Chapter 2

Bottleneck Transportation problems

A study of unbalanced capacitated bottleneck transportation problem and an interval bottleneck transportation problem is carried out in the first and second section of this chapter respectively. Due to certain restrictions, the shipment in both the problems is forced to be done in two stages. The aim is to find a solution that minimizes the sum of first stage and second stage shipment times. Polynomial time algorithms are proposed to solve the problems. At various stages of the algorithms, restricted versions of the standard bottleneck transportation problem are examined and finally a global optimal solution is achieved.

2.1 Capacitated bottleneck transportation problem

A capacitated two stage time minimization transportation problem is under consideration, in which the total availability of a homogeneous product at various sources is more than the total requirement of the same at destinations. Unlike the conventional unbalanced time minimization transportation problem, in the current problem, transportation takes place in two stages such that the minimum requirement of the destinations is satisfied in first stage and the surplus amount is transported in
the second stage. Each time the transportation from sources to destinations is done in parallel and the capacity on each route (source destination link) remains fixed i.e. the total amount transported in both the stages cannot exceed its upper bound. In each stage, the objective is to minimize the shipment time and the overall goal is to find a solution that minimizes the sum of first and second stage times.

2.1.1 Mathematical Formulation

Let \( a_i, \ i \in I \) be the availability of a homogeneous product at \( i^{th} \) source and let \( b_j, \ j \in J \) be the requirement of the same at \( j^{th} \) destination, where \( \sum_{i \in I} a_i > \sum_{j \in J} b_j \).

Since the total availability is more than the total requirement, the shipment is done in two stages. In first stage, the total requirement at destinations is satisfied and the surplus amount is supplied in second stage.

The Stage I problem is formulated as:

\[
(P_{2.1a}) \quad \min_{X=\{x_{ij}\} \in S} \left[ \max_{x_{ij} \in S} \left( t_{ij}(x_{ij}) \right) \right]
\]

where the set \( S \) is given by

\[
S : \begin{cases} 
\sum_{j \in J} x_{ij} \leq a_i & \forall \ i \in I, \\
\sum_{i \in I} x_{ij} = b_j & \forall \ j \in J, \\
0 \leq x_{ij} \leq u_{ij} & \forall \ (i, j) \in I \times J.
\end{cases}
\]

Corresponding to a feasible solution \( X = \{x_{ij}\} \) of Stage I problem, let \( S(X) \) be the set of feasible solutions of Stage II problem which is stated below

\[
(P_{2.2a}) \quad \min_{X=\{x_{ij}\} \in S(X)} \left[ \max_{x_{ij} \in S(X)} \left( t_{ij}(x_{ij}) \right) \right]
\]
where the set $S(X)$ is given by

$$
S(X) : \begin{cases}
\sum_{j \in J} \bar{x}_{ij} = a'_i & \forall i \in I, \\
\sum_{i \in I} \bar{x}_{ij} \geq 0 & \forall j \in J, \\
0 \leq \bar{x}_{ij} \leq \bar{u}_{ij} & \forall (i, j) \in I \times J,
\end{cases}
$$

where $\bar{u}_{ij} = u_{ij} - x_{ij}$ and $a'_i = a_i - \sum_{j=1}^{n} x_{ij} \forall i \in I$.

We aim at finding a feasible solution of Stage I corresponding to which the optimal time of Stage II is such that the sum of shipment times of both the stages is the least. Thus a Two-stage capacitated time minimizing transportation problem is defined as:

$$
(P_{2,3a}) \quad \min_{\chi = \{x_{ij}\} \in S} \left[ \max(t_{ij}(x_{ij})) + \min_{\chi \in S(X)} \left[ \max(t_j(\bar{x}_{ij})) \right] \right]
$$

The set of feasible solutions of the problem $(P_3)$ is union of the sets of feasible solutions of the Stage I and Stage II problem. To find a feasible solution of Stage I problem we define the following equivalent balanced (CTMTP) problem:

$$
(P_{2,1a}) \quad \min_{\gamma = \{y_{ij}\} \in S' \times J'} \left[ \max(t'_{ij}(y_{ij})) \right]
$$

where the set $S'$ is given by

$$
S' : \begin{cases}
\sum_{j \in J} y_{ij} = a_i & \forall i \in I, \\
\sum_{i \in I} y_{ij} = b'_j & \forall j \in J', \\
0 \leq y_{ij} \leq u'_ij & \forall (i, j) \in I \times J'
\end{cases}
$$

and

$$
t'_{ij}(y_{ij}) = t'_{ij} \quad \text{if } y_{ij} > 0, \\
= 0 \quad \text{otherwise}
$$
where,
\[ J' = J \cup \{n + 1\}, \]
\[ J^*_j = b_j \forall j \in J, \]
\[ J^*_n = \sum_{i \in I} a_i - \sum_{j \in J} b_j, \]
\[ t^*_{ij} = t_{ij} \forall (i, j) \in I \times J, \]
\[ t^*_{n+1} = 0 \forall i \in I, \]
\[ u^*_{ij} = u_{ij} \forall (i, j) \in I \times J, \]
\[ u^*_{n+1} = \infty \forall i \in I. \]

The Problem \((P_{2.1a})\) can be looked upon as a capacitated transportation problem with bounds on rim conditions which can be easily solved by the algorithm discussed by Dahiya and Verma [29].

### 2.1.2 Theoretical Development

An (OFS) of Stage 1 problem \((P_{2.1a})\) is obtained from an (OFS) of the related problem \((P_{2.1a}^*)\), as \((P_{2.1a}^*)\) is a balanced (CTMTP) and can be easily solved. In Theorem 2.1.1 & 2.1.2, an equivalence has been established between problems \((P_{2.1a})\) and \((P_{2.1a}^*)\).

**Theorem 2.1.1** Corresponding to every feasible solution of the problem \((P_{2.1a})\) there exists a feasible solution of the problem \((P_{2.1a}^*)\) and vice versa.

**Proof.** Let \(X = \{x_{ij}\}\) be a feasible solution of \((P_{2.1a})\).

Define \(\{y_{ij}\}\) as follows :

\[
\begin{align*}
    y_{ij} &= x_{ij} \forall (i, j) \in I \times J, \\
    y_{n+1} &= a_i - \sum_{j \in J} x_{ij} \forall i \in I.
\end{align*}
\]

Then, for \(i \in I\),

\[
\sum_{j=1}^{n+1} y_{ij} = \sum_{j=1}^{n} y_{ij} + y_{n+1} = \sum_{j=1}^{n} x_{ij} + (a_i - \sum_{j=1}^{n} x_{ij}) = a_i.
\]
For each $j \in J$, 

$$
\sum_{i=1}^{m} y_{ij} = \sum_{i=1}^{m} x_{ij} = b_{j} \quad \because x_{ij} = y_{ij} \forall (i, j) \in I \times J
$$

and

$$
\sum_{i=1}^{m} y_{in+1} = \sum_{i=1}^{m} (a_{i} - \sum_{j=1}^{n} x_{ij})
= \sum_{i=1}^{m} a_{i} - \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij}
= \sum_{i=1}^{m} a_{i} - \sum_{j=1}^{n} b_{j} = b'_{n+1}.
$$

Also for each $(i, j) \in I \times J$, $0 \leq x_{ij} \leq u_{ij}$. 

This implies that $0 \leq y_{ij} \leq u'_{ij} (= u_{ij}) \forall (i, j) \in I \times J$.

