16,p} = -[\mu + C_3 D_{p-3} - (2\mu + C_3 D_{p-3}) v_{p1}^2 \sin^2 \theta_0^{(1)}] \times \exp[i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}}] / (\mu v_{p1}), \]

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\[ a_{16,5} = -[\mu + C_d D_{p-3} - (2\mu + C_d D_{p-3}) V_p^2 \sin^2 \theta_0^{(1)}] \times \exp\{ik_p-3 \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_p^2)\}, \]

\[ a_{16,6} = -(2\mu + C_d D_{p-3}) \sin \theta_0^{(1)} \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)}} \times \exp\{-ik_{p-3} \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_p^2)\}, \]

\[ a_{16,6+6} = -[\mu + C_d D_{p-3} - (2\mu + C_d D_{p-3}) V_p^2 \sin^2 \theta_0^{(1)}] \times \exp\{-ik_{p-3} \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_p^2)\}, \]

\[ a_{16,16} = B_4^{(2)} \sin \theta_0^{(1)} \sqrt{1 - v_{11}^{(2)} \sin^2 \theta_0^{(1)}} \exp\{ik_{11}^{(2)} \sqrt{1 - v_{11}^{(2)} \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} v_{11}^{(2)})\}, \]

\[ a_{16,17} = [(\mu^{(2)} + K^{(2)} + K^{(2)} S_{11}^{(2)}) - (2\mu^{(2)} + K^{(2)} + 2K^{(2)} S_{11}^{(2)} v_{31}^{(2)} \sin^2 \theta_0^{(1)})/(\mu^{(1)} v_{31}^{(2)})] \times \exp\{ik_{31}^{(2)} \sqrt{1 - v_{31}^{(2)} \sin^2 \theta_0^{(1)} D}\}, \]

\[ a_{16,18} = [(\mu^{(2)} + K^{(2)} + K^{(2)} S_{12}^{(2)}) - (2\mu^{(2)} + K^{(2)} + 2K^{(2)} S_{12}^{(2)} v_{41}^{(2)} \sin^2 \theta_0^{(1)})/(\mu^{(1)} v_{41}^{(2)})] \times \exp\{ik_{41}^{(2)} \sqrt{1 - v_{41}^{(2)} \sin^2 \theta_0^{(1)} D}\}, \]

\[ a_{17,\rho} = (\mu + C_d D_{p-3}) \sqrt{1 - V_{31}^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_{31}^2), \]

\[ a_{17,\rho+6} = -(\mu + C_d D_{p-3}) \sqrt{1 - V_{31}^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_{31}^2), \]

\[ a_{17,\rho+6} = (\mu + C_d D_{p-3}) (1 - 2V_{p3}^2 \sin^2 \theta_0^{(1)}) \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)}} \times \exp\{-ik_{p-3} \sqrt{1 - V_p^2 \sin^2 \theta_0^{(1)} D}/(\mu^{(1)} V_p^2)\}, \]

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\[ a_{17,s+6} = -(\mu + C_3 D_{s-3})(1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)}) \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \]
\[ \times \exp(-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D)/(\mu^{(1)} V_{s1}^2). \]
\[ a_{18,4} = [\lambda + (2\mu + C_3 D_{s-3})(1 - V_{s1}^2 \sin^2 \theta_0^{(1)})] \exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} V_{s1}^2). \]
\[ a_{18,p} = (2\mu + C_3 D_{p-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \]
\[ \times \exp[i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} V_{p1}). \]
\[ a_{18,8} = (2\mu + C_3 D_{s-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \]
\[ \times \exp[i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} V_{s1}). \]
\[ a_{18,s+6} = [\lambda + (2\mu + C_3 D_{s-3})(1 - V_{s1}^2 \sin^2 \theta_0^{(1)})] \]
\[ \times \exp(-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D)/(\mu^{(1)} V_{s1}^2). \]
\[ a_{18,p+6} = -(2\mu + C_3 D_{p-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \]
\[ \times \exp(-i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D)/(\mu^{(1)} V_{p1}). \]
\[ a_{18,s+6} = -(2\mu + C_3 D_{s-3}) \sin \theta_0^{(1)} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \]
\[ \times \exp(-i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D)/(\mu^{(1)} V_{s1}). \]
\[ a_{18,16} = -B_2^{(2)} \exp[i k_1^{(2)} \sqrt{1 - \nu_{11}^{(2)} \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} v_{11}^{(2)}). \]
\[ a_{18,17} = -B_1^{(2)} \sin \theta_0^{(1)} \sqrt{1 - \nu_{31}^{(2)} \sin^2 \theta_0^{(1)}} \exp[i k_3^{(2)} \sqrt{1 - \nu_{31}^{(2)} \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} v_{31}^{(2)}). \]
\[ a_{18,18} = -B_1^{(2)} \sin \theta_0^{(1)} \sqrt{1 - \nu_{41}^{(2)} \sin^2 \theta_0^{(1)}} \exp[i k_4^{(2)} \sqrt{1 - \nu_{41}^{(2)} \sin^2 \theta_0^{(1)}} D]/(\mu^{(1)} v_{41}^{(2)}). \]
5.3. Reflection and transmission of waves

t = 4, 5, p = 6, 7, s = 8, 9

where

\[
B_1^{(1)} = 2\mu^{(1)} + K^{(1)}, \quad B_1^{(2)} = 2\mu^{(2)} + K^{(2)},
\]

\[
B_2^{(1)} = \lambda^{(1)} + B_1^{(1)} \cos^2 \theta_0^{(1)}, \quad B_2^{(2)} = \lambda^{(2)} + B_1^{(2)} (1 - \nu_1^{(2)} \sin^2 \theta_0^{(1)}),
\]

\[
S_1^{(1)} = \frac{\omega_0^{(1)}{^2}}{\omega_0^{(1)}{^2} + \omega_1^{(1)}{^2}}, \quad S_1^{(2)} = \frac{\omega_0^{(1)}{^2}}{\omega_0^{(1)}{^2} + \omega_1^{(1)}{^2}},
\]

\[
S_2^{(1)} = \frac{\omega_0^{(2)}{^2}}{\omega_0^{(2)}{^2} + \omega_1^{(2)}{^2}}, \quad S_2^{(2)} = \frac{\omega_0^{(2)}{^2}}{\omega_0^{(2)}{^2} + \omega_1^{(2)}{^2}},
\]

\[
v_{r1}^{(1)} = \frac{V_{r1}^{(1)}}{V_{1}^{(1)}} (r = 3, 4), \quad v_{n1} = \frac{V_{n}}{V_{1}^{(1)}} (n = 1, 2, \ldots, 6), \quad v_{m1}^{(2)} = \frac{V_{n}^{(2)}}{V_{1}^{(1)}} (m = 1, 3, 4),
\]

\[
V_{41} = v_{11}, \quad V_{51} = v_{21}, \quad V_{61} = v_{31}, \quad V_{71} = v_{41}, \quad V_{81} = v_{51}, \quad V_{91} = v_{61}
\]

and the column matrix

\[
[N] = \left[-\sin \theta_0^{(1)}, 0, \cos \theta_0^{(1)}, 0, 0, 0, \sin \theta_0^{(1)} \cos \theta_0^{(1)}, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0\right]^T.
\]

