Chapter 1

Introduction

This introductory chapter on mathematical programming problems comprising of three sections, discusses mathematical programming problems with special emphasis on linear, linear fractional integer programming problem, transportation problems, multilevel programming problems and multiobjective programming problems.

Section 1.1 discusses origin and development in the field of these problems. Section 1.2 gives a brief survey of the related work done by various researchers in the relevant field and finally Section 1.3 gives brief summary of the work carried out in the subsequent chapters of the thesis.

1.1 Development in Mathematical Programming

In a mathematical programming problem, one seeks to minimize or maximize a real valued function of real variables, subject to constraints on these variables. The term mathematical programming refers to the study of these problems: their mathematical properties, the development and implementation of algorithms to solve these problems, and the application of these algorithms to real world problems. In last few decades, mathematical programming has emerged in many areas like economics, industry, military, finance, medicine and has received a lot of attention.

Linear programming is a special class of mathematical programming problems in which the objective function and the constraints can be expressed as linear functions
of the decision variables. In its standard form, a linear programming problem can be stated as:

\[
\begin{align*}
\min z &= c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} & \\
\begin{align*}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1, \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2, \\
& \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m, \\
x_j &\geq 0, \ j = 1, 2, \ldots, n
\end{align*}
\end{align*}
\]

In vector form this problem can be stated as

\[
\min C^T X
\]
subject to

\[
AX = b
\]
\[
X \geq 0
\]

where \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) and \( C, X \in \mathbb{R}^n \). In fact, any problem of maximizing or minimizing a linear function subject to linear equations/inequalities can be easily reduced to its standard form.

The linear programming model has been applied to a large number of areas including military applications, transportation and distribution, scheduling, production and inventory management, telecommunication, agriculture, finance and more. Many problems simply lend themselves to a linear programming solution but in some cases, some ingenuity is required for the modelling. Linear programming also has interesting theoretical applications in combinatorial optimization and complexity theory.

The classical tool for solving the linear programming problem in practice is the class of simplex algorithms proposed and developed by George Dantzig [31]. The method is based on generating a sequence of bases. A basis is a nonsingular
submatrix of $A$ of order $m \times m$. The fundamental characteristic of the method is that at some point a basis is reached which provides a solution to the problem. A suitable basis can certify either that the problem has no solution at all or that it is unbounded; otherwise, a basis will be reached, which defines optimal solutions for both the primal and the dual problems. Due to the problems of degeneracy, special care is needed to guarantee that the method will not cycle. Bland [20] proved that the “least index” rule guarantees no cycling.

Another aspect of mathematical programming is fractional programming problems, which deals with optimization of objective function, which is a ratio of two functions, subject to some constraints. Since the early paper of Isbell and Marlow [56], dealing with a constrained ratio optimization problem, a considerable number of articles have appeared in this field. Applications of fractional programming are widely spread over literature [83, 84]. Ratios to be optimized, often describe some kind of an efficiency measure for a system. In 1963, Gilmore and Gomory [45] discussed a stock cutting problem in paper industry and showed that under the given circumstances, it is important to minimize the ratio of wasted and used amount of raw material. In a cargo-loading problem considered by Kydland [65], profit per unit time is to be maximized. Both loading cost and loading time depend on cargo chosen. In case of linear functions for loading cost and time, one obtains linear fractional programming problem.

An important class of mathematical programming problems, in which all decision variables are integers, is the class of integer programming problems. Integer programming problems is a very vast topic, having applications in the field of scheduling, location, network and selection problems, military, education, health and other environments. Unlike other mathematical programming problems with continuous variables, integer programming problems are difficult to tackle with, mainly because the feasible region is no more convex. Due to its prevalent applications this field has caught a great attention of researchers.

One of the most important and well structured class of linear programming problems is the class of transportation problems. In a classical transportation problem,
popularly known as cost minimization transportation problem (CMTP), one may be interested in finding least expensive way of transporting a homogeneous product from a number of warehouses to a number of destinations. If capacity of each source-destination link is also introduced then it is called capacitated cost minimization transportation problem (CCMTP), which is a generalization of classical transportation problem, where there is no restriction on any source-destination link.

If $I$ denotes the index set of $m$ sources and $J$ denotes the index set of $n$ destinations, then mathematically the transportation problem can be formulated as:

$$\min_{x \in S} \sum_{(i,j) \in I \times J} c_{ij} x_{ij}$$

where

$$S : \begin{cases} \sum_{j \in J} x_{ij} = a_i, & \forall i \in I \\ \sum_{i \in I} x_{ij} = b_j, & \forall j \in J \\ 0 \leq x_{ij} \leq u_{ij}, & \forall (i, j) \in I \times J \end{cases}$$

here $a_i$ is the availability of material at each source $i$, $i \in I = \{1, 2, \ldots, m\}$ and $b_j$ is the requirement of the same at each destination $j \in J = \{1, 2, \ldots, n\}$. $x_{ij}$ denotes the amount transported from source $i$ to destination $j$ and $c_{ij}$ denotes unit cost of transportation from $i^{th}$ source to $j^{th}$ destination. The first formulation of the transportation problems dates back to the late 1930’s and early 1940’s. L.V. Kantorovitch contributed in this field in 1939. Later Hitchcock [54] discussed problem of distribution of a product from several warehouses to numerous localities. The problem of optimal routing of messages in a communication network, contract award problem and routing of aircrafts and ships are important applications of linear programming to the field of transportation problems.