For each $i \in I$, 

$$
y_{m+1} = a_{i} - \sum_{j=1}^{n} y_{ij} = a_{i} - \sum_{j=1}^{n} x_{ij} \leq a_{i} < \infty
$$

and

$$
y_{m+1} = a_{i} - \sum_{j=1}^{n} x_{ij} \geq 0.
$$

Thus $0 \leq y_{m+1} < \infty$ ($= u'_{m+1}$) $\forall i \in I$. Therefore the solution $y_{ij}$ defined by equation (2.1.1) is a feasible solution of $(P_{21a})$. Conversely, let $\{y_{ij}\}_{I \times J'}$ be a feasible solution of $(P_{21a})$. Define $\{x_{ij}\}$ as follows:

$$
x_{ij} = y_{ij} \forall (i, j) \in I \times J. \quad (2.1.2)
$$

Since each $0 \leq y_{ij} \leq u'_{ij}(= u_{ij})$, we have, $0 \leq x_{ij} \leq u_{ij} \forall (i, j) \in I \times J$.

For $i \in I$,

$$
\sum_{j=1}^{n+1} y_{ij} = a_{i},
$$

$$
\Rightarrow \sum_{j=1}^{n} y_{ij} = a_{i} - y_{m+1} \leq a_{i} \; (\because y_{m+1} \geq 0).
$$
\[ \Rightarrow \sum_{j=1}^{n} x_{ij} \leq a_i \quad \forall \ i \in I. \]

Also, \[ \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} y_{ij} = b_j \quad \forall \ j \in J. \]

Therefore \( \{x_{ij}\} \) defined by (2.1.2) is a feasible solution of the problem \( (P_{2.1a}) \).

**Theorem 2.1.2** Corresponding feasible solutions of problem \( (P_{2.1a}) \) and \( (P^*_{2.1a}) \) yield same value of objective functions.

**Proof.** Let \( Y = \{y_{ij}\} \) be a feasible solution of problem \( (P^*_{2.1a}) \) yielding time \( T^*_1 \) and let \( X = \{x_{ij}\} \) be the corresponding feasible solution of the problem \( (P_{2.1a}) \) giving value \( T_1 \).

\[
T^*_1 = \max_{I \times J} (t'_{ij}(y_{ij}) \mid y_{ij} > 0) = \max \left[ \max_{I \times J} (t'_{ij}(y_{ij}) \mid y_{ij} > 0), \max_{I \times J} (t'_{in+1}(y_{in+1}) \mid y_{in+1} > 0) \right] = \max \left[ \max_{I \times J} (t_{ij}(y_{ij}) \mid y_{ij} > 0), 0 \right] = \max_{I \times J} (t_{ij}(y_{ij}) \mid y_{ij} > 0) = \max_{I \times J} (t_{ij}(x_{ij}) \mid x_{ij} > 0) = T_1.
\]

**Corollary 2.1.3** Problems \( (P_{2.1a}) \) and \( (P^*_{2.1a}) \) are equivalent.

**Proof.** Proof is a direct consequence of Theorem-2.1.1 and Theorem-2.1.2.

**Corollary 2.1.4** An optimal feasible solution of Problem \( (P^*_{2.1a}) \) gives minimum time for Stage I.

**Proof.** It follows from Theorem-2.1.1 and Theorem-2.1.2.

If \( \{y_{ij}\} \) is a feasible solution of the problem \( (P^*_{2.1a}) \), then \( X = \{x_{ij}\} \), where \( x_{ij} = y_{ij} \forall (i,j) \in I \times J \), is a feasible solution of Stage I problem and a corresponding (OFS) of Stage II problem \( (P_{2.2a}) \) is obtained, thereby yielding a feasible solution of problem \( (P_{2.3a}) \). In order to find an (OFS) of problem \( (P_{2.3a}) \), an iterative algorithm is proposed which generates a sequence of feasible solutions of Stage I and Stage II.
yielding a pair of Stage I and Stage II times as \((T^k_1, T^k_2)\) at \(k^{th}\) iteration. To avoid a number of undesirable solutions of the problem \((P_{2.3a})\), specific restricted versions of the problem \((P_{2.1a})\) are introduced at each iteration of the algorithm and their optimal feasible solutions are investigated. In each restricted problem, we aim at decreasing the Stage II time.

**Restricted version of problem \((P_{2.1a})\)**

Let at \(k^{th}\) iteration of the algorithm, Stage I time be \(T^k_1\) and the corresponding optimal time for Stage II be \(T^k_2\). The restricted version of the problem \((P_{2.1a})\) is defined at \(T^k_2\) and denoted by \(P_{2.1a}(T^k_2)\). The following restrictions are imposed on the problem \((P_{2.1a})\).

1. For \(i \in I\), put \(t_{m+1} = M\) for which \(\min_{j \in J} (t_{ij}) \geq T^k_2\), where \(M\) is a very large positive number.

2. Set \(t_{ij} = M\) \(\forall (i, j) \in I \times J\) for which \(t_{ij} + T^k_2 \geq \min_{h=0, 1, \ldots, k-1} (T^h_1 + T^h_2)\), where \(T^k_1\) is the lower bound on the Stage II time.

3. Let \(I^k = \{i \in I \mid \min_{j \in J} (t_{ij}) < T^k_2\}\) and for \(i \in I^k\), let \(J^k = \{j \in J \mid t_{ij} < T^k_2\}\). If \(\sum_{j \in J^k} u_{ij} < a_i\), then apply partial sum constraint defined by \(\sum_{j \in J^k} y_{ij} + y_{m+1} \leq \sum_{j \in J^k} u_{ij}\).

Therefore the problem \(P_{2.1a}(T^k_2)\) takes the following form

\[
P_{2.1a}(T^k_2) = \min_{Y = \{y_{ij}\} \in S'_{(k)}} \left[ \max_{Y' = \{y_{ij}'\} \in S'_{(k)}} \right]
\]

where the set \(S'_{(k)}\) is given by

\[
S'_{(k)} = \left\{ \begin{array}{l}
\sum_{j \in J^k} y_{ij} = a_i \quad \forall i \in I,

\sum_{i \in I} y_{ij} = b_j \quad \forall j \in J',

\sum_{j \in J^k} y_{ij} + y_{m+1} \leq \sum_{j \in J^k} u_{ij} \quad \forall i \in I^k,

0 \leq y_{ij} \leq u_{ij}' \quad \forall (i, j) \in I \times J',
\end{array} \right\}
\]
and

\[ J' = J \cup \{n + 1\}, \]

\[ b'_j = b_j \quad \forall j \in J, \]

\[ b'_{n+1} = \sum_{i \in I} a_i - \sum_{j \in J} b_j, \]

\[ t'_{ij} = t_{ij} \quad \forall (i,j) \in L, \]

where \(L = \{(i,j) \mid t_{ij} + T_{ij}^k < \min_{h=0,1,...,k-1} \left[T_{ij}^h + T_{i,j}^k\right]\}. \]

\[ t'_{ij} = M \quad \forall (i,j) \in I \times J \setminus L, \]

\[ t'_{m+1} = 0 \quad \forall i \in I^k, \]

\[ t'_{m+1} = M \quad \forall i \in I \setminus I^k, \]

\[ u'_{ij} = u_{ij} \quad \forall (i,j) \in I \times J, \]

\[ u'_{m+1} = \infty \quad \forall i \in I. \]

**Definition (M-feasible solution).** A feasible solution \(Y = \{y_{ij}\}\) of problem \(P_{2,1a}(T_{ij}^k)\) is an M-feasible solution if \(y_{ij} = 0 \quad \forall (i,j) \in I \times J\) for which \(t_{ij} = M\).

