CHAPTER IV

Hypercentral Units in Alternative Integral Loop Rings
IV. Hypercentral Units in Alternative Integral Loop Rings

In this chapter we study the upper central series of $U_i(ZL)$, the loop of normalized units of integral loop ring for any RA loop $L$. We show that if $L$ is not a Hamiltonian Moufang 2-loop, then $U_i(ZL)$ is of central height 1. In case $L$ is a Hamiltonian Moufang 2-loop, obviously, $U_i(ZL) = L$ (see [25, Theorem 7]) is of central height 2. The central height of unit groups of integral group rings and their hypercentral units have been studied in [2] and [4].

§1. Hypercentral units

The upper central series for a Moufang loop $L$ is defined as

$$\{1\} = Z_0(L) \subseteq Z_1(L) \subseteq Z_2(L) \subseteq \ldots$$

where for $i \geq 0$, $Z_{i+1}(L)/Z_i(L) = Z(L/Z_i(L))$, where $Z(L/Z_i(L))$ denotes the centre of $L/Z_i(L)$ (see [11, Page 265]). This series terminates at $L$ in a finite number of steps if and only if $L$ is nilpotent.

The Moufang loop of normalized units of the alternative loop ring $RL$, where $L$ is an RA loop, will be denoted by $U_i(RL)$; that is,

$$U_i(RL) = \{\alpha \in U(RL) | \epsilon(\alpha) = 1\},$$

where $\epsilon : RL \to R$ is the augmentation map.

Let $L$ be a RA loop and $U_1 = U_1(ZL)$ be the (Moufang) loop of normalized units of the (alternative) integral loop ring $ZL$. Let

$$\{1\} = Z_0(U_1) \subseteq Z_1(U_1) \subseteq Z_2(U_1) \subseteq \ldots$$

be the upper central series of $U_1$, as defined above. Let $\tilde{Z}(U_1) = \bigcup_{n=0}^{\infty} Z_n(U_1)$. Each $Z_n(U_1)$, $n \geq 0$, and hence $\tilde{Z}(U_1)$ is a normal subloop of $U_1$; $\tilde{Z}(U_1)$ will
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be called the hypercentre of \( \mathcal{U}_i \). Let \( T = T(\mathcal{Z}(\mathcal{U}_i)) \) be the set of all torsion elements of \( \mathcal{Z}(\mathcal{U}_i) \).

**Lemma 1.1.** Let \( L \) be an RA loop. Then \( T(\mathcal{Z}(\mathcal{U}_i)) \) is a periodic normal subloop of \( \mathcal{U}_i = \mathcal{U}_i(\mathcal{Z}L) \).

Proof. For \( n \geq 0 \), \( \mathcal{Z}_n(\mathcal{U}_i) \) is a nilpotent Moufang loop [11, Theorem 4A]. Let \( a, b \in T(\mathcal{Z}_n(\mathcal{U}_i)) \), the set of torsion elements of \( \mathcal{Z}_n(\mathcal{U}_i) \). By diassociativity, \( G = \langle a, b \rangle \) is a nilpotent group. Therefore \( ab \) is a torsion element of \( G \) and hence of \( \mathcal{Z}_n(\mathcal{U}_i) \), so that \( T(\mathcal{Z}_n(\mathcal{U}_i)) \) is a subloop of \( \mathcal{U}_i \). Let \( \theta \in \{T(x), R(x, y), L(x, y) \mid x, y \in \mathcal{U}_i\} \), where \( T(x), R(x, y), L(x, y) \) are inner mappings as defined in Chapter II. §1 (also see [12, Page 61]. Since \( \mathcal{U}_i \) is a Moufang loop, \( \theta \) is a pseudo-automorphism of \( \mathcal{U}_i \) with companion, say, \( c \) ([12, page 117]). Hence \( t\theta(t'\theta.c) = (tt'\theta)c \) for all \( t, t' \in \mathcal{U}_i \). By diassociativity it follows that \( (t\theta)^2c = (t^2\theta)c \) and hence, by induction, \( (t\theta)^mc = (t^m\theta)c \) for \( m \geq 1 \). Thus \( T(\mathcal{Z}_n(\mathcal{U}_i))\theta \subseteq T(\mathcal{Z}_n(\mathcal{U}_i)) \), showing that \( T(\mathcal{Z}_n(\mathcal{U}_i)) \) is a normal subloop of \( \mathcal{U}_i \) ([12, page 61]). Now observing that \( T(\mathcal{Z}(\mathcal{U}_i)) = \bigcup_{n=0}^{\infty} T(\mathcal{Z}_n(\mathcal{U}_i)) \), the proof follows. ■

