Chapter 4

Transmission of longitudinal wave at a plane interface between micropolar elastic and chiral solid half-spaces: Incidence from micropolar half-space

4.1 Introduction

In the previous chapters, we have assumed that the material is isotropic. However, some materials are not invariant to coordinate inversions and therefore, possess a kind of anisotropy. This type of anisotropy results in qualitatively different behavior in comparison to isotropic solids. A micropolar solid which is isotropic with respect to coordinate rotations but not with respect to reflections is called as mirror asymmetric, non-centrosymmetric, acentric or Chiral [Weitsman (1967)]. Such materials are also known as hemitropic micropolar solids. Materials can exhibit chirality on atomic scale (quartz, sugar, biological molecules) and also on a larger scale (bones, porous materials, composites containing fibers or inclusions) [Lakes (2001)]. Chiral media, due to lack of geometric symmetry between an object and its mirror image, have been known in optics as optically active materials. They are characterized by an intrinsic left- or right- handedness at optical frequencies, due to helical natural structure and

\[ \text{Journal of Sound and Vibration, 311(3-5), 973-990(2008).} \]
4.1. Introduction

hence, cannot be made to coincide with their mirror image by any operation involving rotations and/or translations. Therefore, waves of various circular polarizations propagate in these media at different phase velocities. The phenomenon of optical activity or mirror asymmetry in substances has been known for almost two centuries and was discovered independently by Arago (1811) and Biot (1812, 1817). It has been extensively utilized by the physical chemists for characterizing the molecular structure. The physical phenomenon of chiral materials was first observed in optical spectrum due to the transverse nature of optical wave [Fresnel (1866)]. Bose (1898) was the first person to develop artificial chiral composite materials. These artificial chiral composites were used to design broadband absorbers and filters [Bose (1898), Ro et al. (1992), Cory and Rosenhouse (1997)]. This gives rise to the idea that the chiral materials may also be applied in the elastic spectrum for designing broadband absorbers and impedance transformers. To attain this goal, the constitutive relations [Lakes and Benedict (1982)] and the governing equations [Nowacki (1986)] for non-centrosymmetric micropolar materials have been proposed. Such a medium undergoes homogeneous deformation and can support couple stresses and spin inertia. Chiral solid (i.e., non-centrosymmetric micropolar elastic solid) has three new elastic constants in addition to the six considered in fully isotropic micropolar solid. Lakhtakia et al. (1988) have shown that acoustically chiral composites can be constructed by embedding chiral microstructures or springs in a host medium due to which the microstructure possesses a screw-like property or a handedness. They derived dispersion equations and shown that there are six independent wavenumbers, two of them represent non-dispersive longitudinal fields, while the remaining four are dispersive circularly polarized transverse fields.

Lakes and Benedict (1982) studied the problem of twisting deformation of a hemitropic micropolar cylinders when subjected to axial tension. Lakhtakia et al. (1990) and Yang and Hsia (1997, 2000) explored the problems of reflection and transmission of an elastic plane wave impinging at a plane discontinuity between the elastic solid and the chiral material in welded contact. The inability of the achiral medium to support the microrotation field and the couple stress dyadic provides the option of two distinct sets of boundary conditions at achiral-chiral interface. It has been observed that both the sets of boundary conditions results in satisfaction of conservation of energy. Yang and Hsia (1995) extended the concept and reported that two transition frequencies of the dispersion equation divide the frequency response of the transverse wavenumbers into
three varying groups and hence the four transverse modes can only be distinguished in a specified frequency range. In this chapter, we have discussed the reflection and transmission phenomena of a plane longitudinal displacement wave impinging obliquely at a plane interface between a micropolar elastic solid half-space and a chiral elastic solid half-space. The incident wave strikes at the plane interface after propagating through the micropolar elastic solid half-space. The reflection and transmission coefficients are obtained by utilizing two possible sets of boundary conditions, for a specific model and their corresponding values are also compared graphically. The effect of chirality parameter on various reflection and transmission coefficients have been noticed and depicted graphically. Results of some earlier workers have also been reduced as special cases from the present formulation.

4.2 Field equations and relations

We consider a model consisting of a micropolar elastic solid half-space and an elastic chiral solid half-space, which are separated by a plane interface. Following Eringen (1966), the constitutive relations and equations of motion in the absence of body force and body couple densities for a uniform micropolar elastic solid are given by

\begin{align}
t_{ij} &= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) + K (u_{j,i} - c_{ijkl} \phi_k), \\
m_{ij} &= \alpha \psi_{k,k} \delta_{ij} + \beta \phi_{i,j} + \gamma \phi_{j,i},
\end{align}

and

\begin{align}
(\lambda + 2\mu + K) \nabla \cdot \mathbf{u} - (\mu + K) \nabla \times \nabla \times \mathbf{u} + K \nabla \times \phi &= \rho \ddot{u}, \\
(\alpha + \beta + \gamma) \nabla \cdot \phi - \gamma \nabla \times \nabla \times \phi + K \nabla \times \mathbf{u} - 2K \phi &= \rho \ddot{\phi}.
\end{align}

Using the expressions of \( \mathbf{u} \) and \( \phi \) from (2.11) and (2.12) into equations (4.3) and (4.4), Parfitt and Eringen (1969) have shown that there exist four basic waves propagating with distinct phase speeds in an infinite micropolar elastic solid. They are as follows.
4.2. Field equations and relations

(i) a longitudinal displacement wave traveling with phase speed $V_1$ given by

$$V_1 = \left[ \frac{\lambda + 2\mu + K}{\rho} \right]^{1/2},$$

(ii) a longitudinal microrotational wave propagating with phase speed $V_2$ given as

$$V_2 = \left[ \frac{\alpha + \beta + \gamma}{\rho j(1 - \Omega)} \right]^{1/2},$$

where $\Omega = \frac{2\omega^2}{\omega_0^2}$, $\omega_0^2 = \frac{K}{\rho j}$ and $\omega = kV$ is defined earlier.