The non-zero elements of the matrix \([a_{i,j}]\) of the matrix equation (5.48) obtained by using Set-II of boundary conditions, that are different from the elements obtained using Set-I are given as follows

\[
a_{4,4} = ((\beta + \gamma)D_{p-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - V_{21}^2 \sin^2 \theta_0^{(1)}}/(V_{21}^2 C_3),
\]

\[
a_{4,p} = \left[ ((\beta + \gamma)D_{p-3} + C_3)V_{p2}^2 \sin^2 \theta_0^{(1)} - \left( \gamma D_{p-3} + C_3 - \frac{C_3 D_{p-3}}{k_{p-3}} \right) \right]/(V_{p2}^2 C_3),
\]

\[
a_{4,s} = \left[ ((\beta + \gamma)D_{s-3} + C_3)V_{s2}^2 \sin^2 \theta_0^{(1)} - \left( \gamma D_{s-3} + C_3 + \frac{C_3 D_{s-3}}{k_{s-3}} \right) \right]/(V_{s2}^2 C_3),
\]

\[
a_{4,t+6} = -((\beta + \gamma)D_{t-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - V_{21}^2 \sin^2 \theta_0^{(1)}}/(V_{21}^2 C_3),
\]

\[
a_{4,p+6} = \left[ ((\beta + \gamma)D_{p-3} + C_3 + \frac{2C_3 D_{p-3}}{k_{p-3}}) V_{p2}^2 \sin^2 \theta_0^{(1)} - \left( \gamma D_{p-3} + C_3 + \frac{C_3 D_{p-3}}{k_{p-3}} \right) \right]/(V_{p2}^2 C_3),
\]

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\[ a_{5,6} = \left( (\beta + \gamma) D_{s-3} + C_3 - \frac{2 C_3 D_{r-3}}{k_{s-3}} \right) V_{s1}^2 \sin^2 \theta_0^{(1)} - \left( \gamma D_{s-3} + C_3 - \frac{C_3 D_{r-3}}{k_{s-3}} \right) \right) / (V_{s1}^2 C_3), \]

\[ a_{5,2} = \gamma (1) \frac{\beta (1)}{k_1} \sqrt{1 - \frac{v_{31}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{s1}^2 C_3),} \]

\[ a_{5,3} = \gamma (1) \frac{\beta (1)}{k_1} \sqrt{1 - \frac{v_{41}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{s1}^2 C_3),} \]

\[ a_{5,4} = D_1 \sin \theta_0^{(1)}/k_1^{(1)}, \]

\[ a_{5,5} = D_2 \sin \theta_0^{(1)}/k_1^{(1)}, \]

\[ a_{5,p} = \left( \gamma D_{p-3} + C_3 - \frac{C_3 D_{p-3}}{k_{p-3}} \right) \sqrt{1 - \frac{V_{p1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{p1}^2 C_3),} \]

\[ a_{5,s} = - \left( \gamma D_{s-3} + C_3 + \frac{C_3 D_{s-3}}{k_{s-3}} \right) \sqrt{1 - \frac{V_{s1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{s1}^2 C_3),} \]

\[ a_{5,10} = D_1 \sin \theta_0^{(1)}/k_1^{(1)}, \]

\[ a_{5,11} = D_2 \sin \theta_0^{(1)}/k_1^{(1)}, \]

\[ a_{5,p+6} = \left( \gamma D_{p-3} + C_3 \right) \left( 1 - 2 V_{p1}^2 \sin^2 \theta_0^{(1)} \right) + \frac{C_3 D_{p-3}}{k_{p-3}} \right) \sqrt{1 - \frac{V_{p1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{p1}^2 C_3),} \]

\[ a_{5,s+6} = - \left( \gamma D_{s-3} + C_3 \right) \left( 1 - 2 V_{s1}^2 \sin^2 \theta_0^{(1)} \right) - \frac{C_3 D_{s-3}}{k_{s-3}} \right) \sqrt{1 - \frac{V_{s1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}/(V_{s1}^2 C_3),} \]

\[ a_{6,f} = [(\alpha + \beta + \gamma) D_{t-3} + C_3 - ((\beta + \gamma) D_{t-3} + C_3) V_{s1}^2 \sin^2 \theta_0^{(1)}]/(V_{s1}^2 C_3), \]

\[ a_{6,p} = ((\beta + \gamma) D_{p-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - \frac{V_{p1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}}/(V_{p1} C_3), \]

\[ a_{6,s} = ((\beta + \gamma) D_{s-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - \frac{V_{s1}^2}{V_{s1}^2} \sin^2 \theta_0^{(1)}}/(V_{s1} C_3), \]

\[ a_{6,f+6} = [(\alpha + \beta + \gamma) D_{t-3} + C_3 - ((\beta + \gamma) D_{t-3} + C_3) V_{s1}^2 \sin^2 \theta_0^{(1)}]/(V_{s1}^2 C_3), \]

\[ a_{6,p+6} = -((\beta + \gamma) D_{p-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - \frac{V_{p1}^2}{V_{p1}^2} \sin^2 \theta_0^{(1)}}/(V_{p1} C_3), \]

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\[
s_{x+6} = -((\beta + \gamma)D_{x-3} + C_3) \sin \theta_0^{(i)} \sqrt{1 - V_{i1}^2 \sin^2 \theta_0^{(i)}}/(V_{i1} C_3),
\]

\[
s_{13,1} = ((\beta + \gamma)D_{1-3} + C_3) \sin \theta_0^{(i)} \sqrt{1 - V_{i1}^2 \sin^2 \theta_0^{(i)}}
\times \exp[ik_{i1-3}\sqrt{1 - V_{i1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{i1} C_3),
\]

\[
a_{13,p} = \left[ ((\beta + \gamma)D_{p-3} + C_3) V_{p1}^2 \sin^2 \theta_0^{(i)} - \left( \gamma D_{p-3} + C_3 \frac{C_3 D_{p-3}}{k_{p-3}} \right) \right]
\times \exp[ik_{p-3}\sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{p1} C_3),
\]

\[
a_{13,s} = \left[ ((\beta + \gamma)D_{s-3} + C_3) V_{s1}^2 \sin^2 \theta_0^{(i)} - \left( \gamma D_{s-3} + C_3 + \frac{C_3 D_{s-3}}{k_{s-3}} \right) \right]
\times \exp[ik_{s-3}\sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{s1} C_3),
\]