Another important class of transportation problems in terms of its prevalent applications is the time minimizing transportation problem (TMTP) which usually arises in connection with the transportation of perishable commodities that have to be distributed as quickly as possible. This problem was first addressed by
Hammer[48]. Later on, many researchers found interest in this problem and proposed various solution techniques. A capacitated time minimization transportation problem (CTMTP), where capacity of each source destination link is fixed, can be mathematically formulated as:

$$\min_{X \in \mathcal{S}} \left[ T(X) = \max_{(i,j) \in I \times J} t_{ij}(x_{ij}) \right]$$

where $t_{ij}(x_{ij}), (i, j) \in I \times J$, the shipment time from source $i$ to destination $j$ is defined as

$$t_{ij}(x_{ij}) = \begin{cases} t_{ij}, & \text{if } x_{ij} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Bansal et al. [8] proved that $T(X)$ is a concave function. Thus (CTMTP) involves minimization of a concave function over a transportation polytope and hence it belongs to the class of concave minimization problems. Due to the concavity of the objective function, the search for an optimal solution is restricted to the set of basic feasible solutions only. Almost all the techniques for solving (TMTP) involve an ordinary (CMTP) for which strongly polynomial algorithms are known to exist [61, 67, 94, 95]. Therefore, it follows that a time minimization transportation is also solvable in strongly polynomial time. Solvability in strongly polynomial time means that there exists an algorithm which solves the problem in number of steps that is bounded by a polynomial function of $m = |I|$ and $n = |J|$ only.

Another important class of mathematical programming problems is the class of multilevel programming problems. Multilevel programming has been proposed for dealing with decision processes, involving multiple decision makers within a hierarchical structure. In a hierarchical structure, specialized management functions are assigned to different levels and decision making is shared by several interacting decision centers rather than limited to the single highest rank of the hierarchy. However, the multiplicity of decision centers often gives rise to conflicting objectives within the same organization. The tendency of the decision makers at each level of
the hierarchy to achieve their own goal and value systems may make them lose sight of how their activities mesh with the objectives of their superiors; what is best for one level, is frequently detrimental to another, so they may end up working at cross purposes. Multilevel mathematical programming emerged as a model to improve the goals of the overall organization by solving the problem of coordinating the decision making process in a decentralized system.

Let $X$ be the vector of decision variables controlled by the highest level, $F$ the corresponding objective function, which is also the objective of the overall system, $X_1$ and $F_1$ the decision variables vector and objective function, respectively, of the first lower level decision maker (the highest with respect to the other lower levels and the second highest with respect to the whole organization), $X_2$ and $F_2$ those of second lower level, $X_i$ and $F_i$ those of lowest level, and $G$ be the vector of constraint functions. The general form of multilevel mathematical programming can be defined as:

$$\max_x F(X, X_1, X_2, \ldots, X_i)$$
where $X_1, X_2, \ldots, X_i$ solve:

$$\max_{X_1} F(X, X_1, X_2, \ldots, X_i)$$

where $X_2, X_3, \ldots, X_i$ solve:

$$\max_{X_2} F(X, X_1, X_2, \ldots, X_i)$$

where $X_3, X_4, \ldots, X_i$ solve:

$$\max_{X_3} F(X, X_1, X_2, \ldots, X_i)$$

such that

$$G(X, X_1, X_2, \ldots, X_i) \leq 0$$

(1.1.1)
Multilevel programming is a class of mathematical programming problems, which deals with the optimization of the objective of the highest level of a hierarchical organization, taking into account the tendency of the lower levels of the hierarchy to improve their own objectives. The decisions of the lower levels are not dictated by their superiors; however, their reactions to the upper levels’ actions are perfectly known.

A large class of multilevel mathematical programs involve only two levels and are called bilevel mathematical programs. Bilevel programming problem (BLP) is a nested optimization problem with two decision makers, the leader and the follower, each controlling a part of the variables and having his/her own objective function. The general form of (BLP) can be defined as:

$$\max_{X_1} F_1(X_1, X_2)$$

where $X_2$ solves the following problem,

$$\max_{X_2} F_2(X_1, X_2) \quad \text{(for a given } X_1)$$

such that

$$(X_1, X_2) \in S \quad (1.1.2)$$

where $S = \{(X_1, X_2) \mid G(X_1, X_2) \leq 0, \ X_1, X_2 \geq 0\}$ is the set of common constraint region. Bilevel programming problem (BLP) can be visualized as an organizational hierarchy in which two decision makers have to improve their strategies from a jointly dependent set $S$.

In the past few year (BLP) has received a lot of attention. Bard [14] and Dempe [33] are good general references on this topic. Due to its structure, BLP is nonconvex and quite difficult to deal with, even when all functions involved are linear. One of the main characteristics of (BLP) is that, unlike general mathematical problems, the (BLP) may not possess a solution even when $F_1$ and $F_2$ are continuous and $S$ is compact. In order to make sure that the (BLP) is well posed, it is usually assumed
that, for each value of the upper level variables $X_i$, there is a unique solution of the lower level problem.

In the literature, equivalent terms such as “outer”, “level one”, “leader” or “policy” are sometimes used instead of “upper”, and similarly “inner”, “level two”, “follower” or “behavioural” are used instead of “lower”. In addition, since (BLP) is concerned with the optimization of the upper objective, the latter is often referred to as the objective of the whole problem, while the lower objective is considered as just a constraint.

A mathematical programming problem with single objective function is called a single objective (or scalar) programming problem. A vector minimum (or maximum) problem is an optimization model with two or more objective functions. Such models are also called multiobjective programming problems. Often the different objectives are conflicting in nature. For example, it may not be possible to obtain a solution of a transportation problem which would minimize the transportation cost as well as the time, to meet the demands. Mathematically a multiobjective programming problem can be stated as:

$$\min_{X \in S} (f_1(X), f_2(X), \ldots, f_n(X))$$

where $S = \{X \in R^n | G(X) \leq 0, X \geq 0\}$ is the set of common constraint region. If all the functions $f_i(X)$, $i = 1, 2, \ldots, n$ are linear and the set $S$ is polyhedral then this problem falls in the class of multiobjective linear programming problems. The existence of multiple objectives leads to many interesting questions, which do not arise in single objective models. It is difficult to obtain a unique solution, since these problems rarely have feasible points that simultaneously minimize (or maximize) all the objectives. Suppose we want to buy a new car and have identified four models we like: a VW Golf, an Opel Astra, a Ford Focus and a Toyota Corolla. The decision will be made according to price, petrol consumption, and power. We prefer a cheap and powerful car with low petrol consumption. In this case, we face a decision problem with four alternatives and three criteria. How do we decide, which of the
four cars is the best alternative, when the most powerful car is also the one with
the highest petrol consumption, so that we cannot buy a car that is cheap as well
as powerful and fuel efficient. However, we observe that with any one of the three
criteria alone the choice is easy. Therefore, it has been well recognized that many real
world problems should have multiple objectives and the concept of optimal solution
in multiobjective optimization problems is clearly related to the preference attitude
of the decision maker. A good decision is based on the principle that there is no
other alternate that can be better in some aspect of consideration. Such a point
is called an efficient solution and is also known as noninferior or nondominated
or Pareto optimal solution. The origin of the vector minimum problem can be
traced to early developments in utility theory in economics. Pareto [77] began
the study of multiobjective programming problems by reducing them to a single
objective. However, the problem was first explicitly defined and studied by Kuhn and
Tucker [64]. To eliminate certain anomalous efficient solutions, they also proposed
a slightly restricted definition of efficiency, called proper efficiency. Later, Geoffrion
[44] modified this concept and called an efficient solution to be properly efficient
if the ratio of gain (in every objective) to loss (in at least one other objective) is
always finite. He also derived necessary and sufficient conditions for properly efficient
solution of convex multiobjective programming problems. His work motivated many
researchers in this field.

1.2 Literature survey

Wide ranging literature is available to solve conventional cost as well time minimiza-
tion transportation problem. The transportation problem discussed in the thesis,
belongs to the class of bottleneck linear programming problems. Bottleneck linear
programming problems deal with minimization of a concave bottleneck objective
function or with maximization of a convex bottleneck objective function over a con-
 vex region bounded by hyperplanes. Many authors have addressed these problems
and numerous solution techniques have been proposed [2, 8, 18, 42, 48, 57]. A time
minimization transportation problem (TMTP), which is a special case of bottleneck linear programming problem, has been studied by [5, 6, 18, 42, 48, 57, 78, 97]. Time minimization transportation problem with mixed constraints and flow constraints has been studied by Khanna and Puri [69].

Various algorithms available for (TMTP) can be easily extended to (CTMTP). Satyaparkash (1982) obtained an optimal feasible solution (OFS) of (TMTP) by solving a related cost minimization transportation problem in the following way:

If \( T_1 > T_2 > \ldots > T_p \) are \( p \) pairwise distinct entries in the \( m \times n \) time matrix \( \{t_{ij}\} \) sorted decreasingly and \( \lambda_1, \lambda_2, \ldots, \lambda_p \) are positive weights attached to the sets \( B_k = \{(i, j) \in I \times J \mid t_{ij} = T_k\} \), \( k = 1, 2, \ldots, p \) respectively, then the cost minimizing transportation problem (CMTP) whose optimal feasible solution will provide an optimal feasible solution of (TMTP) is:

\[
\min_{X \in \hat{S}} \sum_{k=1}^{p} \lambda_k \left( \sum_{(i,j) \in B_k} x_{ij} \right)
\]

where the set \( \hat{S} \) is given by

\[
\hat{S} : \quad X = (x_{ij}) \in \mathbb{R}^{mn} \left\{ \begin{array}{l}
\sum_{j \in J} x_{ij} = a_i \quad \forall \ i \in I, \\
\sum_{i \in I} x_{ij} = b_j \quad \forall \ j \in J, \\
x_{ij} \geq 0 \quad \forall \ (i,j) \in I \times J,
\end{array} \right.
\]

where \( \lambda_k >> \lambda_{k+1}, \ k = 1, 2, \ldots, p - 1 \). For numerical specification of weights \( \lambda_1, \lambda_2, \ldots, \lambda_p \), one may refer to Sherali [88] and Mazzola et al. [73]. Similarly an (OFS) of (CTMTP) can be obtained by solving the associated capacitated cost minimization transportation problem (CCMTP), which falls into the class of capacitated transportation problems. Capacitated transportation problems have been studied by various authors. Kassay [68] gave an operator method for solving capacitated transportation problem. Hassain et al. [50] studied probabilistic analysis of capacitated transportation problem, assuming that the capacities are random.
variables and proved asymptotic conditions on supplies and demands, which assure that a feasible solution exists. Later on, Bit et al. [19], Zheng et al. [110], Rachev and Olkin [80] worked on capacitated transportation problem. Unbalanced classical transportation problem can be tackled by solving a related balanced transportation problem. Dahiya and Verma [29] discussed method for solving various unbalanced capacitated transportation problems.

Mathematical programming techniques have been used for some time to model hierarchical systems, especially multidivisional or multilevel organizations where planning problems require the synthesis of decisions of several, interacting individuals or agencies. Often, these groups are arranged within a hierarchical administrative structure, each with independent, and perhaps conflicting objectives. Multilevel decision-making has always been regarded as an important, although sometimes overlooked aspect of the planning process. Frequently, the impact of directives from superiors and the reactions of subordinates have been viewed as externalities, beyond the control of a planner. However, there have been attempts to model the ability of one planner to indirectly influence the decisions of others to his own benefit.

Because of the pervasive nature of the topic, the term, “multilevel planning” has appeared in a variety of settings, clothed in different mathematical raiments. With few exceptions, multilevel methods for optimizing hierarchical systems have rested heavily on the decomposition method of Dantzig and Wolfe [32]. Multilevel decomposition methods easily lend themselves to an economic interpretation of the algorithmic process. The procedure is viewed as an adjustment phase with the superior planner sending tentative information to the lower level subunits, observing their reactions, and then updating the corporate information. This information can take the form of placing prices on the scarce resources [17, 28], or partitioning the resources among the subunits [41, 63].

The multilevel programming problem was first introduced by Kornai and Liptak [63] in the context of resource allocation among independently working sectors, coordinated by a central planning agency. A formal definition of the problem was provided by Candler and Townsley [24]. Due to complexity, bilevel case is considered
mainly. Bilevel programming (BLP) is an important area of current research.