(iii) two sets of coupled transverse waves propagating with phase speeds $V_{3,4}$ given by

$$V_{3,4} = \left[ \frac{1}{2(1 - \Omega)} \{ \varepsilon \pm \sqrt{\varepsilon^2 - 4c_2^2(1 - \Omega)(c_2^2 + c_4^2)} \} \right]^{1/2},$$

where the ‘+’ and ‘−’ signs correspond to $V_3$ and $V_4$, respectively,

$$\varepsilon = c_2^2 + c_4^2(1 - \Omega) + c_4^2 \left( 1 - \Omega \frac{2}{\Omega} \right), \quad c_2^2 = \frac{\mu}{\rho}, \quad c_3^2 = \frac{K}{\rho}, \quad c_4^2 = \frac{\gamma}{\rho j}$$

and $\Omega$ is defined earlier. Each set of coupled transverse waves consists of two transverse motions; a transverse displacement and a transverse microrotation normal to it.

Following Lakes and Benedict (1982) and Nowacki (1986), the constitutive relations and elastodynamic equations with vanished body force and body couple densities for a linear chiral solid are given by

$$\dot{\epsilon}_{ij} = \lambda \dot{u}_{k,j} \delta_{ij} + \mu \left( \dot{u}_{ij} + \dot{u}_{ji} \right) + C_3 \dot{\phi}_{j,k},$$

$$m_{ij} = \alpha \dot{\phi}_{k,j} \delta_{ij} + \beta \dot{\phi}_{j,k} + \gamma \dot{\phi}_{j,j} + C_3 \dot{u}_{j,i} - C_3 \delta_{ij} \dot{\phi}_k,$$

and

$$\left( \lambda^* + \mu^* \right) \nabla \cdot \dot{u}' + \mu^* \nabla^2 \dot{u}' + C_3 \nabla^2 \dot{\phi}' = \rho \dot{u}',$$

$$\left( \alpha^* + \beta^* \right) \nabla \cdot \dot{\phi}' + \gamma^* \nabla^2 \dot{\phi}' + 2C_3 \nabla \times \dot{\phi}' + C_3 \nabla^2 \dot{u}' = \rho \dot{\phi}'$$
where \( u' \) and \( \phi' \) are, respectively, the displacement and the microrotation vectors in an elastic chiral medium, \( \sigma'_{ij} \) is the force stress tensor, \( m'_{ij} \) is the couple stress tensor, \( \lambda' \), \( \mu' \), \( \alpha' \), \( \beta' \), \( \gamma' \) are the corresponding elastic parameters in the chiral medium, \( C_3 \) is the hemitropic constant, \( \rho' \) and \( \rho' \alpha' \) are, respectively, the mass density and the moment of inertia per unit volume.

Decomposing the vectors \( u' \) and \( \phi' \) into scalar potentials \( q' \) and \( \xi' \), and vector potentials \( U' \) and \( V' \), respectively, as we have decomposed the vectors \( u \) and \( \phi \) in (2.11) and (2.12) earlier. Then inserting their expressions into the equations (4.7) and (4.8), we obtain the following dispersion relations after substituting the expression of plane harmonic wave [see Lakhtakia et al. (1990)]

\[
V'' - \left[ c_1'^2 + c_2'^2 + c_3'^2 \right] V'' + \left[ c_4'^2 (c_4'^2 + c_5'^2) - c_6'^2 c_7'^2 \right] = 0, \tag{4.9}
\]

\[
\left[ (V'' - c_4'^2) (V'' - c_5'^2) - c_6'^2 c_7'^2 k^2 - (2c_7'^2 c_8'^2 - V'') \right]^2 = 0, \tag{4.10}
\]

where

\[
c_1'^2 = \frac{\lambda' + 2\mu'}{\rho'}, \quad c_2'^2 = \frac{\mu'}{\rho'}, \quad c_3'^2 = \frac{\gamma'}{\rho' \alpha'},
\]

\[
c_4'^2 = \frac{\alpha' + \beta'}{\rho' \beta'}, \quad c_5'^2 = \frac{C_3}{\rho'}, \quad c_6'^2 = \frac{c_6'^2}{\rho'}, \quad c_7'^2 = \frac{C_3}{\rho' \beta'}.
\]

Lakhtakia et al. (1988) have shown that there exist six sets of basic waves traveling with different phase speeds in an infinite effective chiral medium. They are as follows:

(i) two sets of coupled non-dispersive longitudinal waves traveling with phase speeds \( V_1' \) and \( V_2' \). Each set of coupled longitudinal waves consists of two motions; a longitudinal displacement and a longitudinal microrotation. The phase speeds of these sets of coupled longitudinal waves can be obtained from the dispersion relation given in (4.9) and the expressions of square of these phase speeds are given by

\[
V_{1,2}'^2 = \Gamma' \pm \sqrt{\Gamma'^2 - \left\{ c_4'^2 (c_4'^2 + c_5'^2) - c_6'^2 c_7'^2 \right\}}, \quad \Gamma' = \frac{1}{2} (c_4'^2 + c_5'^2 + c_6'^2), \tag{4.11}
\]

(ii) four sets of dispersive coupled transverse waves traveling with phase speeds \( V_i' \) (\( i = 3, 4, 5, 6 \)). Each set of these coupled transverse waves consists of two motions; a trans-
verse displacement and a transverse microrotation. The square of phase speeds $V_i'^2$ $(i = 3, 4, 5, 6)$ are the roots of the dispersion equation (4.10). Out of these four distinct sets of coupled transverse waves, the two sets are coupled right circularly polarized (RCP) waves and the remaining two sets are the coupled left circularly polarized (LCP) waves.

