CHAPTER I

Introduction and Preliminaries
In this work we study alternative loop rings, their unit loops, augmentation ideals and their powers and algebraic elements in such loop rings. In §1 we shall briefly indicate the results obtained in the subsequent chapters. In §2 and §3 we shall describe alternative loop rings and give the characterization of those loops, called RA loops, for which the loop rings for rings of characteristic different from 2 are always alternative rings.

§1. Introduction

The following is a summary of the results obtained in the subsequent chapters.

In Chapter I we discuss the residual properties of unit loops of alternative loop rings. In case $R = \mathbb{Z}$, the ring of integers, or $R$ is a field the conditions for the nilpotency of $U(RL)$, the unit loop of alternative loop ring $RL$, have been studied in [22] and [24]. After defining the residual nilpotency and the residual solvability for Moufang loops, we investigate the solvability, the residual nilpotency and the residual solvability of $U(RL)$, for $R$ as above.

We establish that the residual nilpotence of $U(FL)$, where $F$ is a field, implies its nilpotence in many cases such as (see Propositions II.1.5, II.1.8 and II.1.10)

(i) Characteristic of $F = p > 0$ and $L$ contains an element of order $p$;

(ii) $L$ is a torsion loop;

(iii) $F$ is a finite field.

We have also constructed an example of a loop for which $U(FL)$ is residually nilpotent but not nilpotent. We have shown that for an RA loop $L$ with
finite torsion subloop $T$, $\mathcal{U}(QL)$ is residually nilpotent if and only if $T$ is an abelian group and every subloop of $T$ is normal in $L$. We also prove that for an RA loop $L$, $\mathcal{U}(ZL)$ is always residually nilpotent, whereas it is rarely nilpotent, by [22].

We further prove that for a finite RA loop $L$ and for a field $F$ of characteristic zero, $\mathcal{U}(FL)$ is solvable if and only if $L$ is an abelian group. On decomposing the loop algebra $FL$ of a finite RA loop $L$ into semisimple components, the solvability of $\mathcal{U}(FL)$ naturally leads to the investigation of the derived series of the unit loops of Cayley-Dickson division algebras. We have been able to show that for the unit loop of Cayley-Dickson division algebras over algebraic number fields the lower central series becomes stationary very quickly.

If $A$ is an alternative loop ring and $I$ is an ideal of $A$, then $I^n$, the linear span of all possible products, with any arrangement of parenthesis, of $n$ elements from $I$ is also an ideal of $A$ ([20, Page 147]). We investigate the powers of augmentation ideal $\Delta_R(L)$, for an RA loop $L$, and its nilpotence. We have obtained that $\Delta_R(L)$ is nilpotent if and only if $L$ is a finite 2-loop and $R$ is of characteristic $2^n$ (Proposition II.2.4). We have also studied the residual nilpotence of $\Delta_R(L)$. The analysis eventually reduces to the study of augmentation ideals of RA 2-loops of bounded exponents, which we have attempted, in the nonassociative setup, using methods of B. Hartley [30]. We have obtained that for an RA loop $L$, $\Delta_Z(L)$ is residually nilpotent, if and only if, the centre of $L$ has no element of infinite 2-height.

In this chapter we also define Lie powers of the augmentation ideal in such a way that these powers become two sided ideals of the loop ring. We then
characterize those loops for which the augmentation ideal is Lie nilpotent or residually Lie nilpotent, for \( R = \mathbb{Z} \) or when \( R \) is a field.

In chapter III we investigate the Jordan decompositions. After formulating the concepts of additive and multiplicative Jordan decompositions in the nonassociative setup, we characterize those RA loops for which the additive Jordan decomposition holds in integral loop rings. We also investigate the multiplicative Jordan decomposition for the integral loop rings of all RA loops of order \( \leq 32 \). For some of these loops the multiplicative Jordan decomposition holds in the integral loop rings while for others we have shown that it does not hold. We analyse the multiplicative Jordan decomposition for integral loop rings of loops with cyclic centre in detail. We have also shown that the unit loop \( U(\mathbb{Z}L) \), for a finite RA loop \( L \), is always almost splittable in the sense that it always has a subloop of finite index such that multiplicative Jordan decomposition holds in \( \mathbb{Z}L \) for all elements of this loop.