We shall frequently make use of the following:

**Lemma 1.2.** Let \( L \) be an RA loop and let \( u_1, u_2, v, w \in \mathcal{U}_i(\mathcal{Z}L) \) be such that the associators \( (u_1, v, w), (u_2, v, w), (u_1u_2, v, w) \) and the commutator \( (v, w) \) are in the centre of \( \mathcal{U}_i(\mathcal{Z}L) \). Then

\[ (u_1u_2, v, w) = (u_1, v, w)(u_2, v, w) \]
Proof. By [12, Lemma VII.2.2], the inner map \( R(v, w) \) is a pseudo automorphism with companion \( c = (v, w) \), i.e., \( [xR(v, w)]yR(v, w)c = [xyR(v, w)].c \) for all \( x, y \in ZL \). Since \( c \) is central, by assumption, \( R(v, w) \) is an automorphism. Also, \( u_1R(v, w) = [u_1v,w](vw)^{-1} = [(u_1w)(u_1,v,w)](vw)^{-1} = u_1(u_1,v,w) \), by the centrality of the associator \( (u_1,v,w) \). Similarly \( u_2R(v, w) = u_2(u_2,v,w) \) and \( u_1u_2R(v, w) = u_1u_2(u_1u_2,v,w) \), thus yielding the required result. ■

We shall now prove some basic properties of the second centre \( Z_2(U_1) \), which are needed in the sequel.

**Remark 1.3.** Let \( L \) be an RA loop. For \( x, y, z \in ZL \), let \( [x, y] = xy - yx \) and \( [x, y, z] = (xy)z - x(yz) \) be the ring commutator and ring associator respectively. If \( u \in U(ZL) \), \( l, m \in L \), then

\[
(u, l) - 1 = u^{-1}l^{-1}ul - 1 = u^{-1}l^{-1}[u, l],
\]

\[
(u, l, m) - 1 = ((lm)^{-1}u^{-1})[u, l, m].
\]

By the linearity of the ring commutators and associators in each of their arguments and \( ZL \) being alternative it follows that \( (u, l) = 1 \) for all \( l \in L \) if and only if \( (u, v) = 1 \) for all \( v \in U(ZL) \) and \( (u, l, m) = 1 \) for all \( l, m \in L \) if and only if \( (u, v, w) = 1 \) for all \( v, w \in U(ZL) \).

**Proposition 1.4.** Let \( L \) be any finite RA loop and let \( L' \) be its commutator-associator subloop. Let \( U_1 = U_1(ZL) \) be the loop of normalized units of \( ZL \). Then

(i) For every \( x \in Z_2(U_1) \), \( x^2 \in Z_1(U_1) = Z(U_1) \).
(ii) The sets \( \mathcal{Z}(U_1), [Z_2(U_1), L], [L, Z_2(U_1), L] \) and \([L, L, Z_2(U_1)]\) are contained in \( L' \).

Proof. Let \( x \in Z_2(U_1), l \in L. \) Since \((x, l) \in Z(U_1), (x^2, l) = x^{-1}(x, l)l^{-1}xl = (x, l)^2 = x^{-1}l^{-1}x(x, l) = (x, l)^2 = 1, \) as squares are central in an RA loop.