\[
a_{13,t+6} = -((\beta + \gamma)D_{t-3} + C_3) \sin \theta_0^{(i)} \sqrt{1 - V_{i1}^2 \sin^2 \theta_0^{(i)}}
\times \exp[-ik_{t-3}\sqrt{1 - V_{i1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{i1} C_3),
\]

\[
a_{13,p+6} = \left[ ((\beta + \gamma)D_{p-3} + C_3 + \frac{2C_3 D_{p-3}}{k_{p-3}}) V_{p1}^2 \sin^2 \theta_0^{(i)} - \left( \gamma D_{p-3} + C_3 + \frac{C_3 D_{p-3}}{k_{p-3}} \right) \right]
\times \exp[-ik_{p-3}\sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{p1} C_3),
\]

\[
a_{13,s+6} = \left[ ((\beta + \gamma)D_{s-3} + C_3 - \frac{2C_3 D_{s-3}}{k_{s-3}}) V_{s1}^2 \sin^2 \theta_0^{(i)} - \left( \gamma D_{s-3} + C_3 - \frac{C_3 D_{s-3}}{k_{s-3}} \right) \right]
\times \exp[-ik_{s-3}\sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{s1} C_3),
\]

\[
a_{14,4} = D_1 \sin \theta_0^{(i)} \exp[ik_1\sqrt{1 - v_{i1}^2 \sin^2 \theta_0^{(i)}} D]/k_1^{(i)},
\]

\[
a_{14,5} = D_2 \sin \theta_0^{(i)} \exp[ik_2\sqrt{1 - v_{p1}^2 \sin^2 \theta_0^{(i)}} D]/k_1^{(i)},
\]

\[
a_{14,p} = \left( \gamma D_{p-3} + C_3 - \frac{C_3 D_{p-3}}{k_{p-3}} \right) \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(i)}} \exp[ik_{p-3}\sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(i)}} D]/(V_{p1} C_3).
\]
\[ a_{14,s} = -\left( \gamma D_{s-3} + C_3 + \frac{C_3 D_{s-3}}{k_{s-3}} \right) \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \exp\left[ i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{s1}^3 C_3), \]

\[ a_{14,10} = D_1 \sin \theta_0^{(1)} \exp\left[ -i k_1 \sqrt{1 - V_{11}^2 \sin^2 \theta_0^{(1)}} D \right] / k_1^{(1)}, \]

\[ a_{14,11} = D_2 \sin \theta_0^{(1)} \exp\left[ -i k_2 \sqrt{1 - V_{21}^2 \sin^2 \theta_0^{(1)}} D \right] / k_1^{(1)}, \]

\[ a_{14,p+6} = \left[ \gamma D_{p-3} + C_3 \left( 1 - 2V_{p1}^2 \sin^2 \theta_0^{(1)} \right) + \frac{C_3 D_{p-3}}{k_{p-3}} \right] \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \]

\[ \times \exp\left[ -i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{p1}^3 C_3), \]

\[ a_{14,s+6} = -\left[ \gamma D_{s-3} + C_3 \left( 1 - 2V_{s1}^2 \sin^2 \theta_0^{(1)} \right) - \frac{C_3 D_{s-3}}{k_{s-3}} \right] \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \]

\[ \times \exp\left[ -i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{s1}^3 C_3), \]

\[ a_{14,17} = \gamma^{(2)} \chi_1^{(1)} (1 - 2v_{31}^{(2)} \sin^2 \theta_0^{(1)}) \exp\left[ i k_3^{(2)} \sqrt{1 - v_{31}^{(2)} \sin^2 \theta_0^{(1)}} D \right] / (v_{31}^{(3)} C_3), \]

\[ a_{14,18} = \gamma^{(2)} \chi_2^{(1)} (1 - 2v_{41}^{(2)} \sin^2 \theta_0^{(1)}) \exp\left[ i k_4^{(2)} \sqrt{1 - v_{41}^{(2)} \sin^2 \theta_0^{(1)}} D \right] / (v_{41}^{(3)} C_3), \]

\[ a_{15,s} = \left[ (\alpha + \beta + \gamma) D_{t-3} + C_3 - ((\beta + \gamma) D_{t-3} + C_3) V_{t1}^2 \sin^2 \theta_0^{(1)} \right] \]

\[ \times \exp\left[ i k_{t-3} \sqrt{1 - V_{t1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{t1}^3 C_3), \]

\[ a_{15,p} = ((\beta + \gamma) D_{p-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} \]

\[ \times \exp\left[ i k_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{p1} C_3), \]

\[ a_{15,s} = ((\beta + \gamma) D_{s-3} + C_3) \sin \theta_0^{(1)} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} \]

\[ \times \exp\left[ i k_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0^{(1)}} D \right] / (V_{s1} C_3), \]

\[ a_{15,t+6} = \left[ (\alpha + \beta + \gamma) D_{t-3} + C_3 - ((\beta + \gamma) D_{t-3} + C_3) V_{t1}^2 \sin^2 \theta_0^{(1)} \right] \]
5.4 Limiting cases

\[
x \exp[-\i \k \pm 3 \sqrt{1 - \frac{\sin^2 \theta_0}{\sin^2 \theta_0} D}] / (v_{p1} C_3),
\]

\[
a_{15,p+6} = -((\beta + \gamma) D_{p-3} + C_3) \sin \theta_0 \sqrt{1 - \frac{\sin^2 \theta_0}{\sin^2 \theta_0}} D] / (v_{p1} C_3),
\]

\[
a_{15,s+6} = -((\beta + \gamma) D_{s-3} + C_3) \sin \theta_0 \sqrt{1 - \frac{\sin^2 \theta_0}{\sin^2 \theta_0}} D] / (v_{s1} C_3),
\]

where \( v_{31}^{(1)}, v_{41}^{(1)}, v_{11}, v_{p1}, v_{s1}, v_{31}^{(2)}, v_{41}^{(2)}, s_1^{(1)}, s_2^{(1)}, s_1^{(2)}, s_2^{(2)} \) and the column matrix \([\mathbf{A}]\) same as obtained corresponding to Set-I.

The system of equations given in the matrix equation (5.48) corresponding to Set-I and Set-II separately, will enable us to determine the amplitude ratios of various reflected and transmitted waves.

5.4 Limiting cases

(i) If we assume that the half-spaces \( M^{(1)} \) and \( M^{(2)} \) are free from micropolarity, then we shall be left with the corresponding problem of a chiral slab sandwiched between two achiral solid half-spaces, earlier studied by Elphinstone and Lakhtakia (1994a). In this limiting case, we see that the phase speeds \( v_{p1}^{(1)} \) and \( v_{p1}^{(2)} \) vanish and thus, the waves corresponding to these phase speeds would disappear. Hence, \( A_{21}^{(1)} = A_{4y}^{(1)} = 0 \).