In chapter IV we study the central height and the hypercentral units in alternative loop rings. Let \( U_1 = U_1(\mathbb{Z}L) \) be the (Moufang) loop of normalized units of the (alternative) integral loop ring \( \mathbb{Z}L \). Denoting the terms of upper central series of \( U_1 \) by \( Z_n(U_1) \), \( n \geq 0 \), we define the hypercentre of \( U_1 \) as \( \tilde{Z}(U_1) = \bigcup_{n=0}^{\infty} Z_n(U_1) \). We have shown that \( T(\tilde{Z}(U_1)) \), the set of torsion units of \( \tilde{Z}(U_1) \), is a periodic normal subloop of \( U_1 \). We further show that such periodic normal subloops of \( U_1 \) are rather restrictive. Using these we are able to prove that if \( L \) is a finite RA loop then \( U_1 \) is always of central height 1, except when \( L \) is a Hamiltonian Moufang 2-loop in which case it is 2 (if \( L \) is a Hamiltonian 2-loop \( Z_2(U_1(\mathbb{Z}L)) = L = U_1(\mathbb{Z}L) \)).

In chapter V we study the algebraic elements, especially the idempotents,
I. Introduction and Preliminaries

in complex loop algebra $CL$, of an RA loop $L$. For finite RA loops, we show that if $\alpha = \sum_{l \in L} \alpha_l l$ is an element of the complex loop algebra $CL$, and $\lambda$ is the maximum of the absolute values of the roots of the minimal polynomial $m(X)$ of $\alpha$ over $C$, then

$$\sum_{l \in Z(L)} |\alpha_l|^2 + \frac{1}{4} \sum_{l \in Z(L)} |\alpha_l + \alpha_{ll^i}|^2 \leq \lambda^2,$$

where $Z(L)$ denotes the centre of $L$. Furthermore, the equality holds if and only if $\alpha$ is the sum of a central and a nilpotent element and all the roots of $m(X)$ are of absolute value $\lambda$. We have obtained some results about algebraic elements of complex loop algebras of infinite RA loops following the methods in [1] and [40]. We have also obtained the bounds on traces of idempotents in alternative loop algebras and have investigated the conditions when traces of idempotents attains these bounds.

Finally, we consider another nonassociative alternative algebra, namely the zorn's vector matrix algebra $M(ZG)$ over the integral group ring $ZG$ of finite abelian group $G$ and prove that a version of the Zassenhaus conjecture for torsion units holds in $M(ZG)$.

Some of the results mentioned above will be appearing as [47], [48], [49] and [50].

In §2 and §3 we shall now briefly describe alternative rings, Moufang loops, RA loops and their characterization and the structure of loop algebras of finite RA loops.

§2. Alternative Rings and Moufang Loops

A not necessarily associative ring is a triple $(R, +, \cdot)$, where $(R, +)$ is an abelian group, $(a, b) \rightarrow a \cdot b$ is a binary operation on $R$ such that both
I. Introduction and Preliminaries

distributive laws hold; i.e., \( a(b + c) = ab + ac \), \((a + b)c = ac + bc\), for all \( a, b, c \in R \). If, in addition, \((R, +)\) is a module over a commutative and associative ring \( \Phi \) such that \( \alpha(ab) = (\alpha a)b = a(\alpha b) \) for all \( \alpha \in \Phi \) and for all \( a, b \in R \), then \((R, +, \cdot)\) is said to be a (nonassociative) \( \Phi \)-algebra.