4.3 Reflection and transmission of waves

Let $z = 0$ be the plane interface between the micropolar elastic solid half-space and chiral elastic half-space. Let the upper half-space $R_1$ be occupied by micropolar elastic solid and the lower half-space $R_2$ be occupied by an elastic chiral solid, where $R_1$ and $R_2$ are

$$R_1 = \{(x, z) : -\infty < x < \infty, -\infty < z < 0\}$$

and

$$R_2 = \{(x, z) : -\infty < x < \infty, 0 < z < \infty\}.$$

We consider a train of time harmonic plane longitudinal displacement wave of amplitude $A_0$ propagating with speed $V_1$ through the micropolar elastic solid half-space and striking at the interface $z = 0$ at an angle $\theta_0$ with the $z-$axis. To satisfy the boundary conditions at the interface, we take the following reflected and transmitted waves into consideration.

**Reflected waves:**
(i) a longitudinal displacement wave of amplitude $A_1$ traveling with speed $V_1$ and making an angle $\theta_1$ with the normal,
(ii) two sets of coupled transverse waves propagating with speeds $V_{3,4}$ and making angles $\theta_{3,4}$ with the normal.

**Transmitted waves:**
(i) two sets of coupled longitudinal waves of amplitudes $A_{1,2}'$ traveling with speeds $V_{1,2}'$ and making angles $\theta_{1,2}'$ with the normal,
(ii) four sets of coupled transverse waves of amplitudes $A_i'$ propagating with phase speeds $V_i'$ and making angles $\theta_i'$ $(i = 3, 4, 5, 6)$, with the normal.
Chapter 4

Figure 4.1: Schematic diagram of the problem

The Schematic diagram of the problem is shown in Figure 4.1. We take the following potentials in the region $R_1$,

$$q = A_0 P_0^+ + A_1 P_1^-, \quad (4.12)$$

$$U = \sum_{p=3}^{4} A_{pp} \hat{e}_y P_p^-, \quad (4.13)$$

$$\Pi = \sum_{p=3}^{4} (B_{px} \hat{e}_x + B_{pz} \hat{e}_z) P_p^-,$$  \hspace{1cm} (4.14)

and the relevant potentials in the region $R_2$,

$$q' = \sum_{p=1}^{2} A_{p'p'} P_p'^-,$$  \hspace{1cm} (4.15)
4.3. Reflection and transmission of waves

\[ \xi' = \sum_{p=1}^{2} D_p A_p' P_p'^{+}, \quad (4.16) \]

\[ U' = \sum_{p=3}^{6} (\Delta_1^p \hat{e}_x + \Delta_2^p \hat{e}_y + \Delta_3^p \hat{e}_z) A_p' P_p'^{+}, \quad (4.17) \]

\[ \Pi' = \sum_{p=3}^{6} D_p (\Delta_1^p \hat{e}_x + \Delta_2^p \hat{e}_y + \Delta_3^p \hat{e}_z) A_p' P_p'^{+}, \quad (4.18) \]

where \( P_0^+ = \exp[i(k_1(\sin \theta x + \cos \theta z) - \omega_1 t)], \ P_p^- = \exp[i(k_p(\sin \theta_p x - \cos \theta_p z) - \omega_p t)], \ P_p^{++} = \exp[i(k_p'(\sin \theta_p' x + \cos \theta_p' z) - \omega_p t)], \omega_l = k_l V_l (l = 1, 3, 4) \) and \( \omega'_r = k_r' V'_r (r = 1, 2, ..., 6) \) have been defined earlier, \( \hat{e}_x, \hat{e}_y \), and \( \hat{e}_z \) are the Cartesian unit vectors, \( A_p \) and \( B_p \) \((p = 3, 4)\) are, respectively, the amplitudes of the reflected transverse displacement wave and the reflected transverse microrotation wave.

The coefficients \( A_p \) and \( B_p \) \((p = 3, 4)\) are related to each other through the relation given by [Parfitt and Eringen (1969)]

\[ B_p = \frac{\omega_1^2 A_{py}}{k_p'(c_p^2 + 2\omega_0^2 k_p - \nu_p^2)} (\cos \theta_p \hat{e}_x + \sin \theta_p \hat{e}_z). \quad (4.19) \]

The expressions of coefficients \( D_p \) are [Yang and Hsia (1995)]

\[ D_p = \begin{cases} 
\frac{\rho' \omega'^2 - k_p^2 (\nu' + 2\mu')}{k_p^2 C_3} & p = 1, 2 \\
\frac{\rho' \omega'^2 - k_p^2 \mu'}{k_p^2 C_3} & p = 3, 4, 5, 6.
\end{cases} \quad (4.20) \]