Let (OFS) of the problem \(P_{2,1a}(T_{ij}^k)\) be denoted by \(Y^{k+1}\). Let \(X_{1}^{k+1}\) be the corresponding feasible solution of Stage I problem and \(X_{2}^{k+1}\) be an (OFS) of Stage II problem corresponding to \(X_{1}^{k+1}\), \(T(X_{1}^k) = T_{2}^k\) and \(T(X_{2}^k) = T_{2}^k\). Corresponding to any feasible solution \(Y = \{y_{ij}\}\) of \(P_{2,1a}(T_{ij}^k)\), a feasible solution \(X_{1}^{k+1} = \{x_{ij}\}\) of Stage I is obtained by using (2.1.2) and an (OFS), \(X_{2}^{k+1} = \{\bar{x}_{ij}\}\) of stage II problem corresponding to \(X_{1}^{k+1}\) is defined below.

For each \(i \in I\), let \(J_{i}^{k} = \{j_1, j_2, \ldots, j_{k_i}\} \subseteq J\),

\[
\begin{align*}
\bar{x}_{ij} &= 0 \quad \forall j \notin J_{i}^{k}, \\
\bar{x}_{ijl} &= \min\{\bar{u}_{ijl}, y_{m+1}\}, \\
\bar{x}_{ijl} &= \min\{\bar{u}_{ijl}, y_{m+1} - \sum_{i=1}^{l-1} \bar{x}_{ijl}\}, \quad l = 2, 3, \ldots, k_i,
\end{align*}
\]

where \(\bar{u}_{ijl} = u_{ij} - \bar{x}_{ijl}\) and \(p < q \Rightarrow t_{ijp} \leq t_{ijq}\) for \(p, q \in \{1, 2, \ldots, k_i\}\). The existence of this (OFS) is justified in the next Theorem. Also, it has been discussed that at each iteration the Stage II time is strictly decreasing.

**Theorem 2.1.5** At an M-feasible optimal solution of the problem \(P_{2,1a}(T_{ij}^k)\), an (OFS) \(X_{2}^{k+1}\) of the corresponding Stage II problem exists such that \(T(X_{2}^{k+1}) < \).
\( T(\tilde{X}^k) \).

**Proof.** Let \( Y^{k+1} \) be an M-feasible solution, such that it is an (OFS) of \( P_{2in}(T^I) \). Let \( X^{k+1}_1 \) be the corresponding feasible solution of Stage I problem defined by (2.1.2) and \( \tilde{X}^{k+1}_2 = \{ \tilde{x}_{ij} \} \) as defined by (2.1.3).

**Claim 1.** \( \tilde{X}^{k+1}_2 \) is a feasible solution of Stage II problem.

Let if possible, \( \tilde{X}^{k+1}_2 \) defined by (2.1.3) is not a feasible solution of Stage II problem. Then there exists some \( i \in I \), such that

\[
\sum_{j \in J} \tilde{x}_{ij} \neq y_{in+1} \left( = a_i - \sum_{j \in J} x_{ij} \right).
\]

\[
\Rightarrow \sum_{j \in J} \tilde{x}_{ij} < y_{in+1}. \tag{2.1.4}
\]

Note that \( \sum_{j \in J} \tilde{x}_{ij} \neq y_{in+1} \), otherwise it contradicts the choice of \( \{ \tilde{x}_{ij} \} \).

Using (2.1.3) and (2.1.4), we have

\[
\sum_{j \in J} \tilde{x}_{ij} < y_{in+1}, \tag{2.1.5}
\]

\[
\Rightarrow \tilde{x}_{ij} = \tilde{u}_{ij}, \quad l = 1, 2, \ldots, k_i.
\]

Therefore, from equation (2.1.5),

\[
\sum_{j \in J_i} \tilde{u}_{ij} < y_{in-1},
\]

\[
\Rightarrow \sum_{j \in J_i} (u_{ij} - \tilde{u}_{ij}) < y_{in+1},
\]

\[
\Rightarrow \sum_{j \in J_i} (u_{ij} - y_{ij}) < y_{in+1},
\]

\[
\Rightarrow \sum_{j \in J_i} y_{ij} + y_{in+1} > \sum_{j \in J_i} u_{ij}.
\]

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This implies \{y_i\} is not a feasible solution of \(P_{2,1e}(T_2^k)\), which is a contradiction. Therefore \{x_i\} defined by (2.1.3) is a feasible solution of Stage II problem.

**Claim 2.** \(X_{k+1}^l\) is an (OFS) of Stage II problem corresponding to the feasible solution \(X_{k+1}^l\) of Stage I problem.

Let \(X = \{x_{ij}'\}\) be any feasible solution of Stage II problem corresponding to a feasible solution \(X_{k+1}^l\) of Stage I, i.e., \(\{x_{ij}'\} \in S(X_l^k)\). Let for each \(i \in I\), \(\max_{j \in J}(t_{ij}(x_{ij})) = t_i^l\) and \(\max_{j \in J}(t_{ij}(x_{ij}')) = t_i^{l'}\). Let \(\{j_1, j_2, \ldots, j_n\}, j_i \in J\) be indexed in such a way that \(p < q\) iff \(t_{ij_p} < t_{ij_q}\). We claim that for each \(i \in I\), \(t_i^{l'} \leq t_i^l\). Let if possible, \(t_i^{l'} < t_i^l\) for some \(i \in I\).

Let \(t_i^{l'} = t_{ij}, t_i^l = t_{ij'}, j_i, j_s \in J\). Then

\[
t_{ij} < t_{ij'} \Rightarrow r < s.
\]  
(2.1.6)

Since \(x_{ij}' \leq u_{ij} \quad \forall (i, j) \in I \times J,\)

\[
\Rightarrow \sum_{i=1}^{r} x_{ij}' \leq \sum_{i=1}^{r} u_{ij}.
\]  
(2.1.7)

Also,

\[
\sum_{j=1}^{n} x_{ij}' = \sum_{i=1}^{r} x_{ij}' = a_i'.
\]  
(2.1.8)

By choice of \(\bar{X} = \{\bar{x}_{ij}\}\), as defined in (2.1.3), \(\bar{x}_{ij} = \bar{u}_{ij}, l = 1, 2, \ldots, s - 1\) and \(0 < \bar{x}_{ij} \leq \bar{u}_{ij}\).

Since

\[
\sum_{l=1}^{s} \bar{x}_{ij} = \sum_{l=1}^{s-1} \bar{u}_{ij} + \bar{x}_{ij} = a_i',
\]

\[
\Rightarrow \sum_{l=1}^{s-1} \bar{u}_{ij} < a_i'.
\]  
(2.1.9)

From (2.1.6),

\[
\sum_{l=1}^{r} \bar{u}_{ij} \leq \sum_{l=1}^{s-1} \bar{u}_{ij}.
\]  
(2.1.10)
From (2.1.7)-(2.1.10), we have

$$a_i' \leq \sum_{i=1}^{r} u_{ij} \leq \sum_{i=1}^{s-1} u_{ij} < a_i,$$

which is a contradiction. Hence, $t^*_i \leq t^*_i \forall i \in I$.

$$\Rightarrow \max_{i \in I}(t^*_i) \leq \max_{i \in I}(t^*_i),$$

$$\Rightarrow T(X^{k+1}) \leq T(X^*).$$

Thus, $X^{k+1}$ is an (OFS) of Stage II problem corresponding to a feasible solution $X_1^{k+1}$ of Stage I. Also by construction of $X_2^{k+1}$, it is clear that $T(X^{k+1}) < T_2^k$. Therefore, $T(X^{k+1}) < T(X^*_k)$.

**Theorem 2.1.6** Let an (OFS) of the problem $P_{2,1a}(T_2^k)$ is an M-feasible solution. Then $T(X_1^{k+1}) \geq T(X^*_k)$.