If \( a, b, c \) are three elements of a ring, the ring commutator \([a, b]\) and the ring associator \([a, b, c]\) are defined respectively as

\[
[a, b] = ab - ba,
\]

\[
[a, b, c] = (ab)c - a(bc),
\]

both of which are linear functions of each of their arguments. The nucleus \( \mathcal{N}(R) \) and the centre \( Z(R) \) of \( R \) are the subrings of \( R \) given by

\[
\mathcal{N}(R) = \{ a \in R | [a, x, y] = [x, a, y] = [x, y, a] = 0 \ \forall \ x, y \in R \}.
\]

\[
Z(R) = \{ a \in \mathcal{N}(R) | [a, x] = 0 \ \forall \ x \in R \}.
\]

**Definition 2.1.** A ring \( R \) is said to be an alternative ring if \( x(xy) = x^2y \) and \((xy)y = xy^2\) for all \( x, y \in R \).

For an alternative ring \( R \), \([x, x, y] = [x, y, y] = 0\), for all \( x, y \in R \). From this, it can be easily shown ([42, Page 27]) that the ring commutator is an alternating function of its arguments, i.e.,

\[
[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = sgn(\sigma)[x_1, x_2, x_3]
\]

where \( \sigma \) is a permutation and \( sgn(\sigma) \) denotes the sign of \( \sigma \).

One of the most fundamental property of alternative rings is diassociativity. In fact the generalized theorem of Artin ([20, Page 12]) yields that if
I. Introduction and Preliminaries

$a, b, c$ are elements of an alternative ring $R$ which associate in some order, then the subring generated by $a, b, c$ is an associative ring and hence the subring generated by any two elements of $R$ is associative.

An important example of an alternative ring (which is not associative) is the ring of Cayley numbers whose underlying set is

$$C = \mathbb{H} + \mathbb{H}l = \{a + bl \mid a, b \in \mathbb{H}\},$$

where $\mathbb{H}$ denotes the real quaternion algebra and $l$ is an indeterminate. Addition and multiplication in $C$ is given by

$$(a + bl) + (c - dl) = (a + c) + (b + d)l,$$

$$(a + bl)(c - dl) = (ac - db) + (da - bc)l,$$

where $a, b, c, d \in \mathbb{H}$ and $\bar{x}$ denote the conjugate of the quaternion $x$ (if $x = a_1 + a_2i + a_3j + a_4k$ then $\bar{x} = a_1 - a_2i - a_3j - a_4k$). The Cayley numbers form an 8-dimensional algebra over $\mathbb{R}$, the field of real numbers, with basis $\{1, i, j, k\} \cup \{l, li, lj, lk\}$, where $\{1, i, j, k\}$ is the usual basis for $\mathbb{H}$.

More general than the Cayley numbers are the Cayley-Dickson algebras, which we now describe (see [42, Chapter III, §4]). Let $B$ be an associative algebra over a field $F$ of characteristic not equal to 2 with an involution (an automorphism of order 2) $b \to \bar{b}$ such that $b + \bar{b}$ and $b\bar{b}$ are in $F$ for all $b \in B$. Let $l$ be an indeterminate and $\alpha \in F$ and let $A$ be the vector space direct sum $B \oplus Bl$. Define addition and multiplication in $A$ by

$$(a + bl) + (c + dl) = (a + c) + (b + d)l,$$

$$(a + bl)(c + dl) = (ac + \alpha db) + (da - bc)l.$$
Then $A$ becomes an alternative algebra over $F$, known as Cayley-Dickson algebra. Every Cayley-Dickson algebra is a simple algebra. It is a division algebra if and only if for all nonzero $a \in A$, $a\bar{a} \neq 0$. Thus, if a Cayley-Dickson algebra over a field $F$ is not a division algebra, then it has zero divisors and it can be shown that up to isomorphism there is a unique such algebra. The Cayley-Dickson algebra over $F$ which is not a division algebra is called the split Cayley algebra.

**Definition 2.2.** A loop is a set $L$ together with a (closed) binary operation $(a, b) \rightarrow ab$ for which there is a two-sided identity element $1$ and such that the right and the left translation maps

$$R_x : a \rightarrow ax \quad \text{and} \quad L_x : a \rightarrow xa$$

are bijections for all $x \in L$. This requirement implies that for any $a, b \in L$, the equations $ax = b$ and $ya = b$ have unique solutions.