When a RCP or LCP plane wave propagates in the \( x - z \) plane, the presentation of \( \Delta_1^p, \ \Delta_2^p \) and \( \Delta_3^p \) can be specified as

\[ \Delta_1^p : \Delta_2^p : \Delta_3^p = \pm \cos \theta_p' : 1 : \mp \sin \theta_p', \]

where the upper signs '+' in \( \Delta_1^p \) and '-' in \( \Delta_3^p \) refer to the RCP plane waves and the lower signs '-' in \( \Delta_1^p \) and '+' in \( \Delta_3^p \) refer to the LCP plane waves.
We see that there are nine unknown in the equations (4.12)-(4.18), thus we need nine linearly independent boundary conditions at the plane interface \( z = 0 \). Classical elasticity theory provides us six conditions, i.e., corresponding to the continuity of displacement and surface traction [Achenbach (1973)]

At the interface \( z = 0 \)

\[
\begin{align*}
\mathbf{u}^\text{(tr)} &= \mathbf{u}^\text{(ref)} + \mathbf{u}^\text{(inc)}, \\
\mathbf{u}^\text{(tr)}_2 &= \mathbf{u}^\text{(ref)}_2 + \mathbf{u}^\text{(inc)}_2, \\
\mathbf{u}^\text{(tr)}_3 &= \mathbf{u}^\text{(ref)}_3 + \mathbf{u}^\text{(inc)}_3, \\
\mathbf{t}^\text{(tr)}_{zz} &= \mathbf{t}^\text{(ref)}_{zz} + \mathbf{t}^\text{(inc)}_{zz}, \\
\mathbf{t}^\text{(tr)}_{zy} &= \mathbf{t}^\text{(ref)}_{zy} + \mathbf{t}^\text{(inc)}_{zy}, \\
\mathbf{t}^\text{(tr)}_{zz} &= \mathbf{t}^\text{(ref)}_{zz} + \mathbf{t}^\text{(inc)}_{zz}.
\end{align*}
\]  

(4.21)

(4.22)

In micropolar continuum, in addition to the above conditions, the following three boundary conditions on microrotation field are required at \( z = 0 \), i.e.,

\[
\phi^\text{(tr)}_1 = \phi^\text{(ref)}_1, \quad \phi^\text{(tr)}_2 = \phi^\text{(ref)}_2, \quad \phi^\text{(tr)}_3 = \phi^\text{(ref)}_3.
\]  

(4.23)

Equations (4.21)-(4.23) constitute a set (say Set-I) of boundary conditions and are suffice to solve the boundary value problem. However, there is another set (say Set-II) of boundary conditions possible as well at the interface which comprise of the continuity of the normal component of the couple stress, i.e., at \( z = 0 \)

\[
\begin{align*}
\mathbf{m}^\text{(tr)}_{zx} &= \mathbf{m}^\text{(ref)}_{zx} + \mathbf{m}^\text{(inc)}_{zx}, \\
\mathbf{m}^\text{(tr)}_{zy} &= \mathbf{m}^\text{(ref)}_{zy} + \mathbf{m}^\text{(inc)}_{zy}, \\
\mathbf{m}^\text{(tr)}_{zz} &= \mathbf{m}^\text{(ref)}_{zz} + \mathbf{m}^\text{(inc)}_{zz}.
\end{align*}
\]  

(4.24)

together with the conditions mentioned in (4.21) and (4.22).

Since the basic difference between the micropolar elasticity and that of the classical elasticity is the introduction of microrotation vector \((\phi)\) and couple stress tensor \((m_{kl})\). Therefore, in a micropolar-chiral problem, both the microrotation and couple stress must be continuous at the interface. It can be seen that whichever set of the boundary conditions is used, a system of nine equations is obtained [see Elphinstone and Lakhtakia (1994a, 94b)]. Here, we have obtained the solution for the unknown using both the possible sets (Set-I and Set-II) of boundary conditions given through equations (4.21)-(4.23) and (4.21), (4.22) and (4.24) respectively.
4.3. Reflection and transmission of waves

Inserting the expressions of potentials given in (4.12)-(4.18) into the Set-I of the boundary conditions and assuming

\[ k_i \sin \theta_i = k_1 \sin \theta_1 = k'_i \sin \theta'_i \quad \text{and} \quad \omega_l = \omega'_l \ (l = 1, 3, 4; \ r = 1, 2, ..., 6), \ \text{at} \ z = 0, \]

we obtain

\[
\begin{align*}
\sin \theta_0 k_1 A_0 + \sin \theta_1 k_1 A_1 + \sum_{p=3}^{4} \cos \theta_p k_p A_{py} &= \\
&= \sum_{p=1}^{2} \sin \theta'_p k'_p A'_p - \sum_{p=3}^{6} \cos \theta'_p k'_p A'_p, \quad (4.25)
\end{align*}
\]

\[
\sum_{p=3}^{4} k'_p A'_p - \sum_{p=5}^{6} k'_p A'_p = 0, \quad (4.26)
\]

\[
\begin{align*}
\cos \theta_0 k_1 A_0 - \cos \theta_1 k_1 A_1 + \sum_{p=3}^{4} \sin \theta_p k_p A_{py} &= \\
&= \sum_{p=1}^{2} \cos \theta'_p k'_p A'_p + \sum_{p=3}^{6} \sin \theta'_p k'_p A'_p, \quad (4.27)
\end{align*}
\]

\[
(2\mu + K) \sin \theta_0 \sin k_1^2 A_0 - (2\mu + K) \sin \theta_1 \cos k_1^2 A_1 - \sum_{p=3}^{4} (\mu \cos 2\theta_p + K \cos^2 \theta_p)
\]