The non convexity and non differentiability even in linear case makes the problem difficult to tackle with. Bard [11] proved that the bilevel linear programming problem is a NP-Hard problem and even it is NP-Hard problem to search for locally optimal solution of the bilevel linear programming problem [99]. Therefore, it is very difficult to solve bilevel programming problem because of its non-convex and non-continuous structure. So most of the researches on algorithms for bilevel programming problem are limited to special structure of this problem or obtaining the locally optimal solutions. Many researchers have made deep contribution to this field, including the monographs [10, 16, 24, 33], and bibliographies about the theory, algorithm and application of bilevel programming [15, 34, 99, 108].

Many classes of (BLP) have now dedicated solution algorithms and researchers have now started to solve more complicated instances, like bilevel programming with integer variables, the discrete bilevel programming, the mixed integer linear/nonlinear bilevel programming problems [13, 37, 75, 96]. For nonlinear bilevel programming problems, Bard [10], Bard and Moore [12] have proposed exact algorithms for convex quadratic and linear quadratic problems respectively, while some local algorithms have been proposed for general nonlinear case [10, 26, 82, 105]. Although not being an exhaustive list, the available methods range over sequential unconstrained optimization (Bracken and McGill [21]), implicit search technique (Candler and Townsley [24]), mixed integer programming (Fortuny and McCarl [40]), parametric complementary pivot algorithm (Bialas and Karwan [16], Judice and Faustino [59]), branch and bound method (Bard and Falk [9], Bard and Moore [12]), hybrid algorithm (Wen and Bialas [107]), and bicriteria programming (Ünlü [98]).

Multiobjective integer and mixed-integer programming (MOIP/MOMIP) is very useful in many areas of application, as any model that incorporates discrete phenomena requires the consideration of integer variables (such as, for modeling investment choices, production levels, fixed charges, logical conditions or disjunctive constraints). However, research on methods for the general multiobjective
integer/mixed-integer model has been rather limited, when compared with multiobjective programming problems with continuous variables.

In the past few decades, multiobjective integer and mixed integer programming problems have become an important area of research, as many real world phenomena require discrete representations by integer variables and decision makers have to deal with several objectives. However, research on methods for the general multiobjective integer/mixed integer model has been limited when compared with multiobjective programming problems with continuous variables. One possible reason for this is the introduction of discrete variables, which make these problems much more difficult to tackle with, even if the objective functions are linear. The feasible set becomes non convex and other difficulties arise with the incorporation of objective functions which are not linear. Further, there are multiobjective approaches designed for all-integer problems that do not apply to the mixed-integer case. Therefore multiobjective integer programming problems are theoretically challenging, as most of them, even their single objective versions, fall into the class of computationally intractable problems.

In the recent past, several methods have been developed to solve multiobjective integer linear programming problems (MOILP), in the sense of finding the whole efficient set, or at least some significant part of it [3, 38, 111]. Villareal and Karwan [103] developed a procedure involving dynamic programming recursive equations to obtain a set of efficient solutions for (MOILP) and an alternate method is produced by combining this procedure with branch and bound rules. Further, Hannon and Klein [60] and Sylva and Crema [90] also discussed algorithms for (MOILP). Multiobjective mixed integer linear program has been discussed by Sylva and Crema [91]. In addition to linear case, various algorithms for linear fractional multiobjective integer programs have been discussed by many authors, among them Metev [74] used the property of strict quasiconvexity of linear fractional functions to find weakly efficient points and then to improve the obtained value of the chosen criterion. Caballero and Hernández [23] introduced a new method to estimate the weakly efficient set for multiobjective linear fractional programming problems.

An interactive reference direction algorithm for solving multiobjective convex
nonlinear integer programming problem is discussed by Vessilev et al. [102], where at each iteration the decision maker sets his preferences as aspirations levels of the objective functions, the modified aspirations points and the solution found at the previous iteration define the reference direction. The techniques of diverse nature including branch and bound [72] and other methods [1, 4, 58, 70, 76] have been discussed for multiobjective integer programming problems.

Saad et al. [81] developed a linearization technique to solve fuzzy multiobjective integer nonlinear fractional programming problem, where each objective function involves fuzzy parameters and is a ratio of quadratic and linear functions over a set of linear constraints with decision variables as integers.

1.3 Summary of thesis

This section briefly surveys the research work carried out in Chapter 2, Chapter 3, Chapter 4 and Chapter 5.

Chapter 2 discusses the research work done on multistage time minimization transportation problem. This chapter is further divided into two sections. In Section 2.1, a variant of time minimization transportation problem is studied, in which the total availability of a homogeneous product at various sources is more than the total requirement of the same at destinations. It is different from the conventional time minimization transportation problem in the sense that minimum requirement of destinations is satisfied in first stage and the surplus amount from sources to destinations is transported in second stage. Each time the transportation from sources to destinations is done parallel and the capacity of each route (source destination link) remains fixed i.e. cannot exceed its upper bound. In each stage, the objective is to minimize the shipment time and the overall goal is to find a solution that minimizes the sum of first and second stage times. Such situations arise when in some circumstances, due to storage constraint, destinations are unable to receive the quantity in excess of their minimum demand. For example, problem of transporting perishable items like food packets and medicines to areas hit by natural calamities or
sending military equipments to different posts during wars, where the commodities are available in abundance but due to shortage of storage capacity at destinations, over supply may not be possible, it is preferable to transport the product in two stages.

This problem was originally studied by Sonia and Malhotra [86]. In their methodology an iterative algorithm was proposed to solve a two stage time minimization transportation problem by examining lexicographic optimal solutions of restricted version of a related standard time minimization transportation problem, thereby obtaining feasible solutions of Stage I and Stage II iteratively such that the sum of Stage I and Stage II shipment time is the least. In their proposed methodology, after obtaining a feasible solution of Stage I in each restricted problem, an (OFS) of the corresponding Stage II problem is obtained by allocating the surplus amount at each source to the routes of minimum duration. But this approach does not yield an optimum solution to their suggested problem, a counter example has been discussed in this regard at the end of Section 2.1.

If in the two stage time minimization transportation problem, capacity constraint is introduced on each route, then the problem becomes more meaningful and complicated too. Section 2.2 discusses a two stage time minimization transportation problem with capacity constraint on each route, which is a generalized form of the problem discussed by Sonia and Malhotra [86], thereby suggesting a different approach which inturn yield a better solution to the problem discussed by Sonia and Malhotra [86].