Given $a, b, c$ in a loop $L$, the loop commutator $(a, b)$ and the loop associator $(a, b, c)$ are defined respectively by

$$ab = ba(a, b),$$

$$(ab)c = [a(bc)](a, b, c).$$

The commutator subloop of a loop is the subloop generated by the set of all commutators and the associator subloop is the subloop generated by the set of all associators. The nucleus and the centre of a loop $L$ are the subloops $\mathcal{N}(L)$ and $\mathcal{Z}(L)$ defined respectively by

$$\mathcal{N}(L) = \{a \in L \mid (a, x, y) = (x, a, y) = (x, y, a) = 1 \forall x, y \in L\}$$

$$\mathcal{Z}(L) = \{a \in \mathcal{N}(L) \mid (a, x) = 1 \forall x \in L\}.$$
Definition 2.3 [12, Page 115]. A loop is called a Moufang loop if its elements satisfy any one of the following three identities (which are equivalent in a loop)

\[(xy.x)z = x(y.xz)\]
\[(xy.z)y = x(y.zy)\]
\[xy.zx = (x.yz)x\]

It follows that the Moufang loops are diassociative ([12, Moufang Theorem, Page 117]) and that ([12, Lemma VII.3.1]) a Moufang loop is also an inverse property loop, i.e., for all \(x, y \in L\), \(x^{-1}(xy) = y\) and \((xy)y^{-1} = x\).

We shall now give some examples of Moufang loops (which are not groups).

Example 2.4. \(M(G, *, g_0)\) loops

Let \(G\) be a nonabelian group and \(g_0 \in G\) be a central element. Let \(g \rightarrow g^*\) be any involution of \(G\) such that \(g_0^* = g_0\) and for every \(g \in G\), \(gg^*\) is a central element. Let \(u\) be an indeterminate and let \(L = G \cup Gu\) be the disjoint union of \(G\) and \(Gu\). Define multiplication on \(L\) by

\[g(hu) = (hg)u\]
\[(gu)h = (gh^*)u\]
\[(gu)(hu) = g_0h^*g\]

for all \(g, h \in G\). Then \(L\) is a Moufang loop denoted by \(M(G, *, g_0)\).

Example 2.5. The loop of units of alternative rings

Suppose \(R\) is an alternative ring with unity. An element \(x \in R\) is called invertible or a unit if there exist elements \(y\) and \(z\) such that \(xy = zx = 1\).
Let $\mathcal{U}(R)$ denote the set of all invertible elements of $R$. We observe that $\mathcal{U}(R)$ is a Moufang loop.

If $xy = zx = 1$, then by [20, Proposition 1.1.11], the alternating nature of the associator and by identity (7) of [20, Page 9]

$$[x, y, z] = (xy)[x, y, z] = y(xz, x, y) = y[zx, x, y] = 0,$$

so that $z = (xy) = (zx)y = y$. We shall denote this element by $x^{-1}$.

If $x$ is invertible and $a$ is arbitrary, then proceeding as above

$$[x^{-1}, x, a] = (xx^{-1})[x^{-1}, x, a] = x^{-1}(x^{-1}, x, a) = x^{-1}x^{-1}x, x, a] = 0.$$

By the generalized theorem of Artin $x, x^{-1}$ and $a$ generate an associative subalgebra of $R$. In particular, this shows that if $x$ is an invertible element in an alternative ring then either of the equations $ax = bx$ and $xa = xb$ implies that $a = b$. Also, for any $a, b \in R$, the equations $ax = b$ and $xa = b$ have unique solutions.

For an alternative ring $R$, the Kleinfeld function $f : R^4 \rightarrow R$ is defined by

$$f(w, x, y, z) = [wx, y, z] - x[y, z] - [x, y, z]w.$$

It follows ([20, Proposition I.1.5]) that $f$ is an alternating function of its arguments.

Let $x$ be invertible and $a, b$ be arbitrary elements of $R$. From the above considerations, $f(a, b, x, x^{-1}) = 0$. Thus, if $[x, a, b] = 0$ then

$$0 = [x^{-1}, x, a, b] = x[x^{-1}, a, b] + [x, a, b]x^{-1} + f(x^{-1}, x, a, b) = x[x^{-1}, a, b],$$

so that $[x^{-1}, a, b] = 0$. Suppose now that $x$ and $y$ are invertible in $R$. Since $[x, y, xy] = 0$, we get that $[x^{-1}, y, xy] = 0$ and hence $[x^{-1}, y^{-1}, xy] = 0$. 