\[
- \frac{K\omega_0^2}{k_1^2 (c_1^2 + 2\omega_0 k_1^2 - V_p^2)} k_1^2 A_{py} = \sum_{p=1}^{2} (2\mu' + C_3 D_p) \sin \theta'_p \cos \theta'_p k'_p A'_p
\]

\[
- \sum_{p=3}^{6} (\mu' \cos 2\theta'_p + C_3 \cos^2 \theta'_p D_p) k'_p A'_p,
\]

\[
\sum_{p=3}^{4} (\mu' + C_3 D_p) \cos \theta'_p k'_p A'_p - \sum_{p=5}^{6} (\mu' + C_3 D_p) \cos \theta'_p k'_p A'_p = 0, \quad (4.29)
\]

\[
[\lambda + (2\mu + K) \cos^2 \theta_0] k_1^2 A_0 + [\lambda + (2\mu + K) \cos^2 \theta_1] k_1^2 A_1
\]

\[
- (2\mu + K) \sum_{p=3}^{4} \cos \theta_p \cos k_1^2 A_{py} = \sum_{p=1}^{2} [\lambda' + (2\mu' + C_3 D_p) \cos^2 \theta'_p] k'_p A'_p
\]

\[
+ \sum_{p=3}^{6} (2\mu' + C_3 D_p) \sin \theta'_p \cos \theta'_p k'_p A'_p. \quad (4.30)
\]
The equations (4.25)-(4.33) obtained by utilizing Set-I of boundary conditions can be written in the matrix form as

$$[M][X] = [N],$$

where $[M] = [a_{ij}]$ is a $9 \times 9$ matrix, $[N]$ is a $9 \times 1$ matrix and $[X] = [Z_1, Z_2, Z_3, Z'_1, Z'_2, Z'_3, Z'_4, Z'_5, Z'_6]'$. $Z_1 = A_1/A_0$ is the amplitude ratio corresponding to the reflected wave propagating with phase speed $V_1$, $Z_2 = A_3p/A_0$ and $Z_3 = A_4p/A_0$ are the amplitude ratios corresponding to the reflected waves propagating with phase speeds $V_3$ and $V_4$, respectively, $Z'_r = A'_r/A_0$ ($r = 1, 2, ..., 6$) are the amplitude ratios corresponding to the transmitted waves propagating with phase speeds $V'_r$, for an incident set of longitudinal displacement wave traveling with phase speed $V_1$.

The non-zero coefficients of the matrix $[M]$ occurring in the matrix equation (4.34) are given as

$$a_{11} = \sin \theta_0,$$

$$a_{12} = \sqrt{1 - v_{31}^2} \sin^2 \theta_0/v_{31},$$

$$a_{13} = \sqrt{1 - v_{41}^2} \sin^2 \theta_0/v_{41},$$

$$a_{14} = -\sin \theta_0,$$

$$a_{1p} = \sqrt{1 - V_p^2} \sin^2 \theta_0/V_p.$$
4.3. Reflection and transmission of waves

\[ a_{14} = \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / V_{s1}''} \]
\[ a_{2p} = 1 / V_{p1}' \]
\[ a_{2s} = -1 / V_{s1}' \]
\[ a_{31} = \cos \theta_0 \]
\[ a_{32} = a_{33} = - \sin \theta_0 \]
\[ a_{34} = \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / V_{s1}''} \]
\[ a_{3p} = a_{3s} = a_{44} = \sin \theta_0 \]
\[ a_{45} = D_2 \sin \theta_0 / D_1 \]
\[ a_{4p} = -D_{p-3} \sqrt{1 - V_{p1}^2 \sin^2 \theta_0 / V_{p1}'} \]
\[ a_{4s} = -D_{s-3} \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / V_{s1}'} \]
\[ a_{52} = S_1 k_3 / D_1 v_{s3} \]
\[ a_{53} = S_2 k_3 / D_1 v_{s4} \]
\[ a_{5p} = -D_{p-3} / D_1 V_{p1}' \]
\[ a_{5s} = D_{s-3} / D_1 V_{s1}' \]
\[ a_{64} = \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / v_{s4}'} \]
\[ a_{65} = D_2 \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / D_1 V_{s1}''} \]
\[ a_{6p} = D_{p-3} \sin \theta_0 / D_1, \]
\[ a_{6s} = D_{s-3} \sin \theta_0 / D_1, \]
\[ a_{71} = \sin \theta_0 \cos \theta_0, \]
\[ a_{72} = (\mu + K - \Delta_2 v_{31}^2 - \sin^2 \theta_0, \]
\[ a_{73} = (\mu + K - \Delta_2 v_{41}^2 - \sin^2 \theta_0, \]
\[ a_{74} = (2\mu' + C_3 D_{t-3}) \sin \theta_0 \sqrt{1 - V_{p1}^2 \sin^2 \theta_0 / \Delta_2 V_{11}''}, \]
\[ a_{7p} = -[\mu'(1 - 2V_{p1}^2 \sin^2 \theta_0) + C_3 D_{p-3}(1 - V_{p1}^2 \sin^2 \theta_0)] / \Delta_2 V_{p1}''; \]
\[ a_{7s} = -[\mu'(1 - 2V_{s1}^2 \sin^2 \theta_0) + C_3 D_{s-3}(1 - V_{s1}^2 \sin^2 \theta_0)] / \Delta_1 V_{s1}''; \]
\[ a_{8p} = (\mu' + C_3 D_{p-3}) \sqrt{1 - V_{p1}^2 \sin^2 \theta_0 / \mu V_{p1}^2}; \]
\[ a_{8s} = -(\mu' + C_3 D_{s-3}) \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / \mu V_{s1}^2}; \]
\[ a_{91} = 1, \quad a_{92} = -\Delta_2 \sin \theta_0 \sqrt{1 - V_{31}^2 \sin^2 \theta_0 / \Delta_3 v_{31}}, \]
\[ a_{93} = -\Delta_2 \sin \theta_0 \sqrt{1 - V_{41}^2 \sin^2 \theta_0 / \Delta_3 v_{41}}, \]
\[ a_{94} = -[\lambda' + (2\mu' + C_3 D_{t-3})(1 - V_{11}^2 \sin^2 \theta_0)] / \Delta_3 V_{11}''; \]
\[ a_{9p} = -(2\mu' + C_3 D_{p-3}) \sin \theta_0 \sqrt{1 - V_{p1}^2 \sin^2 \theta_0 / \Delta_3 V_{p1}'}, \]
\[ a_{9s} = -(2\mu' + C_3 D_{s-3}) \sin \theta_0 \sqrt{1 - V_{s1}^2 \sin^2 \theta_0 / \Delta_3 V_{s1}'}, \]