In general, a capacitated two stage time minimization transportation problem can be formulated as:

$$\min_{X=\{x_{ij}\} \in S} \left[ \max_{i,j} (t_{ij}(x_{ij})) + \min_{X \in \mathcal{S}(X)} \left[ \max_{i,j} (t_{ij}(\bar{x}_{ij})) \right] \right]$$
where the set $S$ is given by

$$S : \begin{cases} \sum_{j \in J} x_{ij} \leq a_i \quad \forall i \in I, \\ \sum_{i \in I} x_{ij} = b_j \quad \forall j \in J, \\ 0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in I \times J. \end{cases}$$

where corresponding to a feasible solution $X = \{x_{ij}\}$ of Stage I problem, $S(X)$ be the set of feasible solutions of Stage II problem

$$S(X) : \begin{cases} \sum_{j \in J} \bar{x}_{ij} = a_i - \sum_{j=1}^{n} x_{ij} \forall i \in I \\ \sum_{i \in I} \bar{x}_{ij} \geq 0 \quad \forall j \in J, \\ 0 \leq \bar{x}_{ij} \leq u_{ij} - x_{ij} \quad \forall (i,j) \in I \times J, \end{cases}$$

where $\bar{u}_{ij} = u_{ij} - x_{ij}$ and $a'_i = a_i - \sum_{j=1}^{n} x_{ij} \forall i \in I$.

In the present work, an iterative algorithm is proposed which finds a sequence of feasible solutions of two stage (CTMTP) by solving its restricted versions at each iteration, thereby solving Stage I and Stage II problems. At each iteration, the restricted problem is solved by considering equivalent balanced (CTMTP) with partial sum constraint which is further solved by considering related (CCMTP). In the proposed algorithm, the number of (CCMTP) required to be solved are at the most $(r - s + 1)$ for some $r, s \in \{1, 2, \ldots, p\}$, where $p$ is the number of distinct entries in the time matrix $\{t_{ij}\}$. The best polynomial running time for (CCMTP) is $o(m\log n(m + n\log n))$, where $m = |I|, n = |J|$ (Orlin [67]). Hence the proposed algorithm is a polynomial time algorithm.

The next section of second chapter presents the study of another two stage bottleneck transportation problem in which the availability at each source is known to lie in a specified interval. A practical situation corresponding to this type of problem is the example of the production of maintenance-free-sealed industrial batteries. Pro-
duction is a continuous process depending upon the available resources. However, each battery has a certain shelf life and batteries need to be periodically re-charged, else the whole lot becomes dead resulting in the loss of the finished goods. Often due to lack of recharging facilities on the production floor, each batch of manufactured batteries is transported immediately to the destinations, this corresponds to the first stage. In the second stage, just enough maintenance-free-sealed batteries from the sources are shipped in order to satisfy the industrial users’ demands at the destinations. Shipment is done in such a way so as to minimize the overall transportation time. The mathematical model of this two stage interval time minimization transportation problem is:

$$\min_{r=\{u_i\} \in S'} \left[ (T_1(Y)) + \min_{Z \in S''(Y)} [(T_2(Z))] \right]$$

where the set $S'$ is given by

$$S' : \begin{cases} 
\sum_{j \in J} y_{ij} = a_i & \forall i \in I, \\
\sum_{i \in I} y_{ij} \leq b_j & \forall j \in J, \\
y_{ij} \geq 0 & \forall (i,j) \in I \times J.
\end{cases}$$

Corresponding to a feasible solution $Y = \{y_{ij}\}$ of stage-1 problem, $S'(Y)$ denotes the set of feasible solutions of Stage-2 problem:

$$S'(Y) : \begin{cases} 
\sum_{j \in J} z_{ij} \leq a'_i - a_i & \forall i \in I, \\
\sum_{i \in I} z_{ij} = b_j - \sum_{i \in I} y_{ij} & \forall j \in J, \\
z_{ij} \geq 0 & \forall (i,j) \in I \times J.
\end{cases}$$

The availability at the source $i$ belongs to $[a_i, a'_i]$ and the demand at the destination $j$ is $b_j$. In first stage, the sources ship all of their on-hand material to the demand points and the second stage shipment covers the demand that is not fulfilled in first stage. In each stage, aim is to minimize the duration of transportation and the
overall goal is to minimize the sum of two stage shipment times.

This problem was first discussed by Sonia et al. [87]. In their methodology, two sequences of Stage I and Stage II time are generated. One of the sequences consists of generating pairs of the form \((T_1(\cdot), T_2(\cdot) : T_1(\cdot) > T_2(\cdot))\) by solving time minimization transportation problem of the form \(PLB(T_2(\cdot))\) and cost minimization transportation problem of the form \(CP_{LB}(T_1(\cdot), T_2(\cdot))\) where the problem \(PLB(T_2(\cdot))\) reduces the on hand shipment time for Stage II, and the problem \(CP_{LB}(T_1(\cdot), T_2(\cdot))\) gives the minimum shipment time for Stage II corresponding to the Stage I shipment time obtained from \(PLB(T_2(\cdot))\). Similarly the sequence of two stage shipment time of the form \((T_1(\cdot), T_2(\cdot) : T_1(\cdot) < T_2(\cdot))\) is obtained by solving the problems \(PUB(T_1(\cdot))\) and \(CP_{UB}(T_2(\cdot), T_1(\cdot))\), where these problems play a similar role as played by \(PLB(T_2(\cdot))\) and \(CP_{LB}(T_1(\cdot), T_2(\cdot))\) with their role for Stage I and Stage II reversed. Further it has been established theoretically that the global minimum value of this problem is obtained out of these generated pairs.

