Chapter 5

Multiobjective transportation problem

The present chapter deals with a trade off between cost and pipeline at a given time in a transportation problem. The time lag between commissioning a project and the time when the last consignment of goods reaches the project site is an important factor. This motivates the study of a bi-criteria transportation problem at a pivotal time $T$. An exhaustive set $E$ of all independent cost-pipeline pairs (called efficient pairs) at time $T$ is constructed in such a way that each pair corresponds to a basic feasible solution and in turn gives an optimal transportation schedule. A convergent algorithm has been proposed to determine non-dominated cost pipeline pairs in a criteria space instead of scanning the decision space, where the number of such pairs is large as compared to those found in the criteria space.

5.1 Mathematical formulation of the problem

Consider the following cost minimization transportation problem:

$$(P_{b1}) \quad \min \sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij}$$
subject to the following constraints:

\[
\begin{align*}
\sum_{j \in J} x_{ij} &= a_i, \quad a_i > 0, \quad i \in I, \\
\sum_{i \in I} x_{ij} &= b_j, \quad b_j > 0, \quad j \in J, \\
x_{ij} &\geq 0, \quad \forall (i, j) \in I \times J
\end{align*}
\]  

(5.1.1)

where \( I \) is the index set of supply points, \( J \) is the index set of destinations, \( x_{ij} \) is the amount of the product transported from \( i \)th supply point to \( j \)th destination, \( c_{ij} \) is the per unit of cost of transportation on \((i, j)\)th route, \( a_i \) is the availability of the product at \( i \)th supply point and \( b_j \) is the requirement of the same at \( j \)th destination. In time minimization transportation problem, the transportation of goods from sources to destinations is done in parallel and its prime aim is to supply to destinations with the required quantity within a shortest possible time. Mathematically, this problem can be formulated as follows:

\[
\begin{align*}
\min_{X \in S} \{ \max_{i} t_{ij}(x_{ij}) \mid x_{ij} > 0 \},
\end{align*}
\]  

(5.1.2)

where

\[
S = \{ X = \{ x_{ij} \} \mid X \text{ satisfies (5.1.1)} \}
\]  

(5.1.3)

and \( t_{ij} \) is the time of transportation from \( i \)th supply point to \( j \)th destination. For any \( X \in S \), let

\[
T = \max \{ t_{ij} \mid x_{ij} > 0 \}.
\]

At any time \( T \), define following cost minimization transportation problem, whose optimal solution yields minimum transportation cost at the given time \( T \).

\[
\begin{align*}
\min_{X \in S} \sum_{i \in I} \sum_{j \in J} c'_{ij} x_{ij}
\end{align*}
\]

where

\[
c'_{ij} = c_{ij} \text{ if } t_{ij} \leq T, \]
\[
c'_{ij} = \infty \text{ if } t_{ij} > T.
\]
Let the optimal value of the problem \((P_{5.3})\) be denoted by \(Z\). Note that this value is optimal cost of transportation yielded at time \(T\).

**Pivotal time.** Time \(T\) is called pivotal time if for any other time of transportation \(T^*\) (say), 
\[ T^* > T \implies Z^* < Z \quad \text{and} \quad T^* < T \implies Z^* > Z \]
where \(Z\) and \(Z^*\) are minimum transportation costs given by \((P_{5.3})\) at times \(T\) and \(T^*\) respectively.

**Remark 5.1.1** \(T\) is a pivotal time of transportation if in each optimal basic feasible solution \((OBFS)\) of the problem \((P_{5.3})\) there exists a cell \((i,j)\) with \(t_{ij} = T, \ x_{ij} > 0\).

**Pipeline.** The pipeline at pivotal time \(T\) corresponding to an optimal basic feasible solution \(X = \{x_{ij}\}\) of \((P_{5.3})\) is given by
\[ p = \sum_{\{(i,j): t_{ij} = T\}} x_{ij}. \]

Pipeline \(p\) at time \(T\) is obtained by considering the related problem \((RP - T)\) as:
\[
(RP - T) \quad \min_{X \in S} \sum_{i \in I} \sum_{j \in J} c_{ij}^* x_{ij}
\]
where
\[ c_{ij}^* = 0 \quad \text{if} \quad t_{ij} < T, \]
\[ = 1 \quad \text{if} \quad t_{ij} = T, \]
\[ = \infty \quad \text{if} \quad t_{ij} > T. \]

**Remark 5.1.2** The optimal value of the problem \((RP-T)\) yields minimum pipeline at time \(T\).

So mathematically the problem of cost pipeline trade off at a pivotal time \(T\) can be formulated as
\[
\min_{X \in S} (Z, \ p), \quad (5.1.4)
\]
where \(Z\) is the minimum cost of transportation problem and \(p\) is the minimum pipeline at time \(T\), yielded by optimal solution of \((P_{5.3})\) and \((RP-T)\) respectively.

**Pair.** A cost pipeline pair at given time of transportation \(T\) is denoted by \((T : Z, \ p)\).
Dominated pair. A pair \((T : Z, p)\) is called dominated pair at pivotal time \(T\) if there exists a pair \((T : Z', p')\) such \((Z, p) \geq (Z', p')\) i.e. \(Z \geq Z'\) and \(p \geq p'\) with strict inequality holding least at one place.