\[ t = 4,5, \quad p = 6,7, \quad s = 8,9, \]

where
4.3. Reflection and transmission of waves

\[ \Delta_2 = 2\mu + K, \quad \Delta_3 = \lambda + \Delta_2 \cos^2 \theta_0, \]

\[ S_1 = \omega_0^2/(\sigma_4^2 k_3^2 + 2\omega_0^2 - \omega^2), \quad S_2 = \omega_0^2/(\sigma_4^2 k_4^2 + 2\omega_0^2 - \omega^2), \]

\[ v_{31} = \frac{V_3}{V_1}, \quad v_{41} = \frac{V_4}{V_1}, \quad v_{r1} = \frac{V_r}{V_1} \quad (r = 1, 2, \ldots, 6), \quad V_{41} = v_{41}, \]

\[ V_{31} = v_{31}, \quad V_{61} = v_{31}, \quad V_{71} = v_{41}, \quad V_{41} = v_{51}, \quad V_{91} = v_{61} \]

and the column matrix

\[ [N] = [-\sin \theta_0, 0, \cos \theta_0, 0, 0, 0, \sin \theta_0 \cos \theta_0, 0, -1]^T. \]

A non-homogeneous system of equations similar to (4.25)-(4.33) can be obtained by utilizing Set-II of boundary equations and they can be written in the matrix form similar to the matrix equation (4.34). The non-zero elements of the matrix \( [M] \) of (4.34) obtained by using Set-II of boundary conditions, that are different from the elements obtained using Set-I are given as follows

\[ a_{41} = ((\beta' + \gamma')D_{r-3} + C_3) \sin \theta_0 \sqrt{1 - V_{r1}^2 \sin^2 \theta_0/V_{41}^2 C_3}, \]

\[ a_{4p} = \left[ ((\beta' + \gamma')D_{p-3} + C_3)V_{p1}^2 \sin^2 \theta_0 - \left( \gamma' D_{p-3} + C_3 - \frac{C_3 D_{p-3}}{k_{p-3}} \right) \right]/V_{p1}^2 C_3, \]

\[ a_{4s} = \left[ ((\beta' + \gamma')D_{s-3} + C_3)V_{s1}^2 \sin^2 \theta_0 - \left( \gamma' D_{s-3} + C_3 + \frac{C_3 D_{s-3}}{k_{s-3}} \right) \right]/V_{s1}^2 C_3, \]

\[ a_{52} = \gamma S_1 k_1 \sqrt{1 - v_{31}^2 \sin^2 \theta_0/v_{41}^2 C_3}, \]

\[ a_{53} = \gamma S_2 k_1 \sqrt{1 - v_{41}^2 \sin^2 \theta_0/v_{41}^2 C_3}, \]

\[ a_{54} = D_1 \sin \theta_0/k_1, \]

\[ a_{55} = D_2 \sin \theta_0/k_1, \]

\[ a_{5p} = \left( \gamma' D_{p-3} + C_3 - \frac{C_3 D_{p-3}}{k_{p-3}} \right) \sqrt{1 - V_{p1}^2 \sin^2 \theta_0/V_{p1}^2 C_3}, \]
Chapter 4

\[ a_{s_3} = - \left( \gamma' D_{s,-3} + C_3 + \frac{C_3 D_{s,-3}}{k_{s,-3}} \right) \sqrt{1 - V_{s_1}^2 \sin^2 \theta_0 / V_{s_1'}^2} C_3, \]

\[ a_{r_3} = \left[ \alpha' D_{r,-3} + ((\beta' + \gamma') D_{r,-3} + C_3)(1 - V_{s_1}^2 \sin^2 \theta_0) \right] / V_{s_1'}^2 C_3, \]

\[ a_{t_3} = ((\beta' + \gamma') D_{t,-3} + C_3) \sin \theta_0 \sqrt{1 - V_{s_1}^2 \sin^2 \theta_0 / V_{s_1'}^2} C_3, \]

\[ a_{d_3} = ((\beta' + \gamma') D_{d,-3} + C_3) \sin \theta_0 \sqrt{1 - V_{s_1}^2 \sin^2 \theta_0 / V_{s_1'}^2} C_3, \]

\[ t = 4, 5, \quad p = 6, 7, \quad s = 8, 9, \]

where \( v_{s_1}, v_{t_1}, v_{p_1}, v_{d_1}, s_1, s_2 \) and column matrix \([N] \) is the same as for Set-I.