Thus \((xy)(y^{-1}x^{-1}) = 1\). Thus \(U(R)\) is closed under multiplication and consequently is a loop. Since Moufang identities hold in \(R\), \(U(R)\) becomes a Moufang loop.

**Example 2.6. The general linear loop**

Let \(R\) be a commutative and associative ring with unity. The *zorn’s vector matrix algebra* over \(R\) is defined as the set \(M(R)\) of all \(2 \times 2\) "vector matrices" of the form

\[
\begin{bmatrix}
a & x \\
y & b
\end{bmatrix},
\]

where \(a, b \in R\), \(x, y \in R^3\). The addition is defined componentwise and the multiplication is defined by

\[
\begin{bmatrix}
a_1 & x_1 \\
y_1 & b_1
\end{bmatrix}
\begin{bmatrix}
a_2 & x_2 \\
y_2 & b_2
\end{bmatrix} =
\begin{bmatrix}
a_1a_2 + x_1y_2 & a_1x_2 + b_2x_1 - y_1 \times y_2 \\
a_2y_1 + b_1y_2 + x_1 \times x_2 & b_1b_2 + y_1 \times x_2
\end{bmatrix}
\]

where \(\cdot\) and \(\times\) denote respectively the dot and cross products in \(R^3\).

By direct calculations (using the properties of dot and cross products) it follows that \(M(R)\) is an alternative algebra. The function (determinant) \(\det : M(R) \to R\) defined by

\[
\det \begin{bmatrix} a & x \\ y & b \end{bmatrix} = ab - x \cdot y
\]

turns out to be multiplicative. An element

\[
A = \begin{bmatrix} a & x \\ y & b \end{bmatrix}
\]

of \(M(R)\) is invertible if and only if its determinant is an invertible element of \(R\), in which case

\[
A^{-1} = (\det A)^{-1} \begin{bmatrix} b & -x \\ -y & a \end{bmatrix}.
\]
By Example 2.5, the loop of invertible elements in $\mathcal{M}(R)$ is a Moufang loop, called the general linear loop.

§3. Alternative loop rings and RA loops

Let $L$ be a loop and let $R$ be a commutative and associative ring with 1. The loop ring of $L$ with coefficients in $R$ is the free $R$-module $RL$ with basis $L$ and multiplication given by extending, via the distributive laws, the multiplication in $L$. Thus, the elements of $RL$ are formal sums of the type $\sum g \in L$, where $\alpha_g \in R$, almost all $\alpha_g$'s are zero and $\sum \alpha_g g = \sum 3_g g$ implies that $\alpha_g = \beta_g$ for all $g \in L$.

By an alternative loop ring, we mean a loop ring which also happens to be alternative. Such (nonassociative) loops for which $RL$ is alternative actually exist was first shown by E. G. Goodaire [19].

Definition 3.1. An RA (ring alternative) loop is a loop whose loop ring $RL$ over some commutative and associative ring $R$ with unity and of characteristic different from 2 is alternative, but not associative.

Since an alternative ring satisfies Moufang identities and since a loop ring $RL$ contains $L$, it is clear that an RA loop is a Moufang loop.

From [16, Corollary 2.4] it follows that if $L$ is an RA loop then for every ring $R$ (including rings of characteristic 2), $RL$ is an alternative ring. The loops for which the loop rings for a ring of characteristic 2 are alternative have been characterized in [16] and are called $\text{RA}_2$ loops.

The next result summarizes some of the most fundamental properties of
an RA loop.

**Theorem 3.2** [19, Theorem 3]. If $L$ is an RA loop then

(i) $x^2 \in \mathcal{N}(L) \forall x \in L$;

(ii) $\mathcal{N}(L) = \mathcal{Z}(L)$;

(iii) $(g, h) = 1$ for $g, h \in L$ if and only if $(g, h, L) = 1$;

(iv) $(g, h, k) \neq 1$ implies $(g, h, k) = (g, h) = (g, k) = (h, k)$ is a central element of order 2;

(v) $L'$, the commutator and associator subloop of $L$ is an abelian group.