Hence by [20, Corollary III.4.2], \( x^2 \in Z(ZL) \supseteq Z(U_1). \) Thus for every \( v \in U_1, (x, v)^2 = (x^2, v) = 1, \) using Remark 1.3. Thus, \((x, v)\) being a central torsion unit in \( ZL \) is trivial by [25, Theorem 6], i.e., \((x, v) \in ZL).\)

Considering the natural ring homomorphism \( \epsilon_{L'} : ZL \to Z(L/L'), \epsilon_{L'}(x, v) = (\epsilon_{L'}(x), \epsilon_{L'}(v)) = 1, \) so that \((x, v) \in L'.\)

Let \( x \in Z_2(U_1), l, m \in L. \) By Lemma 1.2 it follows that \((l, x, m)^2 = (l^2, x, m) = 1, \) i.e., \((l, x, m)\) is a torsion central unit. Thus \((l, x, m) \in ZL\) and as before, going modulo \( L', \) \((l, x, m) \in L'.\) Similarly \((l, m, x) \in L'\) for all \( x \in Z_2(U_1) \) and \( l, m \in L. \) Since \( U_1(ZL) \) is Moufang and \((l, x, m)\) is central, by [12, Lemma VII.5.4], \((l, x, m)^{-1} = (l, x, m)^{-1} \in L'.\) This completes the proof of the Proposition. ■

Having seen that \( T(\bar{Z}(U_1)) \) is a periodic normal subloop of \( U_1(ZL), \) we now observe that such subloops of \( U_1(ZL) \) are rather restrictive. Recall that for a Moufang loop \( M, \) the inner mappings \( T(x), x \in M, \) is the mappings given by \( hT(x) = x^{-1}hx, \) for all \( h \in M. \) We have:

**Proposition 1.5.** Let \( L \) be a finite RA loop and \( N \) be a periodic normal subloop of \( U_1(ZL). \) Then \( N \subseteq L \) and every cyclic subloop of \( N \) is invariant under the inner mappings \( T(u), u \in U_1(ZL); \) in particular, every subloop of \( N \) is normal in \( L. \) Moreover, if \( N \) is not central, then \( L \) must be a Hamiltonian 2-loop.
Proof. Let $\alpha = \sum_{l \in L} \alpha_l l \in N$ and let $\alpha_l \neq 0$ for some $l \in L$. Suppose that the order of $l$ is $m$. Now $U_i(ZL)$ being diassociative,

$$(al^{-1})^m = \alpha(l^{-1}al)(l^{-2}a^2l^2) \cdots (l^{-(m-1)}a^{m-1})(l^{-m})l^{-m} = \alpha' \in N,$$

as $N$ is a normal subloop of $U_i(ZL)$. Since $N$ is periodic, $al^{-1}$ is of finite order and hence, by [21, Proposition 2.1], $\alpha = l \in L$.

Let $h \in N$ be of order $n$ and suppose, if possible, that $hT(x^{-1}) = xhx^{-1} = h' \notin < h >,$ for some $x \in U_i(ZL)$. For every $m \in Z, m \neq 0,$ $y_m = 1 + m(1 + h + \cdots + h^{n-1})x(1 - h)$ is a non-trivial unit in $ZL$ and

$$y_m^{-1}hy_m = \frac{[1 - m(1 + h - h^2 + \cdots - h^{n-1})x(1 - h)]h}{[1 + m(1 + h - h^2 + \cdots - h^{n-1})x(1 - h)]}$$

$$= h + m(1 + h + \cdots + h^{n-1})x(1 - h)^2 \in N.$$

Choosing $m$ so that the coefficient of $h$ on the right hand side is different from zero, it follows that $(1 - h - \cdots + h^{n-1})(1 - h')^2 = 0$, yielding $h' \notin < h >$, which is not possible. Hence $hT(x^{-1}) = xhx^{-1} \notin < h >$. Moreover, it follows that every cyclic subloop of $N$ is normal in $L$, as by [20, Corollary IV.1.11] a subloop of an RA loop $L$ is normal if and only if it is invariant under the inner mappings of the type $T(x), x \in L$.