Using this and substituting the constants corresponding to micropolarity equal to zero into the system of equations given in the matrix equation (5.48), one can obtain the corresponding reflection and transmission coefficients. It has been seen that the results of the problem investigated by Hsia and Yang (1999) for the case of normal incidence of longitudinal displacement wave are recovered.

(ii) If the chirality and the microrotation effects are removed from the chiral slab \( M \), then the problem will reduce to an elastic solid slab sandwiched between two micropolar elastic solid half-spaces. In the chiral medium \( M \), if we consider the waves
propagating with phase speeds $V_3$ and $V_4$ as LCP waves and the waves traveling with phase speeds $V_5$ and $V_6$ as RCP waves, then in this limiting case there will exist three waves in medium $M$, namely, a longitudinal displacement wave traveling with speed $V_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and two transverse circularly polarized waves, involving a LCP wave and a RCP wave, propagating with equal speeds, i.e., $V_3 = V_5 = \sqrt{\mu/\rho}$ (say).

After making the required substitutions into the equations (5.31), (5.34), (5.40) and (5.43), we obtain $A_3 = A_5 = A/2$ (say), $B_3 = B_5 = B/2$ (say). Also, we have seen that the equations (5.36), (5.38), (5.45) and (5.47) are identically satisfied. Thus, the equations (5.30)-(5.47) reduce to

\begin{align*}
\sin \theta_0^{(1)} k_1^{(1)} A_0^{(1)} &+ \sin \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \cos \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \sin \theta_1 k_1 A_1 \\
&+ \sin \theta_1 k_1 B_1 - \cos \theta_3 k_3 A + \cos \theta_3 k_3 B, \quad (5.49)
\end{align*}

\begin{align*}
\cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} &- \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \sin \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \cos \theta_1 k_1 A_1 \\
&- \cos \theta_1 k_1 B_1 + \sin \theta_3 k_3 A + \sin \theta_3 k_3 B, \quad (5.50)
\end{align*}

\begin{align*}
(2\mu^{(1)} + K^{(1)}) \sin \theta_0^{(1)} \cos \theta_0^{(1)} k_1^{(1)} k_1 A_0^{(1)} - (2\mu^{(1)} + K^{(1)}) \sin \theta_1^{(1)} \cos \theta_1^{(1)} k_1^{(1)} k_1 A_1^{(1)} \\
- \sum_{p=3}^{4} \left[ \mu^{(1)} \cos 2\theta_p^{(1)} + K^{(1)} \cos^2 \theta_p^{(1)} - \frac{K^{(1)} \omega_0^{(1)} k_0^{(1)} A_0^{(1)}}{c_4^{(1)} k_p^{(1)} A_p^{(1)}} \right] k_p^{(1)} A_p^{(1)} = 2\mu \sin \theta_1 \cos \theta_1 k_1^2 A_1 - 2\mu \sin \theta_1 \cos \theta_1 k_1^2 B_1 - \mu \cos 2\theta_3 k_3^2 A - \mu \cos 2\theta_3 k_3^2 B, \quad (5.51)
\end{align*}

\begin{align*}
[\lambda^{(1)} + (2\mu^{(1)} + K^{(1)}) \cos^2 \theta_0^{(1)}] k_0^{(1)} A_0^{(1)} &+ [\lambda^{(1)} + (2\mu^{(1)} + K^{(1)}) \cos^2 \theta_1^{(1)}] k_1^{(1)} A_1^{(1)} \\
- (2\mu^{(1)} + K^{(1)}) \sum_{p=3}^{4} \sin \theta_p^{(1)} \cos \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = [\lambda + 2\mu \cos^2 \theta_1] k_1^2 A_1 \\
+ [\lambda + 2\mu \cos^2 \theta_3] k_3^2 B_1 + 2\mu \sin \theta_3 \cos \theta_3 k_3^2 A - 2\mu \sin \theta_3 \cos \theta_3 k_3^2 B, \quad (5.52)
\end{align*}

\begin{align*}
\sum_{p=3}^{4} \frac{\omega_0^{(1)} k_0^{(1)} A_0^{(1)}}{c_4^{(1)} k_p^{(1)} A_p^{(1)}} = 0. \quad (5.53)
\end{align*}
5.4. Limiting cases

\[ \sin \theta_1 k_1 \exp[i k_1 \cos \theta_1 D] A_1 + \sin \theta_1 k_1 \exp[-i k_1 \cos \theta_1 D] B_1 \]

\[ - \cos \theta_3 k_3 \exp[i k_3 \cos \theta_3 D] A + \cos \theta_3 k_3 \exp[-i k_3 \cos \theta_3 D] B \]

\[ = \sin \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} - \sum_{p=3}^{4} \cos \theta_p^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}, \quad (5.54) \]

\[ \cos \theta_1 k_1 \exp[i k_1 \cos \theta_1 D] A_1 - \cos \theta_1 k_1 \exp[-i k_1 \cos \theta_1 D] B_1 \]

\[ + \sin \theta_3 k_3 \exp[i k_3 \cos \theta_3 D] A + \sin \theta_3 k_3 \exp[-i k_3 \cos \theta_3 D] B \]

\[ = \cos \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} + \sum_{p=3}^{4} \sin \theta_p^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}, \quad (5.55) \]

\[ 2 \mu \sin \theta_1 \cos \theta_1 k_1^2 \exp[i k_1 \cos \theta_1 D] A_1 - 2 \mu \sin \theta_1 \cos \theta_1 k_1^2 \exp[-i k_1 \cos \theta_1 D] B_1 \]

\[ = (2 \mu^{(2)} + K^{(2)}) \sin \theta_1^{(2)} k_1^{(2)} \cos \theta_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} \]

\[ - \sum_{p=3}^{4} \left[ \mu_p^{(2)} \cos \theta_p^{(2)} + K^{(2)} \cos^2 \theta_p^{(2)} + \frac{K^{(2)} \omega_0^{(2)} \cos \theta_p^{(2)}}{k_p^{(2)} \omega_0^{(2)} + 2 \omega_p^{(2)} - \omega_p^{(2)} - \omega_p^{(2)}} \right] \]

\[ \times k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}, \quad (5.56) \]

\[ [\lambda + 2 \mu \cos^2 \theta_1] k_1^2 \exp[i k_1 \cos \theta_1 D] A_1 + [\lambda + 2 \mu \cos^2 \theta_1] k_1^2 \exp[-i k_1 \cos \theta_1 D] B_1 \]

\[ + 2 \mu \sin \theta_3 \cos \theta_3 k_3^2 \exp[i k_3 \cos \theta_3 D] A - 2 \mu \sin \theta_3 \cos \theta_3 k_3^2 \exp[-i k_3 \cos \theta_3 D] B \]

\[ = [\lambda^{(2)} + (2 \mu^{(2)} + K^{(2)}) \cos^2 \theta_1^{(2)} k_1^{(2)} \exp[i k_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} \]

\[ +(2 \mu^{(2)} + K^{(2)}) \sum_{p=3}^{4} \sin \theta_p^{(2)} \cos \theta_p^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D] A_p^{(2)}, \quad (5.57) \]

\[ \sum_{p=3}^{4} \frac{\omega_0^{(2)} \cos \theta_0^{(2)} k_p^{(2)} \exp[i k_p^{(2)} \cos \theta_p^{(2)} D]}{k_p^{(2)} \omega_0^{(2)} + 2 \omega_0^{(2)} - \omega_p^{(2)} - \omega_p^{(2)}} A_p^{(2)} = 0. \quad (5.58) \]
These equations will enable us to provide the reflection and transmission coefficients corresponding to the relevant problem.