The following theorem, which is due to O. Chein and E. G. Goodaire [15] characterizes RA loops and also gives their construction.

**Theorem 3.3** [15, §3] (see also [20, Theorem IV.3.1]). A loop $L$ is an RA loop if and only if it is not commutative and for any two elements $a$ and $b$ of $L$ which do not commute the subloop of $L$ generated by its centre together with $a$ and $b$ is a group $G$ such that

(i) for any $u \notin G$, $L = G \cup Gu$ is the distinct union of $G$ and the coset $Gu$;

(ii) $G$ has a unique non identity commutator $s$, which is necessarily central and of order 2;

(iii) the map

$$g \rightarrow g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases}$$

is an involution of $G$.
(iv) multiplication in $L$ is defined by

$$g(hu) = (hg)u$$

$$(gu)h = (gh^*)u$$

$$(gu)(hu) = g_0h^*g$$

where $g, h \in G$ and $g_0 = u^2$ is a central element of $G$ such that $g_0^2 = g_0$. Thus $L$ is a loop of the type $M(G, *, g_0)$.

Conversely (by Theorem IV.3.1 and Proposition III.3.6 of [20]) if $G$ is a group such that $G/Z(G) \cong C_2 \times C_2$, then the loop $L = M(G, *, g_0)$, where $g_0$ is any central element of $G$ and $*$ is the involution as defined above, is an RA loop.

If $N$ is a normal subloop of an RA loop $L$ and $R$ is a commutative and associative ring with unity, there is a natural homomorphism $\epsilon : RL \to R(L/N)$ whose kernel, denoted by $\Delta_R(L, N)$ is an ideal of $RL$ ([20, Page 149]). If $N = L$, we write $\Delta_R(L)$ instead of $\Delta_R(L, L)$ and call it the augmentation ideal of $L$. By [20, Lemma VI.1.1],

$$\Delta_R(L, N) = \sum_{n \in N} RL(1 - n) = \sum_{n \in N} (1 - n)RL$$

If $L = M(G, *, g_0)$ is an RA loop and $N$ is a normal subloop of $L$ contained in $G$, then

$$\Delta_R(L, N) = \Delta_R(G, N) + \Delta_R(G, N)u.$$
field such that characteristic of $F$ does not divide the order of $L$, then $FL$ is a direct sum of ideals which are simple rings.

G. Leal and C. P. Milies [34] and later L. G. X. Barros [5] and [6] obtained the decomposition of semisimple alternative loop algebras, which we now describe briefly.

Let $L = M(G, \ast, g_0)$ be an RA loop and let $F$ be a field such that characteristic of $F / |L|$. Writing $\hat{G}' = \frac{1+\ast}{2}$ it follows that

$$FG = F(G/G') \oplus \Delta_F(G, G') \text{ and } FL = F(L/L') \oplus \Delta_F(L, L').$$

Moreover, $F[G/G'] \cong FG\hat{G}'$ and $\Delta_F(G, G') \cong FG(1 - \hat{G}')$; $FL\hat{G}' \cong FL.\hat{G}'$ and $\Delta_F(L, L') \cong FL(1 - \hat{G}')$.

Also, $FL\hat{G}' = FL\frac{1+\ast}{2}$ is a direct sum of components which are both associative and commutative (since $L/L'$ is an abelian group) and the simple components of $FL(1 - \hat{G}') = FL\frac{1+\ast}{2}$ are neither associative nor commutative. In fact $\Delta_F(G, G') \cong \oplus_{i=1}^{n} B_i$, where each $B_i$ is an algebra of generalized quaternions and $\Delta_F(L, L') \cong \oplus_{i=1}^{n} A_i$, where each $A_i$ is a Cayley-Dickson algebra over some suitable extension of $F$.

The basic properties of RA loops and alternative loop rings are given in [20].