Suppose that $N$ is not central. Let $a \in N \subseteq L$ be a noncentral element. We first claim that $L$ must be a Hamiltonian Moufang loop. To this end, we show that if $g \in L$ is a noncentral element then $< g >$ is a normal subloop of $L$. By [20, Corollary IV.1.11], it is enough to show that $s \notin < g >$, where $s$ is the unique non-identity commutator in $L$. Now, $v = 1 - 3(1 - g)a\tilde{g}$,

$$\tilde{g} = 1 + g + \cdots + g^{m-1},$$

where $m = \text{order of } g$, is a unit of $ZL$ and using
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diassociativity,

\[ aT(v) = v^{-1}av = [1 + 3(1 - g)a\delta]a[1 - 3(1 - g)a\delta] \]

\[ = a + 3[(1 - g)a\delta a - a(1 - g)a\delta] - 9(1 - g)a\delta a(1 - g)a\delta \]

is in \( < a > \), which in turn implies that either \( aga = a^2g^4 \) or \( aga = gag^3a \). so that \( (a, g) = g^k \), for some \( k \). If \( (a, g) \neq 1 \), then \( s = (a, g) = g^k \in < g > \).

Suppose now that \( g \) commutes with \( a \). Since \( a \) is noncentral, there exists \( y \in L \) such that \( (a, y) = s \). By Theorem I.3.2, \( a, g \) and \( y \) associate in all orders. As above, considering the unit \( w = 1 - 3(1 - g)y^2 \) and observing that

\[ aT(w) = w^{-1}aw \in < a > \],

it follows that either \( ay = yg'a \) or \( ay = agyg' \) and consequently \( s \in < g > \), unless \( gy = yg \). Hence, either \( < g > \) is normal in \( L \) or \( g \) commutes with \( a \) as well as \( y \), which is not possible by the structure of RA loops. Indeed, let \( G = < a, y, Z(L) > \). As in Theorem I.3.3 \( L = G - Gu \). where \( u \in L \) is such that \( (a, y, u) \neq 1 \). Now, \( g \) cannot be in \( G \), as then \( g \) commuting with \( a \) and \( y \) gives that \( g \in Z(G) = Z(L) \). On the other hand, if \( g = hu \in Gu \), then by Lemma 1.2, \( (g, a, y) = (h, a, y)(u, a, y) = s \) implies (Theorem I.3.2) that \( (g, a) = s \), a contradiction. Hence \( L \) is a Hamiltonian loop. By [37, Theorem VI], \( L \cong C_{16} \times E \times A \), where \( C_{16} \) is the Cayley loop. \( E \) is a (possibly trivial) elementary abelian 2-group and \( A \) is a (possibly trivial) abelian group, all of whose elements are of odd order.

We next observe that under the given conditions \( A \) must be trivial. Let \( a' \in L \) be such that \( < a' > \) is invariant under the maps \( T(u) . u \in U(ZL) \) and write \( a' = let \), where \( l \in C_{16} \) is a non-central element of order 4, \( e \in E \) and \( t \in A \) is of odd order \( m \). Then \( a = (let)^m = (le)^m = le \) or \( l^{-1}e \) is noncentral element of order 4 and \( < a > \) is also invariant under the maps
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$T(u)$, $u \in \mathcal{U}(\mathbb{Z}L)$. In case $A$ is not trivial, choose $x \in A$ of order $p$, where $p$ is an odd prime. Let $b \in C_{16}$ be such that $(a, b) \neq 1$. Then by [12, Theorem IV.7.2] $l$ and $b$ generate a group isomorphic to the quaternion group of order $8$ and $a^2 = l^2 = b^2 = s$, where $s$ is the unique nonidentity commutator of $L$.

Also $y = xb = bx$ is of order $4p$.

We now conjugate $a$ by suitable Bass cyclic units (see [20, Page 201]) to arrive at a contradiction. In case $p = 3$, we consider the unit

$$u = (1 + y + y^2 - y^3 + y^4)^4 + \frac{1 - 5^4}{12}(1 + y + \cdots + y^{11})$$

$$= -16 - 32x^2b^2 - 16x - 16b^2 + 33x^2 + 16b^2 x + 28(1 - x - b^2 + b^2 x)b.$$ It is easy to see that $au - ua = 56(1 - x)(1 - b^2)ab \neq 0$, $au - ua^3 = a(-32 - 32x - 65x^2)(1 - b^2) \neq 0$ so that neither $au = ua$ nor $au = ua^3$, contradicting the fact that $u^{-1}au \in < a >$. In case $p > 3$, we consider the unit