The present work aims at reducing the computational complexity of the method discussed by Sonia et al. [87], thereby suggesting a different approach to solve this problem, which adopts only one sequence of Stage I and Stage II problems in contrast to the two way procedure adopted by Sonia et al. [87]. This two stage interval bottleneck transportation problem, where the total availability of the product at the source lies in the specified intervals, is shown to be related to an ordinary interval (TMTP), which is further shown to be equivalent to a standard (TMTP). Feasible solutions of the Stage I and Stage II transportation problems are derived from a feasible solution of this standard (TMTP). Moreover, it has also been shown that in the developed algorithm a finite number of cost minimization transportation problems (CMTP) are solved to generate different pairs of Stage I and Stage II shipment times.

Next Chapter discusses bilevel programming problems. Two problems have been considered on a class of integer bilevel programming problems. First problem discussed in Section 3.1, deals with a class of linear fractional bilevel programming problem (LFBLP) with upper level objective function as linear fractional function.
and lower level objective function as linear function. It is worth mentioning that the objective functions, which are ratios of two functions appear frequently, for example, when an efficiency measure of a system is to be optimized or in optimizing return on investment in resource allocation problem. In the present work, we have considered a more complicated instance where in addition, decision variables at upper level are integers. Such problems find their relevance in literature [34]. Mathematically this problem can be stated as follows:

\[
(P_{3.1}) \quad \max_{X_1} F(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha}{d_{11}X_1 + d_{12}X_2 + \beta}
\]

where \( X_2 \) solves the following problem,

\[
(P_{3.2}) \quad \max_{X_2} f(X_1, X_2) = c_{21}X_1 + c_{22}X_2 \quad \text{(for a given } X_1)\]

such that

\[
A_1X_1 + A_2X_2 \leq b,
\]

(2.1)

\[
X_1, X_2 \geq 0 \text{ and } X_1 \text{ is an integer vector,}
\]

where

\( c_{11}, d_{11}, c_{21} \in R^{1 \times n_1} \) and \( c_{12}, d_{12}, c_{22} \in R^{1 \times n_2} \), \( A_1 \in R^{m \times n_1} \), \( A_2 \in R^{m \times n_2} \), \( b \in R^{m \times 1} \), \( \alpha, \beta \) scalars, \( X_1 \in R^{n_1 \times 1} \), \( X_2 \in R^{n_2 \times 1} \) are the decision variables under the control of upper level and lower level respectively. In this bilevel programming problem the upper level variables are integers and lower level variables are continuous. Then for each fixed \( X_1 \), dual of the lower level programming problem can be defined as:

\[
DP_{3.2}(X_1) \quad \min_w (b - A_1X_1)^t w
\]

subject to

\[
A_2^t w \geq c_{22}^t
\]

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The algorithm discussed is such that after finding all extreme points of the dual of lower level programming problem, the given bilevel programming problem is converted into a single level linear fractional programming problem. Then at each extreme point of the dual problem, a linear fractional programming problem with linear constraints is solved and the algorithm justifies that an optimal solution is obtained from one of these problems. The algorithm converges in finite number of steps due to the fact that the number of extreme points of the dual problem is finite.

Section 3.2 discusses an integer bilevel programming problem with bounded variables in which the objective function of the first level is linear fractional, the objective function of the second level is linear and the common constraint region is a polyhedron and the decision variables at both the levels are taken as integers. Mathematically this problem can be formulated as:

$$(P_{3.16}) \quad \max_{X_1, X_2} F(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha}{d_{11}X_1 + d_{12}X_2 + \beta} = \frac{C^1X + \alpha}{D^1X + \beta}$$

where $X_2$ solves the following problem,

$$(P_{3.2b}) \quad \max_{X_2} f(X_1, X_2) = c_{21}X_1 + c_{22}X_2 = C^2X \quad \text{(for a given } X_1)$$

subject to

$$AX \leq b, \quad L \leq X \leq U, \quad X \text{ is an integer vector}$$

where

\begin{align*}
&c_{11}, d_{11}, \text{ and } c_{21} \in R^{n_1 \times n_1}, \quad X_1, \quad L_1 \text{ and } U_1 \in R^{n_1 \times 1} \\
c_{12}, d_{12}, \text{ and } c_{22} \in R^{n_2 \times n_2}, \quad X_2, \quad L_2 \text{ and } U_2 \in R^{n_2 \times 1}, \quad L = (L_1, L_2)^T, \quad U = (U_1, U_2)^T, \\
&C^1 = (c_{11}, c_{12}), \quad D^1 = (d_{11}, d_{12}), \quad C^2 = (c_{21}, c_{22}), \quad A_1 \in R^{m \times n_1}, \quad A_2 \in R^{m \times n_2}, \\
&b \in R^{m \times 1}, \quad A = (A_1, A_2), \quad X = (X_1, X_2)^T, \quad \alpha, \beta \text{ are scalars and each component of } \\
&L \text{ is strictly less than the corresponding component of } U.
\end{align*}
Consider the relaxed problem of the leader

\[
(P_{3.13}^*) \max_{X_1, X_2} F(X_1, X_2) = \frac{c_{11}X_1 + c_{12}X_2 + \alpha}{d_{11}X_1 + d_{12}X_2 + \beta}
\]

subject to

\[
AX \leq b,
\]
\[
L \leq X \leq U.
\]

This problem was originally discussed by Thirwani and Arora [96], without any bound on variables. They suggested an algorithm for this problem to obtain bilevel feasible points by ranking the integer feasible solutions of the leader’s objective function. Indeed, the method discussed by them ranks the set of feasible points but it is silent about all other alternate feasible solutions of leader’s objective function, which is a serious concern, as missing out these points may result into an inferior bilevel feasible solution or the problem may come out to be infeasible, which is actually feasible. An example discussed at the end of Section 3.2 clearly elaborates this point.

The problem becomes more realistic from application point of view if bound on each decision variable is introduced. Motivated by such a case, we have extended the work of Thirwani and Arora [96] to more general case by introducing bounds on variables. For this we have derived additional cuts for the integer fractional programming problem, which successively rank the integer feasible solutions in the decreasing order of the value of leader’s objective function. Further with the repeated applications of edge truncating cut, all alternate integer points to each ranked solution which was actually missed by Thirwani and Arora [96], are scanned. Scanning of the alternate integer feasible solutions is significant in the sense that it enables the leader to make a number of choices giving same value of his objective function, and out of these choices he can offer the one which results in minimum/maximum gain of the follower depending upon his relationship with the follower. Moreover in the absence of scanning, there is a possibility that (BLP) may come out to be infeasible.
although it is not, or the conflict may resolve at an inferior solution.