The matrix equation (4.34) will enable us to determine the amplitude ratios of various reflected and transmitted waves.

### 4.4 Limiting cases

(i) If we assume that the half-space \( R_1 \) is free from micropolarity, then we will be left with the relevant problem at an achiral-chiral interface. In this limiting case, we see that \( B_p = 0 \). Also, the phase speed \( V_4 \) would be zero [Tomar and Gogna (1992)]. Thus, the wave propagating with phase speed \( V_4 \) will not appear in the medium \( R_1 \). Hence, after making the required substitutions, the equations (4.25)-(4.33) will reduce to the equations that are analogous to the equations obtained by Lakhtakia et al. (1990).

(ii) If micropolar effect is removed from the half-space \( R_1 \) and the chirality and microrotation effects are removed from the half-space \( R_2 \), then the problem will reduce to an elastic-elastic solid interface. In this case, the roots of equation (4.9) are given by \( V_1'^2 = (\lambda' + 2\mu') / \rho' \) and \( V_2'^2 = 0 \), while equation (4.10) have two non-zero equal roots, one of them corresponds to a LCP wave and the other corresponds to a RCP wave propagating with the same phase speed \( \sqrt{\mu' / \rho'} \). Here, the boundary equations will be corresponding to the continuity of displacement and normal component of force stress only. With these considerations, the equations given in the matrix equation (4.34) obtained by utilizing both the possible sets of boundary conditions exactly match with the equations obtained by Miklowitz (1978) for the relevant problem.
4.5 Numerical results and discussion

We have computed the square of the phase speeds of all the coupled transverse plane waves existing in an elastic chiral solid medium and the modulus of amplitude ratios of various reflected and transmitted waves using two possible sets of boundary conditions, for a model having the following numerical values of the relevant parameters

<table>
<thead>
<tr>
<th>Medium $R_1$</th>
<th>Medium $R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 75900 \times 10^6$ N/m$^2$</td>
<td>$\lambda' = 500 \times 10^6$ N/m$^2$</td>
</tr>
<tr>
<td>$\mu = 13500 \times 10^6$ N/m$^2$</td>
<td>$\mu' = 300 \times 10^6$ N/m$^2$</td>
</tr>
<tr>
<td>$K = 149 \times 10^6$ N/m$^2$</td>
<td>$C_3 = 20.0 \times 10^6$ N/m</td>
</tr>
<tr>
<td>$j = 0.00000196$ m$^2$</td>
<td>$j' = 0.01$ m$^2$</td>
</tr>
<tr>
<td>$\rho = 2200$ kg/m$^3$</td>
<td>$\rho' = 1200$ kg/m$^3$</td>
</tr>
<tr>
<td>$\alpha = 0.01 \times 10^6$ N</td>
<td>$\alpha' = 2.0 \times 10^6$ N</td>
</tr>
<tr>
<td>$\beta = 0.015 \times 10^6$ N</td>
<td>$\beta' = 4.0 \times 10^6$ N</td>
</tr>
<tr>
<td>$\gamma = 0.0268 \times 10^6$ N</td>
<td>$\gamma' = 5.0 \times 10^6$ N</td>
</tr>
</tbody>
</table>

and $\omega/\omega_0 = 10$.

**Figure 4.2:** Variation of $V_3^2$ (dashed curve) and $V_5^2$ (solid curve) with frequency ratio ($\omega/\omega_0$)

**Figure 4.3:** Variation of $V_4^2$ (dashed curve) and $V_6^2$ (solid curve) with frequency ratio ($\omega/\omega_0$)
Figures 4.2 and 4.3 depict the variation of the square of phase speeds \( V_i'^2 \) \((i = 3, 4, 5, 6)\) of the coupled transverse waves existing in the chiral elastic medium with the frequency ratio \( \omega/\omega_0 \) lying in the range \( 0 \leq \omega/\omega_0 \leq 5000 \). It can be seen that in the initial range of frequency ratio, the quantities \( V_3'^2 \) and \( V_4'^2 \) increase, while the quantities \( V_5'^2 \) and \( V_6'^2 \) decrease with increase in the frequency ratio. However, at higher values of the frequency ratio, the square of the phase speeds corresponding to each set of these coupled transverse waves approach towards certain constant values. The variation (increase or decrease) of square of these phase speeds in the initial frequency range are very small from their constant values at higher frequency.

In Figure 4.4, we have plotted the modulus values of the reflection coefficients obtained by utilizing Set-I of the boundary conditions, as a function of the angle of incidence. The reflection coefficient \( Z_1 \) begins with the value 0.8587 near the normal incidence, then it decreases with increase in the angle of incidence \( \theta_0 \) lying in the range \( 1^0 \leq \theta_0 \leq 54^0 \), thereafter, the reflection coefficient \( Z_1 \) increases to attain its maximum value at grazing incidence, which is equal to 1. The reflection coefficients \( Z_2 \) and \( Z_3 \) increase with increase in the angle of incidence till \( \theta_0 = 48^0 \), then their corresponding values decrease and approach to zero as \( \theta_0 \) approaches grazing incidence. We have plotted the reflection coefficient \( Z_2 \) by magnifying its original value by the factor 10.
4.5. Numerical results and discussion