$$v = (1 + y - y^2)^{2p-2} + \frac{1 - 3^{2p-2}}{4p} y,$$

$\hat{y} = 1 + y - \cdots - y^{4p-1}$, of $\mathbb{Z}L$. Putting $k = (1 - 3^{2p-2})/4p$ we see that $v = (\alpha_1 + k\beta) + (\alpha_2 + k\beta)y$, where $\alpha_1, \alpha_2, \beta$ are given by

$$\alpha_1 = (1 + y^2)^{2p-2} - 2^{2p-2} - 2^{2p-2} C_2(1 + y^2)^{2p-4} y^2 + \cdots - y^{2p-2},$$

$$\alpha_2 = 2^{p-2} C_2(1 - y^2)^{2p-3} + 2^{p-2} C_3(1 + y^2)^{2p-5} y^2 + \cdots + 2^{p-2} C_{2p-3}(1 + y^2)^{2p-4},$$

$$\beta = 1 + y^2 + y^4 + \cdots + y^{4p-2} = (1 + y^2 + \cdots + y^{2p-2})(1 - y^{2p})$$

$$= (1 - y^2 + \cdots + y^{2p-2})(1 + a^2)$$

as $y^{2p} = (xb)^{2p} = b^{2p} = b^2 = a^2$ (here $2^{p-2} \mathcal{C}_r$ etc. are binomial coefficients).

As squares are central in an RA loop, it follows that $\alpha_1, \alpha_2$ and $\beta$ are central
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elements of $ZL$. Hence $va - av = \alpha_2 y(a - a^3)$ and $va - a^3v = \alpha_1 (a - a^3)$, as $\beta(a - a^3) = 0$. Using that $y^{2p} = b^2 = a^2$ it is easy to see that $y^r a^3 = y^{2p-r}a$, $1 \leq r \leq 4p - 2$, if and only if $r = 4p - 3$. The coefficient of $y^{2p-4}$ in $\alpha_2$ is nonzero and the term $y^{2p-4}$ does not occur in $\alpha_2$, therefore, it follows that $\alpha_2 y a \neq \alpha_2 y a^3$ and hence $va \neq av$; similarly (looking at the coefficient of $y^{2p-2}$ in $\alpha_1$) it can be shown that $va \neq a^3v$. This contradicts the fact that $<a>$ is invariant under the mapping $T(v)$. Hence $L$ must be a Hamiltonian 2-loop. ■

We are now ready to prove the main result, that is:

**Theorem 1.6.** Let $L$ be a finite RA loop. Then $Z_2(U_L(ZL)) = Z_2(U_L(ZL))$, i.e., the hypercentre of $U_L(ZL)$ is equal to its centre. unless $L$ is a Hamiltonian 2-loop (in case $L$ is a Hamiltonian 2-loop, $Z_2(U_L(ZL)) = L = U_L(ZL)$).

**Proof.** By the structure of RA loops, we may assume that $L = G \cup G u$, where $G = <a, b, Z(L)>$, $(a, b) = s = (a, b, u)$, $s$ is the unique nonidentity commutator-associator in $L$.

We first observe that given any $x \in Z_2(U_L(ZL))$, there exists an element $l \in L$ such that $(z, a, b) = (l, a, b)$, $(x, a, u) = (l, a, u)$ and $(x, b, u) = (l, b, u)$. Indeed, by Proposition 1.4, the associators on the left hand side are either 1 or $s$. Thus all possible values of the triple $\{(x, a, b), (x, a, u), (x, b, u)\}$ are $(1, 1, 1), (1, 1, s), (1, s, 1), (s, 1, 1)(1, s, s), (s, 1, s), (s, s, 1)$ and $(s, s, s)$. Taking $l = 1, a, b, u, ab, au, bu$ and $(ab)u$ respectively yields all the eight possibilities.