(iii) If we assume that the micropolar elastic solid medium $M^{(2)}$ is removed from the model of the problem and we extend the thickness of the chiral slab to infinite, then the problem will reduce to the incidence of a longitudinal displacement wave at a perfectly bonded interface between a micropolar/chiral elastic solid half-spaces. In this limiting case, the reflected waves propagating with phase speeds $V_s$ having amplitudes $B_s$ ($s = 1, 2, ..., 6$) will not appear in the medium $M$ as there is no other boundary. Hence, the system of equations given in the matrix equation (5.48) obtained separately corresponding to both the sets of boundary conditions, will reduce to the equations obtained in Section 4.3 for the relevant problem.

(iv) If we make the half-spaces $M^{(1)}$ and $M^{(2)}$ free from micropolarity and the chiral slab $M$ free from chirality and microrotation effects, then the problem will reduce to the model consisting of a uniform elastic solid slab interposed between two uniform elastic solid half-spaces. Here, the waves propagating with phase speeds $V_s^{(1)}$ and $V_s^{(2)}$ will not appear in the media $M^{(1)}$ and $M^{(2)}$, respectively. In the chiral medium $M$, if we take the waves propagating with phase speeds $V_3$ and $V_4$ as LCP waves and the waves traveling with phase speeds $V_5$ and $V_6$ as RCP waves, then we shall have only a longitudinal wave with phase speed $V_1 = \sqrt{(\lambda + 2\mu)/\rho}$ along with two transverse polarized waves, i.e., a LCP wave and a RCP wave, propagating with equal speeds $V_3 = V_5 = \sqrt{\mu/\rho}$ (say). After making the required changes into the equations (5.31), (5.34), (5.40) and (5.43), we obtain $A_3 = A_5 = A/2$ (say) and $B_3 = B_5 = B/2$ (say). The boundary conditions at the two interfaces $z = 0$ and $z = D$ are only corresponding to the continuity of all components of displacement and normal component of force stress tensor. Hence, the equations (5.30)-(5.35) and (5.39)-(5.44) reduce to the following equations

\[
\sin \theta_0^{(1)} k_1^{(1)} A_0^{(1)} + \sin \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \cos \theta_3^{(1)} k_3^{(1)} A_3^{(1)}
\]

\[
= \sin \theta_1 k_1 (A_1 + B_1) - \cos \theta_3 k_3 (A - B), \quad (5.59)
\]

\[
\cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sin \theta_3^{(1)} k_3^{(1)} A_3^{(1)}
\]

\[
= \cos \theta_1 k_1 (A_1 - B_1) + \sin \theta_3 k_3 (A + B), \quad (5.60)
\]
5.4. Limiting cases

\[ 2\mu^{(1)} \sin \theta_0^{(1)} \cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - 2\mu^{(1)} \sin \theta_1^{(1)} \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} - \mu^{(1)} \cos 2\theta_3^{(1)} k_3^{(1)} A_{3y}^{(1)} \]

\[ = 2\mu \sin \theta_1 \cos \theta_1 k_1^2 (A_1 - B_1) - \mu \cos 2\theta_3 k_3^2 (A + B), \quad (5.61) \]

\[ [\lambda^{(1)} + 2\mu^{(1)} \cos^2 \theta_0^{(1)}] k_1^{(1)} A_0^{(1)} + [\lambda^{(1)} + 2\mu^{(1)} \cos^2 \theta_1^{(1)}] k_1^{(1)} A_1^{(1)} - 2\mu^{(1)} \sin \theta_3^{(1)} \cos \theta_3^{(1)} k_3^{(1)} A_{3y}^{(1)} \]

\[ = [\lambda + 2\mu \cos^2 \theta_1] k_1^2 (A_1 + B_1) + 2\mu \sin \theta_3 \cos \theta_3 k_3^2 (A - B) , \quad (5.62) \]

\[
\begin{align*}
&\text{sin} \theta_1 \kappa_1 \exp [k_1 \cos \theta_1 D] A_1 + \text{sin} \theta_1 \kappa_1 \exp [-ik_1 \cos \theta_1 D] B_1 \\
&- \cos \theta_3 k_3 \exp [k_3 \cos \theta_3 D] A + \cos \theta_3 k_3 \exp [-ik_3 \cos \theta_3 D] B
\end{align*}
\]

\[ = \sin \theta_1^{(2)} \kappa_1^{(2)} \exp [ik_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} - \cos \theta_3^{(2)} k_3^{(2)} \exp [ik_3^{(2)} \cos \theta_3^{(2)} D] A_{3y}^{(2)} , \quad (5.63) \]

\[
\begin{align*}
&\text{cos} \theta_1 \kappa_1 \exp [k_1 \cos \theta_1 D] A_1 - \cos \theta_1 \kappa_1 \exp [-ik_1 \cos \theta_1 D] B_1 \\
&+ \sin \theta_3 k_3 \exp [k_3 \cos \theta_3 D] A + \sin \theta_3 k_3 \exp [-ik_3 \cos \theta_3 D] B
\end{align*}
\]

\[ = \cos \theta_1^{(2)} \kappa_1^{(2)} \exp [ik_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} + \sin \theta_3^{(2)} k_3^{(2)} \exp [ik_3^{(2)} \cos \theta_3^{(2)} D] A_{3y}^{(2)} , \quad (5.64) \]

\[ 2\mu \sin \theta_1 \cos \theta_1 k_1^2 \exp [k_1 \cos \theta_1 D] A_1 - 2\mu \sin \theta_1 \cos \theta_1 k_1^2 \exp [-ik_1 \cos \theta_1 D] B_1 \\
- \mu \cos 2\theta_3 k_3^2 \exp [k_3 \cos \theta_3 D] A - \mu \cos 2\theta_3 k_3^2 \exp [-ik_3 \cos \theta_3 D] B
\]

\[ = 2\mu^{(2)} \sin \theta_1^{(2)} \cos \theta_1^{(2)} k_1^{(2)} \exp [ik_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} , \quad (5.65) \]

\[ \begin{align*}
&[\lambda + 2\mu \cos^2 \theta_1] k_1^2 \exp [k_1 \cos \theta_1 D] A_1 + [\lambda + 2\mu \cos^2 \theta_1] k_1^2 \exp [-ik_1 \cos \theta_1 D] B_1 \\
&+ 2\mu \sin \theta_3 \cos \theta_3 k_3^2 \exp [k_3 \cos \theta_3 D] A - 2\mu \sin \theta_3 \cos \theta_3 k_3^2 \exp [-ik_3 \cos \theta_3 D] B
\end{align*}
\]

\[ = \begin{align*}
&[\lambda^{(2)} + 2\mu^{(2)} \cos^2 \theta_1^{(2)}] k_1^{(2)} \exp [ik_1^{(2)} \cos \theta_1^{(2)} D] A_1^{(2)} \\
&+ 2\mu^{(2)} \sin \theta_3^{(2)} \cos \theta_3^{(2)} k_3^{(2)} \exp [ik_3^{(2)} \cos \theta_3^{(2)} D] A_{3y}^{(2)} . \quad (5.66)
\end{align*}

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These equations enable us to provide the reflection and transmission coefficients for the relevant problem.