Further an extension of (BLP) is discussed in the form of constrained bilevel programming problem (CBLP). Such problems arise frequently, as in many practical situations, integer solutions are required, which apart from satisfying primary constraints are also required to satisfy some secondary constraints. For example, consider a regulatory problem in which the government would like to control the behavior or output of a particular industry. Here, the government assumes the role of leader and exercises its power through a combination of taxes, fines and quotas. The primary constraints are taxes and fines that the industry must pay to government and secondary constraints are contributions of industry towards the welfare of society by initiating some kind of scholarships or charitable institutions etc. In general, secondary constraints may be nonlinear in nature and hence their inclusion leads to a complicated problem which may not be easily solvable. We are able to solve this problem by using the application of ranking as well as scanning of integer feasible solutions of the common constraint region without actually using any nonlinear programming technique.

Chapter 4 discusses a class of multiobjective integer programming problems. This chapter is further divided into two sections. The first section of this chapter deals with a class of biobjective integer nonlinear fractional programming problems, where the objective functions involved are the ratios of the two quadratic functions and the constraints are linear and decision variables are integers. Mathematically the problem can be stated as follows:

\[(P_{4.1e}) \quad \min_{X \in \Omega} (f(X), h(X))\]

where

\[f(X) = \frac{C^TX + XTDX + \alpha}{E^TX + XTFX + \beta}\]

\[h(X) = \frac{R^TX + XTSX + \gamma}{Q^TX + XTWX + \delta}\]

\[\Omega = \{X \in \mathbb{R}^n \mid AX = b, X \geq 0 \text{ is an integer vector}\}, D, F, S, W \text{ are } n \times n\]
symmetric matrices, $C, E, R, Q \in \mathbb{R}^{n \times n}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$ and $ETX + XTFX + \beta > 0$, $QTX + XTWX + \delta > 0$ for all $X \in \Omega$.

For the purpose of finding the set of efficient points of $(P_{4.1a})$, following non linear fractional programming problem $(P_{4.2a})$ constructed

$$(P_{4.2a}) \quad \min_{X \in \Omega} f(X)$$

Related to $(P_{4.2a})$, an integer programming problem $(P_{4.3a})$ is formulated as

$$(P_{4.3a}) \quad \min_{X \in \Omega} g(X)$$

where $g(X)$ is a linear function or a linear fractional function (as discussed in following cases), bearing different relationships with the function $f(X)$. Define vectors $U = (U_1, U_2, \ldots , U_n)^T$ and $V = (V_1, V_2, \ldots , V_n)^T$, we have following observations.

**Case 1.** If $CX + X^TDX + \alpha \geq 0$ for all $X \in \Omega$.

Define

$$U_j = \min_{X \in \Omega} X^TD_j$$

and

$$V_j = \max_{X \in \Omega} X^TF_j$$

where $D_j$ and $F_j$ are $j$th columns of $D$ and $F$ respectively. In this case

$$f(X) \geq g(X), \text{ for all } X \in \Omega,$$

where $g(X) = \frac{(C + U)^T X + \alpha}{(E + V)^T X + \beta}$

**Case 2.** If $CX + X^TDX + \alpha \leq 0$ for all $X \in \Omega$, define

$$U_j = \min_{X \in \Omega} X^TD_j$$

and

$$V_j = \min_{X \in \Omega} X^TF_j$$

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where $D_j$ and $F_j$ are the $j$th columns of $D$ and $F$ respectively. Then we have

following possibilities depending upon the sign of $(E + V)^T X + \beta$

**Subcase (2.a).** If $(E + V)^T X + \beta < 0$, for all $X \in \Omega$, it is shown that, $f(X) \leq g(X)$, for all $X \in \Omega$, where $g(X) = \frac{(C + U)^T X + \alpha}{(E + V)^T X + \beta}$.

**Subcase (2.b).** If $(E + V)^T X + \beta > 0$, for all $X \in \Omega$, we get, $f(X) \geq g(X)$, for all $X \in \Omega$, where $g(X) = \frac{(C + U)^T X + \alpha}{(E + V)^T X + \beta}$.

**Subcase (2.c).** If $(E + V)^T X + \beta$ is changing sign on $\Omega$, then for $H(X) = (E + V)^T X + \beta$ and $M' = \max_{X \in \Omega} H(X)$, $M' > 0$. Define $G(X) = H(X) - (M' + 1)$. We get $f(X) \leq g(X)$, where

$$
g(X) = \frac{(C + U)^T X + \alpha}{G(X)},
$$

which is again a linear fractional function.

**Case 3.** If $CX + X^T DX + \alpha$ is changing sign on $\Omega$.

Then define

$$
U_j = \max_{X \in \Omega} X^T D_j
$$

and

$$
V_j = \min_{X \in \Omega} X^T F_j
$$

where $D_j$ and $F_j$ are the $j$th columns of $D$ and $F$ respectively. Then we have the following possibilities

**Subcase (3.a).** If $(E + V)^T X + \beta > 0$, for all $X \in \Omega$. We get $f(X) \leq g(X)$, where $g(X) = \frac{(C + U)^T X + \alpha}{(E + V)^T X + \beta}$.

**Subcase (3.b).** If $(E + V)^T X + \beta < 0$, for all $X \in \Omega$. We get $f(X) \leq g(X)$, where $g(X)$ in this case is a linear function of the form

$$
g(X) = \frac{1}{\zeta}((C + U)^T X + \alpha)
$$
and \( \zeta = \min_{X \in \Omega} EX + X^T FX + \beta \).