Also, proceeding as in Remark 1.3 and using that in an alternative ring the ring associator is an alternating function (i.e., $[x, y, z] = -[z, y, x]$ etc.) (see Chapter 1, §1), it follows from Proposition 1.4 that if $g, k \in L$ and $x$ is
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either in \( Z_2(U_1(ZL)) \) or in \( L \), then

\[
(x, g, k) = (x, k, g) = (g, x, k) = (k, x, g) = (k, g, x) . \quad (\ast)
\]

Indeed, \((x, g, k) \in \{1, s\}\). If \((x, g, k) = 1\), then

\[
0 = (x, g, k) = 1 - ((gk) x^{-1})[x, g, k] ,
\]

so that \([x, g, k] = 0\) and hence \([x, k, g] = 0\). Thus

\[
(x, k, g) - 1 = ((kg)^{-1} x^{-1})[x, k, g] = 0 .
\]

which gives that \((x, k, g) = 1\). Similarly it can be shown that either all associators of \((\ast)\) are 1 or each one is \( s \).

Thus, using Lemma 1.2, we observe that for \( g, h, k \in L \) and for \( x \in Z_2(U_1(ZL)) \) (or \( x \in L \)),

\[
(x, gh, k) = (x, g, k)(x, h, k)
\]

and

\[
(x, g, hk) = (x, g, h)(x, g, k) .
\]

From the above observations it is easy to see that given any \( x \in Z_2(U_1(ZL)) \), there exists an \( l \in L \) such that \((x, g, h) = (l, g, h)\), for all \( g, h \in L \) and hence \((xl, g, h) = (x, g, h)(l, g, h) = 1\). We next observe that for such an \( l \in L \), \((x, g) = (l, g)\), for all \( g \in L \). Indeed, suppose \((l, g) = 1\) for some \( g \in L \), \( g \notin Z(L) \). Then by Theorem I.3.2, \((l, g, h) = 1\) for all \( h \in L \). Hence \((x, g, h) = 1\) for all \( h \in L \), so that, by Remark 1.3, \([x, g, \alpha] = 0\) for all \( \alpha \in ZL \). Now if \((x, g) = s\), it follows by a result of Kleinfeld [33] (see also [20, Page 10]) that \([x, xg] = (1 - s)x^2g\) is in the nucleus of \( ZL \). Choose
m, n ∈ L such that (g, m, n) = s. Since $x^2 ∈ Z(\mathcal{U}_1(\mathbb{Z}L))$ by Proposition 1.4, we get

$$[(1 - s)x^2g]m,n = [(1 - s)x^2g].mn$$

and hence $(1 - s)gm.n = (1 - s)g.mn = (1 - s)[sgm.n]$, as $(g, m, n) = s$. But this in turn implies that $1 - s = (1 - s)s$, which is not possible. Thus $(x, g) = 1$.

We now show that if $(x, g) = 1$, then $(x, g, h) = 1$ for all $h ∈ L$. Suppose $(x, g, h) = s$ for some $h ∈ L$. By alternativity, $[h(x + g)](x - g) = h(x + g)^2$ and hence

$$(hg)x - h(gx) + (hx)g - h(xg) = 0 .$$

Since $(x, g, h) = (h, g, x) = s$ and $xg = gx$, the above relation leads to

$$2h(gx)(s - 1) = 0,$$

which is not possible. Hence $(x, g, h) = 1$ for all $h ∈ L$. If $(l, g) = s$, then, for some $h ∈ L$, $(x, g, h) = (l, g, h) = s$. Thus $(x, g)$ has to be equal to $s$, a contradiction. We have shown that for all $g ∈ L$, $(x, g) = (l, g)$. Also, because $x, l, g$ associate, $(xl, g) = [l^{-1}(x, g)]l(l, g) = (x, g)(l, g)$, implying that $(xl, g) = 1$ for all $g ∈ L$. By Remark 1.3, we get that $xl ∈ Z_1(\mathcal{U}_1(\mathbb{Z}L)) ⊆ Z_2(\mathcal{U}_1(\mathbb{Z}L))$, so that, by Lemma 1.1 and Proposition 1.5, $l ∈ T(\mathcal{Z}_1(\mathcal{U}_1(\mathbb{Z}L))) ⊆ Z_1(\mathcal{U}_1(\mathbb{Z}L))$, unless $L$ is a Hamiltonian 2-loop. Thus, $Z_2(\mathcal{U}_1(\mathbb{Z}L)) = Z_1(\mathcal{U}_1(\mathbb{Z}L))$, unless $L$ is a Hamiltonian 2-loop, completing the proof of the theorem. ■