**Proof.** Let if possible $T_1^{k+1} < T^*_k$. Now in the problem $P_{2,1a}(T_2^k)$, by its definition, $t_{ij} < M \forall (i,j) \in I \times J$ for which $t_{ij} = T_1^{k+1}$ and $t_{m+1} = 0$ for $i \in I$ for which $\min(t_{ij}) \leq T_2^k < T_2^{k-1}$ (Theorem-2.1.5). As $T_2^k < T_2^{k-1} \Rightarrow J^k \subset J_1^{k-1}$.

Thus any feasible solution $Y = \{y_{ij}\}$ which satisfies

$$\sum_{j \in J^k} y_{ij} + y_{m+1} \leq \sum_{j \in J^k} u_{ij}$$

will also satisfy

$$\sum_{j \in J_1^{k-1}} y_{ij} + y_{m+1} \leq \sum_{j \in J_1^{k-1}} u_{ij}.$$  

Therefore an (OFS), $Y^{k+1}$ of the problem $P_{2,1a}(T_2^k)$ giving time of transportation of Stage I as $T_1^{k+1}$ is a feasible solution of the problem $P_{2,1a}(T_2^{k-1})$ as it also satisfies partial sum constraints. Thus $Y^{k+1}$ is a feasible solution of the problem $P_{2,1a}(T_2^{k-1})$ with associated time $T_1^{k+1} < T_1^k$. This is a contradiction as $T_1^k$ is the optimal time of transportation for $P_{2,1a}(T_2^{k-1})$. Hence $T_1^{k+1} \geq T_1^k$.  

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Theorem 2.1.7 If $\hat{T}_1 + \hat{T}_2 = \min_{k \geq 0} [T^k_1 + T^k_2]$, where $T^k_1$ and $T^k_2$ are the times of transportation of Stage I and Stage II respectively corresponding to an M-feasible optimal solution of the problem $P_{2,1a}(T^{k-1}_2)$, $k \geq 1$, then $\hat{T}_1 + \hat{T}_2$ is the optimal value of objective function of the problem $(P_{2,2a})$.

Proof. Let if possible, $\hat{T}_1 + \hat{T}_2$ be not the optimal value of the objective function of problem $(P_{2,2a})$, then there exists a pair $(\bar{T}_1, \bar{T}_2)$ yielded by some feasible solution $\bar{Y}$ of problem $(P_{2,2a})$ such that

$$\bar{T}_1 + \bar{T}_2 < \hat{T}_1 + \hat{T}_2. \quad (11)$$

Clearly $\bar{T}_2 \geq T^0_2$. Also $\bar{T}_2 \leq T^0_2$ because if $\bar{T}_2 > T^0_2$ and $\bar{T}_1 + \bar{T}_2 < T^0_1 + T^0_2 \Rightarrow \bar{T}_1 < T^0_1$, which contradicts the optimality of $T^0_1$. Thus $T^0_1 \leq \bar{T}_2 \leq T^0_2$.

If $(T^*_1, T^*_2)$ is a pair such that either the (OFS) of the problem $P_{2,1a}^*(T^*_2)$ is not an M-feasible solution or $T^*_2 = T^0_2$, then $\bar{T}_2 \neq T^*_2$. Hence there exists an index $k \geq 1$ such that $T^k_2 \leq \bar{T}_2 \leq T^k_2$.

Case 1: $\bar{T}_2 > T^k_2$.

Consider the problem $P_{2,1a}^*(T^{k-1}_2)$. Let $L = \{(i,j) \mid t_{ij} + T^k_2 < \min_{h=0,1,\ldots,k-1} [T^h_1 + T^h_2]\}$.

Since $\bar{T}_1 + \bar{T}_2 < T^k_1 + T^k_2$ and $\bar{T}_2 > T^k_2$, we have $\bar{T}_1 < T^k_1$. Therefore, the set $\{(i,j) : t_{ij} = \bar{T}_1\} \subset L$. If $\bar{Y}$ is a feasible solution of $P_{2,1a}^*(T^{k-1}_2)$, then $\bar{Y}$ yields a better time $\bar{T}_1$ as compared to $T^k_1$, which contradicts the optimality of $T^k_1$. If $\bar{Y}$ is a not feasible solution of $P_{2,1a}^*(T^{k-1}_2)$, then for at least one $i \in I^{k-1}$,

$$\sum_{j \in J^{k-1}_i} y_{ij} + y_{i,n+1} > \sum_{j \in J^{k-1}_i} u_{ij}.$$ 

For this $i$, there exists $j \in J$ such that $y_{ij} > 0$ and the corresponding $t_{ij} > T^{k-1}_2$. This implies that $\bar{T}_2 > T^{k-1}_2$. Thus we have a solution of Stage II for which $\bar{T}_2 > T^{k-1}_2$.

Then in the problem $P_{2,1a}^*(T^{k-2}_2)$, we have $\bar{T}_1 < T^{k-1}_1$. Again, either $\bar{Y}$ is a feasible solution of $P_{2,1a}^*(T^{k-2}_2)$ or it violates the partial sum constraint for some $i \in I$.

Continuing like this we get, $\bar{T}_2 > T^0_2$ which implies $\bar{T}_1 < T^0_1$. But $T^0_1$ is the optimal value of $(P_{2,1a})$ (by Corollary-2.1.4). Also by assumption $\bar{T}_2 \leq T^0_2$, which is not true.
Therefore $T_2 \neq T_2^k$.

Case 2: $T_2 = T_2^k$.

In this case also, we get $T_1 < T_1^k$ as discussed in Case 1. Thus, there does not exist $(\hat{T}_1, \hat{T}_2)$ satisfying (11). Hence, $T_1 + \hat{T}_2$ is the optimal value of objective function of the problem $(P_{2,3a})$.

Computing $T_2^k$

Find $\min \{t_{ij}\} = t_{r_1s_1}$. If $b'_{n+1} \leq u_{r_1s_1}$, then $T_2^k = t_{r_1s_1}$ else, find $\min_{i \times J \setminus \{(r_1, s_1)\}} t_{ij} = t_{r_2s_2}$ if $b'_{n+1} \leq u_{r_1s_1} + u_{r_2s_2}$ then $T_2^k = T_{r_2s_2}$. Continuing in this way we get,

$$\min_{i \times J \setminus \{(r_1, s_1), (r_2, s_2), \ldots, (r_k, s_k)\}} t_{ij} = t_{r_{k+1}s_{k+1}}.$$

If

$$b'_{n+1} \leq \sum_{i=1}^{k-1} u_{r_is_i}$$

and

$$\sum_{i=1}^{k} u_{r_is_i} < b'_{n+1},$$

then

$$T_2^k = t_{r_{k+1}s_{k+1}}.$$

2.1.3 Algorithm

**Initial Step:** Obtain an (OFS) of the problem $(P_{2,1a})$ and note the time of Stage I and Stage II say $T_1^0$ and $T_2^0$ respectively. If $T_2^0 = T_2^k$, then stop and go to Terminal Step, else, go to General step.

**General Step:** Let the pairs in hand be $(T_i^g, T_2^g)$ for $g=1, 2, \ldots, k-1$. Construct the problem $P_{2,1a}(T_2^{g-1})$ and find its (OFS). If this is not M-feasible solution then stop and go to terminal step, otherwise read the time $T_1^g$ of Stage I and $T_2^g$ of Stage II. If $T_2^g = T_2^k$, stop and go to terminal step, else repeat the general step for next higher values of $k$.

**Terminal Step:** Declare $\min_{r \geq 0} \{T_1^r + T_2^r\}$ as the optimal value of the objective function.
of the problem \((P_{2,3a})\).

2.1.4 Examples

In this section two examples are presented, the first one gives a overview of the algorithm discussed and the another one is a counter example depicting better time than the one obtained by Sonia et al. [86].