In Figure 4.5, we have shown the variation of the modulus values of the transmission coefficients with the angle of incidence. The transmission coefficients $Z'_1$ and $Z'_2$ begin with their maximum values, i.e., 0.1413 and 0.1176, respectively, near the normal incidence, then they decrease with increase in the angle of incidence. The transmission coefficients $Z'_3$ and $Z'_4$ begin with the value zero near the normal incidence, then both the transmission coefficients increase slowly with increase in the angle of incidence till $\theta_0 = 52^\circ$. Beyond $\theta_0 = 52^\circ$, these transmission coefficients decrease rapidly and approach to zero as $\theta_0$ approaches $90^\circ$. The pattern of the transmission coefficient $Z'_6$ is almost similar to that of the coefficient $Z'_6$ with respect to the angle of incidence. Both these transmission coefficients begin with the value zero near the normal incidence, then both increase sharply with increase in the angle of incidence to attain their respective maximum values at $\theta_0 = 52^\circ$ and decrease afterwards. We have plotted the curves corresponding to the transmission coefficients $Z'_3, Z'_4, Z'_5$ and $Z'_6$ after magnifying their original values with the factors $10, 10, 10^2$ and $10^2$, respectively.

![Figure 4.6](image1.png)  
**Figure 4.6**: Variation of modulus of reflection coefficient $Z'_1$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)

![Figure 4.7](image2.png)  
**Figure 4.7**: Variation of modulus of reflection coefficient $Z'_2$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)

In Figures 4.6-4.14, we have shown the comparison in the respective absolute values of the reflection and transmission coefficients obtained by using Set-I and II of the boundary conditions, against the angle of incidence of longitudinal wave propagating...
with phase speed $V_1$.

**Figure 4.8:** Variation of modulus of reflection coefficient $Z_3$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)

**Figure 4.9:** Variation of modulus of transmission coefficient $Z'_1$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)

**Figure 4.10:** Variation of modulus of transmission coefficient $Z'_2$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)

**Figure 4.11:** Variation of modulus of transmission coefficient $Z'_3$ with angle of incidence, using Set-I (solid curve) and Set-II (dashed curve)
4.5. Numerical results and discussion

In Figure 4.6, it can be seen that at each angle of incidence, the value of the reflection coefficient $Z_1$ obtained using Set-I and II, attains almost equal value.

In Figures 4.7, 4.8, 4.10, 4.11 and 4.12, it can be seen that the modulus values of the amplitude ratios $Z_2, Z_3, Z'_2, Z'_3$ and $Z'_4$ obtained using Set-II of the boundary conditions are bigger than their respective values obtained using Set-I of the boundary conditions, at each angle of incidence.

In Figures 4.9, 4.13 and 4.14, it has been observed that the absolute values of the amplitude ratios $Z'_1, Z'_5$ and $Z'_6$ obtained using Set-I are greater than those obtained by using Set-II, at each angle of incidence.

Next, we have shown the variation of the modulus of reflection and transmission coefficients obtained using Set-I of the boundary conditions with the angle of incidence of longitudinal displacement wave propagating with phase speed $V_1$ at two different values of the hemitropic constant, namely, $C_3 = 20 \times 10^6$ N/m and $80 \times 10^6$ N/m through Figures 4.15-4.23. It is evident that the amplitude ratios depend on the angle of incidence as well as on the chirality parameter. The nature of this dependence on the angle of incidence and the chirality parameter is however different for different reflected and transmitted waves.
In Figure 4.15, we observed that at each angle of incidence lying in the range $1^\circ \leq \theta_0 \leq 80^\circ$, the value of the reflection coefficient $Z_1$ corresponding to the wave propagating with phase speed $V_1$ increases with increase of the value of parameter $C_3$. Beyond $\theta_0 = 80^\circ$, the amplitude ratio $Z_1$ has almost equal values at the two different values of parameter $C_3$. 

Chapter 4
4.5. Numerical results and discussion

In Figures 4.16, 4.17, 4.18, 4.20 and 4.21, the modulus values of the amplitude ratios $Z_2$, $Z_3$, $Z'_2$, $Z'_3$ and $Z'_4$ increase with increase in the value of $C_3$ at each angle of incidence.

**Figure 4.18:** The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_1$ with angle of incidence

**Figure 4.19:** The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_2$ with angle of incidence

**Figure 4.20:** The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_3$ with angle of incidence

**Figure 4.21:** The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_4$ with angle of incidence

In Figures 4.19, 4.22 and 4.23, we see that at each angle of incidence, the modulus
values of the amplitude ratios $Z'_L$, $Z'_G$, and $Z'_S$ decrease with increase in the value of $C_3$.

![Graph](image1)

**Figure 4.22**: The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_L$ with angle of incidence

![Graph](image2)

**Figure 4.23**: The effect of hemitropic constant ($C_3$) on the modulus of transmission coefficient $Z'_S$ with angle of incidence

It can be noticed from Figures 4.15-4.23 that the modulus values of amplitude ratios of various reflected and transmitted waves of an incident longitudinal wave propagating with phase speed $V_1$ are significantly influenced by the hemitropic constant ($C_3$).

### 4.6 Conclusions

In this chapter, we have presented the reflection and transmission phenomena of an incident longitudinal plane wave propagating through the micropolar elastic solid half-space at micropolar/chiral elastic solid interface. We conclude that

1. At normal incidence, the reflection and transmission of only longitudinal wave take place and no set of coupled transverse waves is found to reflect or transmit.
2. At grazing incidence, no reflection or transmission phenomena take place and the same wave propagates along the interface.
3. Amplitude ratios of various reflected and transmitted waves depend upon the angle of incidence and the chirality parameter.
4. Various reflection and transmission coefficients are found to be affected significantly by the chirality parameter ($C_3$).