(v) If thickness of the chiral slab $M$ approaches to zero, then the we shall be left with the reflection and transmission problem of a longitudinal displacement wave impinging towards the plane interface between two micropolar elastic solid half-spaces. With the required substitutions, the equations (5.30)-(5.47) reduce to

\[
\sin \theta_0^{(1)} k_1^{(1)} A_0^{(1)} + \sin \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \cos \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \sin \theta_1^{(2)} k_1^{(2)} A_1^{(2)} - \sum_{p=3}^{4} \cos \theta_p^{(2)} k_p^{(2)} A_p^{(2)},
\]

(5.67)

\[
\cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)} + \sum_{p=3}^{4} \sin \theta_p^{(1)} k_p^{(1)} A_p^{(1)} = \cos \theta_1^{(2)} k_1^{(2)} A_1^{(2)} + \sum_{p=3}^{4} \sin \theta_p^{(2)} k_p^{(2)} A_p^{(2)},
\]

(5.68)

\[
(2\mu^{(1)} + K^{(1)}) \sin \theta_0^{(1)} \cos \theta_0^{(1)} k_1^{(1)} A_0^{(1)} - (2\mu^{(1)} + K^{(1)}) \sin \theta_1^{(1)} \cos \theta_1^{(1)} k_1^{(1)} A_1^{(1)}
\]

\[
- \sum_{p=3}^{4} \left[ \mu^{(1)} \cos 2\theta_p^{(1)} + K^{(1)} \cos^{2} \theta_p^{(1)} - \frac{K^{(1)} \omega_p^{(1)} k_p^{(1)} + \omega_p^{(1)} k_p^{(1)} + \omega_p^{(1)} k_p^{(1)}}{c_4^{(1)} k_p^{(1)} + \omega_p^{(1)} k_p^{(1)}} \right] A_p^{(1)}
\]

\[
= (2\mu^{(2)} + K^{(2)}) \sin \theta_0^{(2)} \cos \theta_0^{(2)} k_1^{(2)} A_0^{(2)} + (2\mu^{(2)} + K^{(2)}) \sin \theta_1^{(2)} \cos \theta_1^{(2)} k_1^{(2)} A_1^{(2)}
\]

(5.69)

\[
\left[ \lambda^{(1)} + \left(2\mu^{(1)} + K^{(1)}\right) \cos^{2} \theta_0^{(1)} k_1^{(1)} A_0^{(1)} + \left[ \lambda^{(1)} + \left(2\mu^{(1)} + K^{(1)}\right) \cos^{2} \theta_0^{(1)} k_1^{(1)} A_0^{(1)}
\]

\[
- \sum_{p=3}^{4} \sin \theta_p^{(1)} \cos \theta_0^{(1)} k_1^{(2)} A_p^{(1)} + \sum_{p=3}^{4} \sin \theta_p^{(2)} \cos \theta_0^{(2)} k_1^{(2)} A_p^{(2)}
\]

\[
= \left[ \lambda^{(2)} + \left(2\mu^{(2)} + K^{(2)}\right) \cos^{2} \theta_0^{(2)} k_1^{(2)} A_0^{(2)} + \left[ \lambda^{(2)} + \left(2\mu^{(2)} + K^{(2)}\right) \cos^{2} \theta_0^{(2)} k_1^{(2)} A_0^{(2)}
\]

(5.70)

\[
+ (2\mu^{(2)} + K^{(2)}) \sum_{p=3}^{4} \sin \theta_p^{(2)} \cos \theta_0^{(2)} k_1^{(2)} A_p^{(2)},
\]

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5.5. Numerical results and discussion

The above equations enable us to provide the reflection and transmission coefficients corresponding to the relevant problem.

5.5 Numerical results and discussion

Here, we have computed the absolute values of various reflection and transmission coefficients corresponding to Set-I and Set-II of the boundary conditions, for a specific model. Comparison in the respective coefficients obtained using the two sets of boundary equations, have been depicted graphically through Figures 5.14-5.17. The effect of chirality parameter \( C_3 \) and the effect of the thickness \( D \) of the chiral slab on the amplitude ratios corresponding to the boundary conditions of Set-I, are depicted through Figures 5.10-5.13 and Figures 5.18-5.35, respectively. The numerical values of material constants used in the numerical computations are given as follows:

<table>
<thead>
<tr>
<th>Medium ( M^{(1)} )</th>
<th>Medium ( M )</th>
<th>Medium ( M^{(2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^{(1)} = 75900 \times 10^6 \text{ N/m}^2 )</td>
<td>( \lambda = 500 \times 10^6 \text{ N/m}^2 )</td>
<td>( \lambda^{(2)} = 78500 \times 10^6 \text{ N/m}^2 )</td>
</tr>
<tr>
<td>( \mu^{(1)} = 13500 \times 10^6 \text{ N/m}^2 )</td>
<td>( \mu = 300 \times 10^6 \text{ N/m}^2 )</td>
<td>( \mu^{(2)} = 13700 \times 10^6 \text{ N/m}^2 )</td>
</tr>
<tr>
<td>( K^{(1)} = 149 \times 10^6 \text{ N/m}^2 )</td>
<td>( j = 0.01 \text{ m}^2 )</td>
<td>( K^{(2)} = 139 \times 10^6 \text{ N/m}^2 )</td>
</tr>
<tr>
<td>( \rho^{(1)} = 2200 \text{ kg/m}^3 )</td>
<td>( \rho = 1200 \text{ kg/m}^3 )</td>
<td>( \rho^{(2)} = 2700 \text{ kg/m}^3 )</td>
</tr>
<tr>
<td>( j^{(1)} = 0.00000196 \text{ m}^2 )</td>
<td>( \alpha = 2 \times 10^6 \text{ N} )</td>
<td>( j^{(2)} = 0.0000212 \text{ m}^2 )</td>
</tr>
<tr>
<td>( \alpha^{(1)} = 0.01 \times 10^6 \text{ N} )</td>
<td>( \beta = 4 \times 10^6 \text{ N} )</td>
<td>( \alpha^{(2)} = 0.01 \times 10^6 \text{ N} )</td>
</tr>
<tr>
<td>( \beta^{(1)} = 0.015 \times 10^6 \text{ N} )</td>
<td>( \gamma = 5 \times 10^6 \text{ N} )</td>
<td>( \beta^{(2)} = 0.015 \times 10^6 \text{ N} )</td>
</tr>
<tr>
<td>( \gamma^{(1)} = 0.0268 \times 10^6 \text{ N} )</td>
<td>( C_3 = 20 \times 10^6 \text{ N/m} )</td>
<td>( \gamma^{(2)} = 0.0288 \times 10^6 \text{ N} )</td>
</tr>
</tbody>
</table>