**Example 1.** Consider the following \(6 \times 4\) transportation problem given in Table-2.1. Table-2.2 gives maximum capacity of each \((i,j)^{th}\) route:

Here \(T_2^L = 3\).

<table>
<thead>
<tr>
<th>(S_i)</th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
<th>(a_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>(S_2)</td>
<td>7</td>
<td>9</td>
<td>12</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>(S_3)</td>
<td>2</td>
<td>8</td>
<td>7</td>
<td>4</td>
<td>45</td>
</tr>
<tr>
<td>(S_4)</td>
<td>11</td>
<td>5</td>
<td>9</td>
<td>8</td>
<td>25</td>
</tr>
<tr>
<td>(S_5)</td>
<td>6</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>50</td>
</tr>
<tr>
<td>(S_6)</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>(b_j)</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>40</td>
<td>-</td>
</tr>
</tbody>
</table>

**Initial Step.** An (OBFS) of the problem \((P_{2,3a})\) gives \(T^0_1 = 7\) and the corresponding \(T^0_2 = 9\). So the current value of \(T^0_1 + T^0_2 = 16\). Since \(T^0_2 > T^0_1\). Go to general step of the algorithm.

**General Step.**

**Iteration 1.** Construct the problem \(P_{2,3a}(T^0_2)\), its (OBFS) yields value \(T^1_1 = 9\) and

<table>
<thead>
<tr>
<th>(S_i)</th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td>10</td>
<td>20</td>
<td>15</td>
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<tr>
<td>(S_2)</td>
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<tr>
<td>(S_3)</td>
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<td>15</td>
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<td>20</td>
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<td>(S_4)</td>
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<td>(S_5)</td>
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<tr>
<td>(S_6)</td>
<td>10</td>
<td>20</td>
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<td>15</td>
</tr>
</tbody>
</table>
the corresponding $T_j^2 = 8$. The current value of $T_1 + T_2$ is $\min(T_1^0 + T_2^0, T_1^1 + T_2^1) = 16$. As $T_j^2 > T_k^2$, solve $P_{2,1a}(T_j^2)$.

**Iteration 2.** $P_{2,1a}(T_j^2)$ yields time $(T_j^2, T_k^2) = (9, 7)$.

**Iteration 3.** $P_{2,1a}(T_j^2)$ yields time $(T_j^3, T_k^3) = (9, 6)$.

**Iteration 4.** $P_{2,1a}(T_j^2)$ yields time $(T_j^4, T_k^4) = (9, 4)$.

**Iteration 5.** $P_{2,1a}(T_j^2)$ yields time $(T_j^5, T_k^5) = (9, 3)$. Since $T_j^5 = T_k^5$, stop and go to terminal step.

**Terminal Step.** The optimal value of the objective function of the problem ($P_{2,3a}$) is given by $\min_{h=0, 1, \ldots, 5} (T_j^h + T_k^h) = 12$. An Optimal feasible solutions of the problem $P_{2,1a}^*(T_j^2)$ is shown in the Table-2.3. In Table-2.3, the entries at the lower right corner of each cell give the time of transportation. Note that the entries in boldface represent the basic cells and the entries of the form $\bar{a}$ represent the nonbasic cells.

<table>
<thead>
<tr>
<th>$a_i \downarrow$</th>
<th>5</th>
<th>15</th>
<th>10</th>
<th>3</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
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<tr>
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<td>3</td>
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</tr>
<tr>
<td>45</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>bj →</th>
<th>40</th>
<th>50</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
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<tr>
<td>50</td>
<td>4</td>
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<tr>
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<td>4</td>
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<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
which are at their upper bounds respectively. Feasible solutions of Stage I and Stage II problems corresponding to the (OFS) of the problem $P^*_{2,1a}(T^4)$ are depicted in Table-2.4 and Table-2.5 respectively.

<table>
<thead>
<tr>
<th></th>
<th>5</th>
<th>6</th>
<th>15</th>
<th>10</th>
<th>3</th>
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<tbody>
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<td>20</td>
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<td>9</td>
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<tr>
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<td>15</td>
<td>8</td>
<td>7</td>
<td>4</td>
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<tr>
<td>11</td>
<td>10</td>
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<tr>
<td>12</td>
<td>4</td>
<td>2</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.4: Solution $X^*_1$

$$b_j \rightarrow \begin{array}{cccc}
40 & 50 & 30 & 40
\end{array}$$
Example 2. Consider a 10 x 5 two stage time minimization transportation problem, discussed by Sonia et al. [86]. Time of transportation along various routes is given in Table 2.6. The optimal solution of this problem by the method suggested by Sonia et al. [86] comes out to be 12, with corresponding time for Stage I and Stage II be 8 and 4 respectively. Whereas the current algorithm, when applied to same problem yields a better solution. The algorithm computes Stage I time as 3, Stage II time as 7 and overall time of transportation as 10. Feasible solutions so obtained for Stage I and Stage II are depicted in Table 2.7 and Table 2.8 respectively. Here the entries in the boldface letters depicts the amount transported on \((i,j)\)th route.
Table 2.6: Time of transportation along various routes

<table>
<thead>
<tr>
<th></th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
<th>(D_5)</th>
<th>(a_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>(S_2)</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>(S_3)</td>
<td>12</td>
<td>25</td>
<td>9</td>
<td>6</td>
<td>26</td>
<td>10</td>
</tr>
<tr>
<td>(S_4)</td>
<td>8</td>
<td>7</td>
<td>9</td>
<td>24</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>(S_5)</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td>(S_6)</td>
<td>11</td>
<td>24</td>
<td>20</td>
<td>12</td>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>(S_7)</td>
<td>11</td>
<td>1</td>
<td>5</td>
<td>4</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
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<td>3</td>
<td>10</td>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>(S_9)</td>
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<td>7</td>
<td>10</td>
<td>5</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>(S_{10})</td>
<td>19</td>
<td>6</td>
<td>16</td>
<td>2</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>(b_j)</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.7: Feasible solution of Stage I

<table>
<thead>
<tr>
<th></th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
<th>(D_5)</th>
<th>(a_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_1)</td>
<td><strong>10</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 20)</td>
</tr>
<tr>
<td>(S_2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 30)</td>
</tr>
<tr>
<td>(S_3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 10)</td>
</tr>
<tr>
<td>(S_4)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 15)</td>
</tr>
<tr>
<td>(S_5)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 15)</td>
</tr>
<tr>
<td>(S_6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>20</strong></td>
<td>(\leq 25)</td>
</tr>
<tr>
<td>(S_7)</td>
<td><strong>10</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 10)</td>
</tr>
<tr>
<td>(S_8)</td>
<td></td>
<td><strong>15</strong></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 25)</td>
</tr>
<tr>
<td>(S_9)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>10</strong></td>
<td>(\leq 10)</td>
</tr>
<tr>
<td>(S_{10})</td>
<td></td>
<td><strong>10</strong></td>
<td></td>
<td></td>
<td></td>
<td>(\leq 15)</td>
</tr>
<tr>
<td>(b_j)</td>
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<td>10</td>
<td>15</td>
<td>10</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

46
Table 2.8: Optimal feasible solution of the corresponding Stage II problem

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$a'_i$</th>
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<tbody>
<tr>
<td>$S_1$</td>
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<td></td>
<td></td>
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<td>$S_2$</td>
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</tr>
<tr>
<td>$S_3$</td>
<td></td>
<td>10</td>
<td></td>
<td>15</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$S_4$</td>
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</tr>
<tr>
<td>$S_5$</td>
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<tr>
<td>$S_6$</td>
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</tr>
<tr>
<td>$S_7$</td>
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<td></td>
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</tr>
<tr>
<td>$S_9$</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$S_{10}$</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$b_i$</td>
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<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
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</tr>
</tbody>
</table>

2.1.5 Concluding Remarks

1. The current approach is not an extension of the algorithm suggested by Sonia et al. [86], rather an entirely different approach has been suggested for a more generalized problem and their problem becomes a particular case of the one discussed in present paper. Also, current approach as pointed out in Example 2 of Section 2.1.4, yields a better value of the objective function than the one obtained by their approach.