**Subcase (3.c)** If \((E+V)^T X + \beta\) is changing sign on \(\Omega\). Let \(H(X) = (E+V)^T X + \beta\) and define

\[
G(X) = H(X) - (\gamma + 1)
\]

where \(\gamma = \max_{X \in \Omega} H(X)\). Again in this case we get \(f(X) \leq g(X)\), where

\[
g(X) = \frac{1}{\zeta}((C + U)^T X + \alpha)
\]

is a linear function.

In the above discussion, it is observed that in each case either \(f(X) \geq g(X)\) or \(f(X) \leq g(X)\), depending upon different sign conditions of \(CX + X^T DX + \alpha\) and \(EX + X^T FX + \beta\). Detailed discussion in this regard is done in Chapter 4. Further, a methodology to find all efficient pairs of the problem \((P_{4.3a})\), is developed by using the relationship of \(f(X) \geq g(X)\). All the cases, where the function \(f(X) \leq g(X)\) on \(\Omega\), are reducible to the case \(f(X) \geq g(X)\), by replacing \(f(X)\) by \(-f(X)\) and \(g(X)\) by \(-g(X)\). A brief discussion in this regard is also done in Chapter 4. In the proposed methodology, one of the objectives is treated as principal objective and other as an additional constraint. By varying the minimum acceptable value of the second objective, a sequence of constrained integer nonlinear fractional programming problems is constructed and optimal solutions of these problems are obtained by ranking the integer feasible solutions of the related linear/linear fractional programming problem \((P_{4.3a})\).

To the best of our knowledge, such bi objective/multiobjective integer nonlinear programming problems have not been studied in literature. Motivated by the idea to obtain an optimal solution of a nonlinear integer programming problem by approximating with the corresponding linear/linear fractional program, as discussed in literature for quadratic programs [46], an attempt is made to discuss an algorithm which not only finds all efficient pairs of the bi objective integer nonlinear programming problem but also finds an optimal solution of the nonlinear integer
The main advantage of the proposed algorithm is that no nonlinear optimization technique is used and it records only efficient pairs. Moreover, as the set of feasible solutions of the problem is assumed to be bounded and value of one of the objective functions is continuously reducing, the algorithm is bound to terminate in a finite number of iterations.

Section 4.2 discusses another variant of multiobjective integer programming problems, where each objective function is a ratio of two quadratic functions and in addition the decision variables are bounded and take integral values. The mathematical model of the problem considered is

\[
(P_{4.1b}) \quad \max_{X \in \Omega} (F_1(X), F_2(X), \ldots, F_t(X))
\]

where

\[
F_i(X) = \frac{(C^i X + \alpha^i)^2}{(D^i X + \beta^i)^2}, \quad i \in H = \{1, 2, \ldots, t\}
\]

subject to

\[
X \in \Omega = \{X \in \mathbb{R}^{n \times 1} \mid AX \leq b, L \leq X \leq U, \ X \text{ is an integer vector}\}
\]

where

\[
\tilde{\Omega} = \{X \in \mathbb{R}^{n \times 1} \mid AX \leq b, L \leq X \leq U\}
\]

is a closed and bounded convex polyhedron over which \(D^i X + \beta^i > 0\) and \(C^i X + \alpha^i \geq 0\) for all \(i \in H\) and \(C^i = (c^i_1, c^i_2, \ldots, c^i_n) \in \mathbb{R}^{1 \times n}, D^i = (d^i_1, d^i_2, \ldots, d^i_n) \in \mathbb{R}^{1 \times n}, \ \alpha^i, \ \beta^i \in \mathbb{R}\) for all \(i \in H\).

In order to find the set of all efficient \(t\)-tuples, the following single objective integer nonlinear fractional programming problem \((P_{4.2b})\) is considered.

\[
(P_{4.2b}) \quad \max_{X \in \Omega} F_1(X) = \frac{(C^1 X + \alpha^1)^2}{(D^1 X + \beta^1)^2}
\]

Let \(X_1\) denote an optimal feasible solution of \((P_{4.2b})\), which may be an extreme
point of $\hat{\Omega}$ or is obtained as an extreme point of the truncated region obtained from $\hat{\Omega}$ by repeated application of cuts developed by Dahiya and Verma [29]. The present procedure, based on cutting plane technique, seeks re-structuring of the set of all feasible solutions of the corresponding continuous version of the relaxed problem (i.e. the problem without integer constraints). Various eligible directions leading to potentially efficient solutions are identified and once a point or a region is scanned, it is truncated from the current region. Assuming that the feasible region of the multiobjective programming problem is closed and bounded, a methodology has been proposed to find all efficient solutions of the problem. The methodology discussed involves the implementation of various cuts, which truncate the region in such a way that the portion or point once scanned does not reappear further, leading to convergence of the algorithm in finite number of steps.

The last chapter discusses a multiobjective transportation problem where a tradeoff between cost and pipeline has been discussed. Cost-pipeline tradeoff relationship in a transportation problem is relevant in project planning, which requires transportation of raw materials/machinery etc. to the site of the project before it starts functioning. Besides taking care of transportation costs, the goods reaching on the last day have also to be taken into consideration because the commissioning of the project is influenced in the sense that some time is consumed even after the last consignment of goods reaches the site, as some formalities are required to be completed before processing them onto the machine. This time lag between the arrival of the last consignment of goods and the time of initiation of the project indirectly means cost to the decision maker. As in some situations, early initiation of project is desired, which is possible when the quantity of goods reaching just before the initiation of the project is very small, but this in turn means high cost of transportation. Therefore the problem of interest is trade-off between total cost of transportation and pipeline at a time $T$ (time of transportation) such that if early initiation of the project is desired, i.e. at some time $T^* (< T)$, means higher cost of transportation, such a time ($T$) is referred to as pivotal time. The proposed algorithm determines all the independent, non-dominated cost-pipeline pairs called
efficient pairs, which correspond to basic feasible solutions (BFS) starting from the minimum cost solution at a pivotal time $T$ chosen. The determination of extreme points of the non-dominated set in objective space in preference to the decision space is justified, as asserted by Aneja and Nair [7], the number of extreme points of the feasible set in objective space is in general lesser than the that in the decision space. The proposed algorithm finds all such pairs in criteria space. Process terminates when no more new efficient extreme points are available.