We have taken \( D = 0.001 \text{ m} \) and the non-dimensional frequency \( \omega_R( = \omega^{(1)} / \omega^{(1)}_0 ) = 10 \). The values of elastic constants in the micropolar elastic solid half-spaces are taken arbitrarily, while those in the chiral slab are taken from Elphinstone and Lakhtakia (1994a). From the above data, the characteristic length \( (\sqrt{\lambda^{(1)}} / \mu^{(1)}) \) in the micropolar...
The variation of modulus of amplitude ratios of various reflected and refracted waves versus angle of incidence of longitudinal displacement wave traveling with phase speed $V_1^{(1)}$ computed using the boundary conditions given in Set-I, are depicted through Figures 5.2-5.13.

In Figure 5.2, the reflection coefficient $Z_1$ begins with the value 0.7385 near normal incidence, then its value decreases slowly to a certain value with increase in the angle of incidence $\theta_0^{(1)}$ lying in the range $0^\circ < \theta_0^{(1)} \leq 50^\circ$. Thereafter, its value increases rapidly and takes the value equal to unity, at grazing incidence. In the range $0^\circ < \theta_0^{(1)} < 90^\circ$, the variation of the reflection coefficients $Z_2$ and $Z_3$ are parabolic of different scaling, but at normal and grazing incidences, both the coefficients vanish. We have plotted the curve corresponding to the amplitude ratio $Z_2$ after magnifying its original value by the factor 10. In fact, the absolute value of the coefficient $Z_3$ is greater than the absolute value of the coefficient $Z_2$ at each angle of incidence.

In Figure 5.3, both the transmission coefficients $Z_4$ and $Z_5$ begin with their maximum values, i.e., 0.1176 and 0.2593, respectively, near normal incidence, then they decrease monotonically with the increase in the angle of incidence and ultimately vanish at polar half-space $M^{(1)}$ is $0.1408 \times 10^{-2}$ m and the wavelength of the incident wave is $0.3683 \times 10^{-2}$ m.
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grazing incidence. All the transmission coefficients corresponding to the transverse circularly polarized waves, namely, $Z_6, Z_7, Z_8$ and $Z_9$, are zero at normal incidence. They increase with increase in the angle of incidence till certain angles to attain their respective maximum values. Thereafter, they decrease to zero as $\theta_0^{(1)}$ approaches to grazing incidence. We have plotted the curves corresponding to the transmission coefficients $Z_6, Z_7, Z_8$ and $Z_9$ after magnifying their original values with the factors $10^2, 10^2, 10$ and 10, respectively.

![Figure 5.4: Variation of modulus of reflection coefficients in medium $M$ with angle of incidence](image)

![Figure 5.5: Variation of modulus of transmission coefficients in medium $M^{(2)}$ with angle of incidence](image)

In Figure 5.4, it can be seen that the reflection coefficients $Z_{10}$ and $Z_{11}$ begin with certain finite values at normal incidence. Afterwards, the modulus value of the coefficient $Z_{10}$ increases as $\theta_0^{(1)}$ increases through the range $0^0 < \theta_0^{(1)} \leq 58^0$, thereafter, its value decreases sharply to achieve the value zero, at grazing incidence, while the coefficient $Z_{11}$ goes on decreasing monotonically and attains the value zero at grazing incidence. We also observe that the reflection coefficients $Z_{12}, Z_{13}, Z_{14}$ and $Z_{15}$ have similar pattern with $\theta_0^{(1)}$. All of them begin with the value zero, then they increase with increase in the angle of incidence till certain angles, where they attain their respective maximum values. Afterwards, these coefficients decrease with further increase in angle of incidence and finally vanish at grazing incidence. We have plotted the curves corresponding to the reflection coefficients $Z_{10}, Z_{12}, Z_{13}, Z_{14}$ and $Z_{15}$ after magnifying their
original values with the factor 10.

In Figures 5.3 and 5.4, we note that the coefficients $Z_6$ and $Z_7$ have same values at each angle of incidence. This fact is also true with the pairs of coefficients $(Z_8, Z_9)$, $(Z_{12}, Z_{13})$ and $(Z_{14}, Z_{15})$.

In Figure 5.5, the transmission coefficient $Z_{16}$ begins with its maximum value, i.e., 0.2373 near normal incidence, then it decreases continuously with increase in the angle of incidence and finally vanishes at grazing incidence. We notice that the behavior of the transmission coefficients $Z_{17}$ and $Z_{18}$ with $\theta_0^{(1)}$ is similar as that of the coefficients $Z_2$ and $Z_3$. The curves corresponding to the transmission coefficients $Z_{17}$ and $Z_{18}$ have been plotted after magnifying their original values with the factors 10 and $10^4$, respectively.

In Figures 5.6-5.9, we have shown the variation of absolute values of various reflection and transmission coefficients obtained from the Set-I, with the non-dimensional frequency $\omega_R$, at $\theta_0^{(1)} = 30^\circ$. We observed that all the coefficients are found to be affected with $\omega_R$, but the coefficients corresponding to transverse waves are found to be affected the most.

**Figure 5.6:** Variation of modulus of reflection coefficients in medium $M^{(1)}$ with frequency ratio ($\omega_R$) at angle of incidence $\theta_0^{(1)} = 30^\circ$

**Figure 5.7:** Variation of modulus of transmission coefficients in medium $M$ with frequency ratio ($\omega_R$) at angle of incidence $\theta_0^{(1)} = 30^\circ$
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![Figure 5.8: Variation of modulus of reflection coefficients in medium $M$ with frequency ratio ($\omega R$) at angle of incidence $\theta_0 = 30^\circ$](image)

![Figure 5.9: Variation of modulus of transmission coefficients in medium $M^{(2)}$ with frequency ratio ($\omega R$) at angle of incidence $\theta_0 = 30^\circ$](image)

![Figure 5.10: Variation of modulus of reflection coefficients in medium $M^{(1)}$ with angle of incidence (solid curve: $C_3 = 80 \times 10^6$ N/m, dashed curve: $C_3 = 20 \times 10^6$ N/m)](image)

![Figure 5.11: Variation of modulus of transmission coefficients in medium $M$ with angle of incidence (solid curve: $C_3 = 80 \times 10^6$ N/m, dashed curve: $C_3 = 20 \times 10^6$ N/m)](image)

Next, we have shown the effect of parameter ($C_3$) on the reflection and transmission coefficients by taking two different values, namely, $C_3 = 20 \times 10^6$ N/m and $C_3 = 80 \times 10^6$ N/m. It is evident from Figures 5.10-5.13 that the coefficients depend on...
Chapter 5

Transmitted waves

In Figure 5.11, it can be seen that the values of the coefficients $Z_5$ and $Z_{11}$ decrease with increase of $C_3$ and all the other coefficients, except the coefficient $Z_4$ (which is almost independent of $C_3$), increase with increase of $C_3$, at each angle of incidence. From Figure 5.13, one can observe that the transmission coefficients in medium $M^{(3)}$ decrease with increase of $C_3$, at each angle incidence, in general.