2. At each iteration we are solving only one problem, as it is depicted in Theorem 2.1.5 and Theorem 2.1.6 that the (OFS) of Stage II problem is obtained by just allocating the surplus amount in each row to the minimum and then to the next minimum time route till all the surplus amount in each row is exhausted.

3. As Stage II time is strictly decreasing at each iteration, and if $T^*_i = T^*$ and $T^*_j = T^*$, for some $r, s \in \{1, 2, \ldots, p\}$, the maximum number of iterations required to solve the problem is $r - s + 1$. Hence, the algorithm converges in a finite number of steps.

4. In the problem $(P_{2,3a})$, the total capacity along $(i, j)^{th}$ route is $u_{ij}$. This condition can be relaxed by assuming the capacity along $(i, j)^{th}$ route at each
stage is $u_{ij}$ i.e. $\bar{u}_{ij} = u_{ij}$ in Stage II problem.

5. Capacitated two-stage time transportation discussed in this section can be further explored in case of multi-stage transportation problem.

### 2.2 Bottleneck interval transportation problem

This section discusses two stage interval time minimization transportation problem, in which the total availability of a homogeneous product is known to lie in a specified interval. This problem was first considered by Sonia et al. [87] where in first stage, the sources ship all of their on-hand material to the demand points and the second stage shipment covers the demand that is not fulfilled in first stage. In each stage, aim is to minimize the duration of transportation and the overall goal is to minimize the sum of two stage shipment times. In their method two sequences of Stage I and Stage II time are generated. One of the sequences consists of generating pairs of the form $(T_1(.), T_2(.): T_1(.) > T_2(\cdot))$ by solving time minimization transportation problem of the form $P_{lb}(T_2(\cdot))$ and cost minimization transportation problem of the form $CP_{lb}(T_1(\cdot), T_2(\cdot))$ where the problem $P_{lb}(T_2(\cdot))$ reduces the on hand shipment time for Stage II, and the problem $CP_{lb}(T_1(\cdot), T_2(\cdot))$ gives the minimum shipment time for Stage II corresponding to the Stage I shipment time obtained from $P_{lb}(T_2(\cdot))$. Similarly the sequence of two stage shipment time of the form $(T_1(.), T_2(.): T_1(.) < T_2(\cdot))$ is obtained by solving the problems $P_{ub}(T_1(\cdot))$ and $CP_{ub}(T_2(\cdot), T_1(\cdot))$, where these problems play a similar role as played by $P_{lb}(T_2(\cdot))$ and $CP_{lb}(T_1(\cdot), T_2(\cdot))$ with their role for Stage I and Stage II reversed. Further it has been established theoretically that the global minimum value of the problem is obtained out of these generated pairs.

The present work aims at reducing the computational complexity of the method discussed by Sonia et al. [87], thereby suggesting a different approach to solve this problem, which adopts only one sequence of Stage I and Stage II problems in contrast to the two way procedure adopted by Sonia et al. [87]. This two stage interval bottleneck transportation problem, where the total availability of the product at
the source lies in the specified intervals, is shown to be related to an ordinary
interval (TMTP), which is further shown to be equivalent to a standard (TMTP).
Feasible solutions of the Stage 1 and Stage II transportation problems are derived
from a feasible solution of this standard (TMTP). Moreover, it has also been shown
that in the developed algorithm a finite number of cost minimization transportation
problems (CMTP) are solved to generate different pairs of Stage I and Stage II
shipment times.

2.2.1 Mathematical Formulation

Let \( a_i \) and \( a'_i \), \( i \in I \) denote respectively the minimum and maximum availability of
a homogeneous product at the source \( i \) and \( b_j \), \( j \in J \) the demand of the same at
destination \( j \), where \( \sum_i a_i < \sum_j b_j < \sum_i a'_i \)

In the first stage of the two stage Interval bottleneck transportation problem,
the quantity \( a_i < a'_i \) is shipped from each source \( i \), \( i \in I \) and after the completion,
enough quantity of the product is dispatched in second stage so as to exactly satisfy
the demand \( b_j \) at the destination \( j \), \( j \in J \). The stage-I problem is thus formulated
as:

\[
(P_{2.1a}) \min_{Y=\{y_{ij}\} \in S'} \max_{I \times J} [t_{ij}(y_{ij})] = \min_{Y \in S'} [T_1(Y)]
\]

where the set \( S' \) is given by

\[
S' : \left\{ \begin{array}{l}
\sum_{j \in J} y_{ij} = a_i \quad \forall \ i \in I, \\
\sum_{i \in I} y_{ij} \leq b_j \quad \forall \ j \in J, \\
y_{ij} \geq 0 \quad \forall \ (i, j) \in I \times J.
\end{array} \right.
\]

Corresponding to a feasible solution \( Y = \{y_{ij}\} \) of stage-1 problem, let \( S'(Y) \) be the
set of feasible solutions of Stage-2 problem which is stated below

\[
(P_{2.2b}) \min_{Z=\{z_{ij}\} \in S'(Y)} \max_{I \times J} [t_{ij}(z_{ij})] = \min_{Z \in S'(Y)} [T_2(Z)]
\]
where the set \( S'(Y) \) is given by

\[
S'(Y) : \{ \sum_{i \in I} z_{ij} \leq a_i' - a_i \quad \forall \ i \in I, \\
\sum_{i \in I} z_{ij} = b_j - b'_j \quad \forall \ j \in J, \\
z_{ij} \geq 0 \quad \forall \ (i, j) \in I \times J, 
\]

and \( b'_j = \sum_{i \in I} y_{ij}, \ j \in J. \)

Thus a two stage time minimization transportation problem can be defined as:

\[
(P_{2.3b}) \quad \min_{Y=(y_{ij}) \in S'} \left( T_1(Y) + \min_{Z \in S'(Y)} \left[ (T_2(Z)) \right] \right)
\]

Closely related to the problem \((P_{2.3b})\), is the interval time minimizing transportation problem \((P_{2.4b})\) defined as:

\[
(P_{2.4b}) \quad \min_{X \in S} \left[ T(X) \right] = \min_{X \in S} \left[ \max_{i \times j} (t_{ij}(x_{ij})) \right]
\]

where

\[
S : \left\{ \begin{array}{ll}
a_i \leq \sum_{j \in J} x_{ij} \leq a_i' \quad \forall \ i \in I, \\
\sum_{i \in I} x_{ij} = b_j \quad \forall \ j \in J, \\
x_{ij} \geq 0 \quad \forall \ (i, j) \in I \times J.
\end{array} \right.
\]

Clearly a feasible solution of \((P_{2.3b})\) provides a feasible solution to the problem \( (P_{2.4b}) \) and conversely. Associated with the problem \((P_{2.4b})\) a balanced transportation problem is defined as:

\[
(P_{2.5b}) \quad \min_{X \in S} \left[ \bar{T}(X) \right] = \min_{X \in S} \left[ \max_{i \times j} (\bar{t}_{ij}(x_{ij})) \right]
\]

where
2.2.2 Theoretical Development

As shipment time in Stage-I and Stage-II are concave functions, two stage interval time minimization transportation problem aims at minimizing a concave function over a polytope. Hence \((P_{2,3})\) is also a concave minimization problem. As the global minimum of a concave minimization problem is attained at an extreme point only, it is desirable to investigate only extreme points of the set of feasible solutions of \(S'\). Let the set of transportation time on various routes is partitioned into a number of disjoint sets, \(B_h, h = 1, 2, \ldots, s\), where \(B_h = \{(i,j) \in I \times J : t_{ij} = t^h\}\) and \(t^j > t^{j+1} \forall j = 1, 2, \ldots, s - 1\). Positive weights say \(\lambda_h, h = 1, 2, \ldots, s\) are attached to these sets where, \(\lambda_{j+1} >> \lambda_j \forall j = 1, 2, \ldots, s - 1\). This yields a standard
cost minimization transportation problem as:

\[
\min \sum_{h=1}^{s} \lambda_h \left( \sum_{(i,j) \in B_h} x_{ij} \right),
\]

where \( X = \{x_{ij}\} \) belongs to the transportation polytope over which original (TMTP) is being studied. To find an (OFS) of the Stage II problem, we define the following problem as:

\((CP)\) \quad \min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij},

where

\[
c_{i,n+1} = M \quad \forall \ i \in I,
\]

\[
c_{m+i,n+1} = 0 \quad \forall \ i \in I,
\]

\[
c_{ij} = 0 \quad \forall \ (i, j) \in I \times J,
\]

\[
c_{m+i,j} = \lambda_{s-h+1} : t_{m+i,j} = t^h, \forall \ (i, j) \in B_h \text{ and } h = 0, 1, \ldots, s.
\]

Let at any given time of Stage I and Stage II say, \( T_1^{k-1}, T_2^{k-1} \) respectively, where \( T_1^{k-1}, T_2^{k-1} \in \{t_1, t_2 \ldots t_s\}, \ k \in \{1, 2 \ldots s + 1\} \). The restricted version of the problem \((CP)\), denoted by \((CP_k)\), \( k \geq 1 \) is defined below:

\[(CP_k) \quad \min \sum_{s \in I} \sum_{j \in J} c_{ij} x_{ij},
\]

where

\[
c_{ij} = M \text{ if } t_{ij} \geq T^{k-1}, \ (i, j) \in I \times J
\]

\[
= 0 \text{ if } t_{ij} < T^{k-1}, \ (i, j) \in I \times J
\]

\[
c_{i,n+1} = M \quad \forall \ i \in I,
\]

\[
c_{m+i,n+1} = 0 \quad \forall \ i \in I,
\]
An OFS of the problem \((CP)\) is denoted by \(Y\) with corresponding stage I time \(T^0\) and stage time by \(T^2\) and let \(Y^k\) be an (OFS) of \((CPk)\) yielding corresponding time of Stage I and stage II by \(T^k_1\) and \(T^k_2\) respectively.

**Theorem 2.2.1** \(T^k_2\) is the minimum time of stage II corresponding to any given time of stage I in the problem \((CPk)\).

**Proof:** Let if possible there exist a pair \((T^1, T^2)\) yielded by some feasible solution \(Y = \{y_{ij}\}\) of \((CPk)\) such that \(T^2 < T^2_k\) and \(T^1 < T^{k-1}_1\) where \(T^2 = t_p\) and \(T^2_k = t_q\) for some \(p, q \in \{1, 2, \ldots, s\}\). Since \(T^2 < T^2_k\), therefore \(p > q\), which implies \(s - p + 1 < s - q + 1\).

Therefore

\[
Z(Y) = \sum_{i \times j} c_{ij}y_{ij} = \sum_{i=1}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij} \right) = \sum_{i=p}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij} \right),
\]

also

\[
Z(Y^k) = \sum_{i \times j} c_{ij}y_{ij}^k = \sum_{i=1}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij}^k \right) = \sum_{i=q}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij}^k \right).
\]

Since \(\lambda_{i+1} \gg \lambda_i, \ i = 1, 2, \ldots, s - 1\)

\[
\Rightarrow \sum_{i=p}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij} \right) < \sum_{i=q}^{s} \lambda_{s-i+1} \left( \sum_{(i,j) \in B_l} y_{ij}^k \right).
\]

\[
\Rightarrow Z(Y) < Z(Y^k)
\]

But this contradict the optimality of \(Y^k\), therefore \(T^k_2 \leq T_2\).

**Theorem 2.2.2** \((CP)\) gives optimal time of Stage II.

**Proof:** It follows on the same lines as proof of Theorem 2.2.1.

**Remark 1.** By construction of \((CPk)\), it is observed that \(T^0_1 > T^1_1 > \cdots > T^s_1\) and \(T^0_2 \leq T^1_2 \cdots \leq T^s_2\).
Proof. The first sequence is clear from the construction of problem $\text{(CP}_k\text{)}. Let if possible $T_{k+1}^2 < T_2^k$, for some $k$. Let $Z_k = Z(Y^k)$, $Z_{k+1} = Z(Y^{k+1})$, since $T_{k+1}^2 < T_2^k$ by reasoning used in the proof of Theorem 1, we see that $Z_{k+1} < Z_k$. Since $T_{k+1}^2 < T_2^k$, implies $Y^{k+1}$ is a feasible solution of $(\text{CP}_k)$ with $Z_{k+1} < Z_k$, contradiction to the fact that $Y^k$ is an (OFS) of $(\text{CP}_k)$.

Remark 2.2.3 Since optimal time of Stage-1 problem is $T_1$, $(\text{OBFS})$ of $(\text{CP}_{i+1})$ is not $M$-feasible.

Remark 2.2.4 Let $T_1^0 = t^*$ for some $r \in \{1, 2, \ldots, s\}$, then the maximum number of iteration required to solve this problem is $s - r + 1$.

Remark 2.2.5 Let $T^\ell(= \ell^*$, $\ell \in \{1, 2, \ldots, s\})$ be the overall time of transportation of the problem $(\text{P}_{2,5b})$ defined by Sonia et al. [87], then the proposed method becomes better if $4r - \ell < 3s - 2$.

Theorem 2.2.6 Let the generated pairs of Stage I and Stage II time be $(T_1^0, T_2^0)$, $k \geq 0$. Then the optimal value of the problem $(\text{P}_{2,5b})$ is given by $\min_{(h=0, 1, \ldots, l)} [T_1^h + T_2^h]$.

Proof: Let if possible, there exists a pair $(Y_1, Y_2)$, yielding Stage I time and Stage II shipment time $(T_1, T_2)$ such that $T_1 + T_2 < \min_{(h=0, 1, \ldots, l)} [T_1^h, T_2^h]$. Since $T_1^0 > T_1 \ldots > T_2$ and $T_2^0 \leq T_2 \ldots \leq T_2$, then the following cases arise:

Case 1: $T_1 > T_1^0$. (3.1)

By construction of $(\text{CP}_1)$, $(Y_1, Y_2)$ is a feasible solution of $\text{CP}_1)$. Since $T_2^0$ is optimal time for $(\text{CP}_1)$, therefore

$$T_2^0 \leq T_2. \quad \text{(3.2)}$$

Combining (3.1) and (3.2), we get, $T_1 + T_2 > T_1^0 + T_2^0 \Rightarrow T_1 + T_2 > \min_{(h=0, 1, \ldots, l)} [T_1^h + T_2^h]$.

Case 2: $T_1 < T_1^0$.

Since $T_1 < T_1^0$, $(Y_1, Y_2)$ is an $M$-feasible solution of $(\text{CP}_1)$, which is a contradiction as this problem is not $M$-feasible.

Case 3: $T_1 \in [T_1^0, T_1^1]$.
In this case, either \( T_1 = T_{f} \) for some \( A_i = 0, 1, \ldots, l \) or \( T_i \in (T_{f}, T_{f-1}) \) \( \Rightarrow T_{f-1} > T_{f} \) 

(i). If \( T_1 = T_{f} \), then by construction of \((CP)\), \((Y_1, Y_2)\) is a feasible solution of \((CP)\).

This implies \( T_2 \geq T_2^{\ast} \), because \((T_2^{\ast})\) is the optimal time of Stage II in \((CP)\).

\[
\Rightarrow T_1 + T_2 \geq T_{f} + T_{2}^{0} \\
\Rightarrow T_1 + T_2 \geq \min_{(h=0, \ 1, \ldots, l)} [T_{h}^{\ast} + T_{2}^{\ast}].
\]

Similarly, the case when \( T_1 = T_{1}^{k}, \ k \in \{1, 2, \ldots, l\} \) it can be shown that

\[
\Rightarrow T_1 + T_2 \geq T_{1}^{k} + T_{2}^{k} \geq \min_{(h=0, \ 1, \ldots, l)} [T_{h}^{k} + T_{2}^{k}].
\]

(ii). If \( T_1 \in (T_{1}^{k}, T_{1}^{k-1}) \), then \((Y_1, Y_2)\) is a feasible solution of \((CP_k)\), \( \Rightarrow T_1 < T_{1}^{k-1} \).

Also \( T_2 \geq T_{2}^{k} \) and \( T_1 > T_{1}^{k} \).

\[
\Rightarrow T_1 + T_2 > T_{1}^{k} + T_{2}^{k} \\
\Rightarrow T_1 + T_2 > \min_{(h=0, \ 1, \ldots, l)} [T_{h}^{k} + T_{2}^{k}].
\]

Therefore there does not exist a feasible solution \( Y = (Y_1, Y_2) \) of \((CP_k)\) yielding time less than \( \min_{(h=0, \ 1, \ldots, l)} [T_{h}^{k} + T_{2}^{k}] \). Thus the optimal value of \((P_{2,3k})\) is given by \( \min_{(h=0, \ 1, \ldots, l)} [T_{h}^{k} + T_{2}^{k}] \).

2.2.3 The Procedure

**Initial Step.** Find an (OBFS) of \((CP)\) and thus obtain the corresponding times \( T_{1}^{0} \) and \( T_{2}^{0} \) of Stage I and Stage II respectively.

**General Step.** If \( k \geq 1 \) at a given pair \((T_{1}^{k-1}, T_{2}^{k-1})\) of Stage I and Stage II times, solve the problem \((CP_k)\). From the (OBFS) of \((CP_k)\) construct the pairs \((T_{1}^{k+1}, T_{2}^{k+1})\).

**Terminal Step.** If (OBFS) of problem \((CP_k)\) is not M-feasible, then Stop. The optimal value of \((P_{2,3k})\) is given by \( \min_{(h=0,1,\ldots,k)} [T_{h}^{k} + T_{2}^{k}] \).
2.2.4 Numerical Illustration

Consider the two stage interval time minimization transportation problem given in Table 1. The problem considered here is same as discussed by Sonia et al [87].

<table>
<thead>
<tr>
<th>Si</th>
<th>Di</th>
<th>a_i</th>
<th>a_i'</th>
</tr>
</thead>
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<tr>
<td>S1</td>
<td>26</td>
<td>23</td>
<td>59</td>
</tr>
<tr>
<td>S2</td>
<td>40</td>
<td>48</td>
<td>20</td>
</tr>
<tr>
<td>S3</td>
<td>26</td>
<td>38</td>
<td>48</td>
</tr>
</tbody>
</table>

The partition of various time routes is given by $t^1(=59) > t^2(=48) > t^3(=40) > t^4(=38) > t^5(=26) > t^6(=23) > t^7(=20) > t^8(=19)$ as $t^8 = t^8 = 59$, therefore $s = 8$.

The corresponding problem $(P_{2.5a})$ is shown in Table 2.

<table>
<thead>
<tr>
<th>bj</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
<th>D6</th>
<th>D7</th>
<th>a_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si</td>
<td>26</td>
<td>23</td>
<td>59</td>
<td>38</td>
<td>19</td>
<td>20</td>
<td>M</td>
<td>6</td>
</tr>
<tr>
<td>S2</td>
<td>40</td>
<td>48</td>
<td>20</td>
<td>19</td>
<td>23</td>
<td>59</td>
<td>M</td>
<td>15</td>
</tr>
<tr>
<td>S3</td>
<td>26</td>
<td>38</td>
<td>48</td>
<td>20</td>
<td>19</td>
<td>40</td>
<td>M</td>
<td>12</td>
</tr>
<tr>
<td>S4</td>
<td>26</td>
<td>23</td>
<td>59</td>
<td>38</td>
<td>19</td>
<td>20</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>S5</td>
<td>40</td>
<td>48</td>
<td>20</td>
<td>19</td>
<td>23</td>
<td>59</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>S6</td>
<td>26</td>
<td>38</td>
<td>48</td>
<td>20</td>
<td>19</td>
<td>40</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>bj</td>
<td>6</td>
<td>9</td>
<td>3</td>
<td>14</td>
<td>10</td>
<td>5</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

An (OBFS) of the problem $(CP)$ yield Stage I time as $T^0_1 = 40$ and and Stage II time as $T^0_2 = 19$, where 19 is optimal time of stage II. Next pair is obtained by solving the time minimization transportation problem $(CP_i)$, an (OBFS) of which yield Stage I time as 38 and Stage II time as 20, where 20 is the minimum time for Stage II corresponding to the stage I time 38. Similarly proceeding in the same
way after solving further restricted problem \((CP_2)\), the pair obtained is \((26,38)\) and 
\((23,40)\) is obtained by solving \((CP_3)\). Algorithm terminates here as \((CP_4)\) is no 
more \(M\)-feasible. Thus \(\min\{40 + 19, 38 + 20, 26 + 38, 23 + 40\} = 58\). Hence the opti-
mal value of problem corresponds to the pair \((38,20)\). The transportation schedule 
which gives this optimal value is shown in the Table 3.

<table>
<thead>
<tr>
<th></th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
<th>(D_4)</th>
<th>(D_5)</th>
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<tr>
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<td>(M)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(M)</td>
</tr>
<tr>
<td>(S_2)</td>
<td></td>
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<td>2</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(M)</td>
<td>(M)</td>
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<td>0</td>
<td>0</td>
<td>(M)</td>
<td>(M)</td>
</tr>
<tr>
<td>(S_3)</td>
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<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>(M)</td>
<td>0</td>
<td>0</td>
<td>(M)</td>
<td>(M)</td>
</tr>
<tr>
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<td>(\lambda_3)</td>
<td>(\lambda_8)</td>
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<td>(\lambda_1)</td>
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</tr>
<tr>
<td>(S_5)</td>
<td>(\lambda_6)</td>
<td>(\lambda_7)</td>
<td>(\lambda_2)</td>
<td>(\lambda_1)</td>
<td>(\lambda_3)</td>
<td>(\lambda_8)</td>
<td>0</td>
</tr>
<tr>
<td>(S_6)</td>
<td>(\lambda_4)</td>
<td>(\lambda_5)</td>
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<td>10</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

An \((OBFS)\) of the problem \((CP_2)\) is depicted in Table 3, where the entries in the 
lower left hand corner represents the associated cost and the highlighted entries
show the values of the basic variables.

2.2.5 Concluding Remark

Present methodology can be described a better than the one discussed by Sonia et al. [87] because of the following reasons:

1. As pointed out in Remark 2.2.5, for certain values of $r$, convergence rate of the proposed algorithm is better than the one discussed by Sonia et al. [87].

2. Only one sequence of Stage I and Stage II pairs is adopted in contrast to the two way procedure discussed by them.

3. Problems such $(P_{lb})$ and $(C_{lb})$ are avoided as there is no need to reduce Stage II time separately corresponding to given the Stage I time.