In Figures 5.14-5.17, we have depicted the comparison of various coefficients computed separately corresponding to Set-I and Set-II, against the angle of incidence. From Figure 5.14, it can be seen that the reflection coefficient $Z_1$ computed corresponding to Set-II is greater than that of computed corresponding to Set-I, at each angle of incidence. The gap between them goes on decreasing as $\theta^{(1)}$ tends to grazing incidence. The pattern of variation of the coefficients $Z_2$ and $Z_3$ are found to be almost same for
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responding to Set-I and Set-II. In Figures 5.15 and 5.16, it can be noted that at each angle of incidence, the coefficients $Z_5, Z_6, Z_9, Z_{11}, Z_{12}, Z_{13}, Z_{14}$ and $Z_{15}$ corresponding to Set-I are greater than those corresponding to Set-II, while the coefficients $Z_4, Z_6, Z_7$ and $Z_{10}$ corresponding to Set-II are greater than those corresponding to Set-I.

Figure 5.14: Variation of modulus of reflection coefficients in medium $M^{(1)}$ with angle of incidence (solid curve: Set-I, dashed curve: Set-II)

Figure 5.15: Variation of modulus of transmission coefficients in medium $M$ with angle of incidence (solid curve: Set-I, dashed curve: Set-II)

Figure 5.16: Variation of modulus of reflection coefficients in medium $M$ with angle of incidence (solid curve: Set-I, dashed curve: Set-II)

Figure 5.17: Variation of modulus of transmission coefficients in medium $M^{(2)}$ with angle of incidence (solid curve: Set-I, dashed curve: Set-II)
From Figure 5.17, we note that there is significant difference between the transmission coefficient $Z_{16}$ computed corresponding to Set-I and that computed corresponding to Set-II, however this difference decreases with increase of the angle of incidence and vanishes at grazing incidence. The gap between the respective values of the transmission coefficients $Z_{17}$ and $Z_{18}$ computed corresponding to Set-I and Set-II, is significant and on similar pattern with the angle of incidence, but it is much appreciable in case of the coefficient $Z_{18}$.

Next, we have shown the variation of the reflection and transmission coefficients obtained by utilizing the boundary conditions of Set-I with the angle of incidence of a longitudinal displacement wave propagating with speed $V_1^{(1)}$ at three different values of thickness of the chiral slab, namely, $D = 0.001$ m, 0.01 m and 0.1 m.

From Figures 5.18-5.35, we can see that the curves corresponding to the variation of respective amplitude ratios are found to be smooth for smaller value of the thickness of the chiral slab, i.e., as the value of the thickness of chiral slab increases, significant fluctuations in all the amplitude ratios with angle of incidence are observed. It is hence verified that the amplitude ratios of various reflected and transmitted waves in all media $M^{(1)}$, $M$ and $M^{(2)}$ depend very much on the thickness as well as on the chirality parameter of the chiral slab.
5.5. Numerical results and discussion

![Figure 5.20](image1.png): Variation of modulus of amplitude ratio $Z_3$ with angle of incidence

![Figure 5.21](image2.png): Variation of modulus of amplitude ratio $Z_4$ with angle of incidence

![Figure 5.22](image3.png): Variation of modulus of amplitude ratio $Z_5$ with angle of incidence

![Figure 5.23](image4.png): Variation of modulus of amplitude ratio $Z_6$ with angle of incidence
Figure 5.24: Variation of modulus of amplitude ratio $Z_7$ with angle of incidence

Figure 5.25: Variation of modulus of amplitude ratio $Z_8$ with angle of incidence

Figure 5.26: Variation of modulus of amplitude ratio $Z_9$ with angle of incidence

Figure 5.27: Variation of modulus of amplitude ratio $Z_{10}$ with angle of incidence
5.5. Numerical results and discussion

**Figure 5.28:** Variation of modulus of amplitude ratio $Z_{11}$ with angle of incidence

**Figure 5.29:** Variation of modulus of amplitude ratio $Z_{12}$ with angle of incidence

**Figure 5.30:** Variation of modulus of amplitude ratio $Z_{13}$ with angle of incidence

**Figure 5.31:** Variation of modulus of amplitude ratio $Z_{14}$ with angle of incidence
Figure 5.32: Variation of modulus of amplitude ratio $Z_{15}$ with angle of incidence

Figure 5.33: Variation of modulus of amplitude ratio $Z_{16}$ with angle of incidence

Figure 5.34: Variation of modulus of amplitude ratio $Z_{17}$ with angle of incidence

Figure 5.35: Variation of modulus of amplitude ratio $Z_{18}$ with angle of incidence
5.6 Conclusions

A detailed theoretical analysis has been carried out to understand the response of a longitudinal displacement wave propagating through a micropolar elastic solid medium and striking on the chiral slab of uniform thickness, which is sandwiched between two different micropolar elastic solid half-spaces. There are two possible sets of boundary conditions at the two bi-material interfaces. The amplitude ratios are obtained independently corresponding to both the sets of boundary conditions. We conclude that

1. At normal incidence, the reflection and transmission of only longitudinal waves take place and no set of coupled transverse waves is found to reflect or transmit, in all the regions. At grazing incidence, no reflection/transmission phenomena take place.

2. Reflection and transmission coefficients are found to dependent upon the frequency, angle of incident wave, chirality and thickness of the interposed slab.

3. At normal incidence, the effect of chirality parameter is found to be maximum on the amplitude ratios of the reflected and transmitted longitudinal waves, which decrease with the increase of angle of incidence. No effect of chirality parameter on any coefficient is observed at grazing incidence. At intermediate angles of incidence, the effect of chirality on various amplitude ratios is found to be the maximum.

4. For a very thin slab, the variation in all the amplitude ratios with the angle of incidence are found to be smooth enough. But as the thickness of the chiral slab becomes significant, we obtain more and more fluctuations in the variation of these coefficients with the angle of incidence.