Non-dominated pair. A pair which is not dominated is called a non-dominated pair.

Efficient point. A solution yielding a non-dominated pair is called an efficient point.

5.2 Notations

\(q^{th}\) Efficient pair \((T : Z_q, p_q)\), \((q \geq 2)\). The \(q^{th}\) efficient pair is that member of \(L_{q-1}\) for which \(Z_q = \min\{Z \mid (T : Z, p) \in L_{q-1}\}\), where the set \(L_q\) for \(q \geq 1\) is defined below.

For \(q \geq 1\), following notations are introduced.

\(X^q\) is the set of all basic feasible solutions yielding the \(q^{th}\) efficient pair = \(\{X^q_h \mid h = 1\text{ to } B^q\}\).

\(B^q\) is the set of basic cells of solution \(X^q\).

\(\Delta_{ij}\) is the relative cost co-efficient for a basic feasible solution of problem \((P_{3,3})\) for cell \((i,j)\).

\(\Delta'_{ij}\) is the relative cost co-efficient for a basic feasible solution of problem \((RP-T)\) for cell \((i,j)\).

\(\hat{X}^q\) is the basic feasible solution derived from \(X^q\) by a single pivot operation such that the corresponding time of transportation remains \(T\). that is

\[
\max_{\{(i,j) \mid x^q_{ij} > 0\}} t_{ij} = T.
\]
For each \( h = 1, 2 \ldots s_q \), define

\[
N^{qh} = \left\{ (i, j) \notin B^{qh} \mid \Delta_{ij} < 0, \Delta_{ij}' > 0, z_{ij}^{qh} = z_{im}^{qh} = 0, (l, m) \in B^{qh} \text{ and } \max_{(i, m) \in B^{qh}, z_{im}^{qh} > 0} t_{rm} = T \right\}
\]

as the collection of those nonbasic cells such that the entry of which into the current basis corresponding to a \( q^{th} \) efficient pair increases the cost of transportation and reduces the pipeline.

\[D^{qh} = \{(T : Z, p) \mid Z = Z_q - \Delta_{ij} x_{im}^{qh}, p = p_q - \Delta_{ij}' x_{im}^{qh}, (l, m) \in B^{qh}, (i, j) \in N^{qh}\},\]

is the collection of all the pairs \( (T : Z, p) \) where \( Z \) is the increased cost and \( p \) is the reduced pipeline obtained by entering nonbasic cells from the set \( N^{qh} \) in a single pivot operation.

\[D_q = \bigcup_{h=1}^{s_q} D^{qh}\]

\[L_q = L_{q-1} - \{(T : Z_q, p_q)\} \text{ for } q \geq 2, \text{ where } L_1 = \emptyset \text{ and the list } L_q \text{ is given by:} \]

\[L_q = L_q' \cup D_q - \{(T : Z, p) \mid (T : Z, p) \text{ is a dominated pair in } L_q' \cup D_q\}\]

\( E \) is the set of efficient pairs \( (T : Z_i, p_i), i = 1 \text{ to } M \).

5.3 Theoretical development

This section discusses the main theoretical results which lead to the convergence of procedure given in Section-5.4.

**Theorem 5.3.1** There exists a basic feasible solution yielding the first efficient pair \( (T : Z_1, p_1) \).

**Proof** : Let \( X_0 \) be an optimal basic feasible solution (OBFS) of problem \( (P_{5.3}) \) giving \( Z_1 \) as the minimum cost at pivotal time \( T \). Let \( B_0 \) be the set of basic cells of
the solution \( X_0 \). Construct set

\[
N_0 = \{(i,j) \notin B_0 \mid \Delta_{ij} = 0, \Delta'_ij > 0\}
\]

If \( N_0 = \emptyset \), then \( X_0 \) gives the first efficient pair \((T : Z_1, p_1)\), else choose \((s,t) \in N_0\) such \( \Delta'_{ij} = \max\{\Delta'_{ij} \mid (i,j) \in N_0\} \) and enter cell \((s,t)\) into basis \( B_0 \). This will reduce the pipeline at the same cost \( Z_1 \). Let the basic feasible solution thus obtained be \( X_1 \) with basis \( B_1 \). Let \( N_1 = \{(i,j) \notin B_1 \mid \Delta_{ij} = 0, \Delta'_ij > 0\} \). Again if \( N_1 = \emptyset \), \( X_1 \) gives the pair \((T : Z_1, p_1)\) and when \( N_1 \neq \emptyset \), entry of cell \((d,e) \in N_1\), into basis \( B_1 \) where \( \Delta'_{de} = \max\{\Delta'_{ij} \mid (i,j) \in N_1\} \) further decreases the pipeline at cost \( Z_1 \). Continuing likewise, a sequence of solutions is constructed till a stage is reached for which \( N_q = \emptyset \).

Denote \( X_q \) by \( X^{11} \). Thus \( X^{11} \) is a basic feasible solution with basis \( B^{11} \), giving pair \((T : Z_1, p_1)\).

**Remark 5.3.2** If set \( H = \{(i,j) \notin B^{11} \mid \Delta_{ij} = 0, \Delta'_ij = 0\} \), then \( X^1 \) is the set of all basic feasible solutions, each obtained as a result of entering a cell of \( H \) into basis \( B^{11} \).

**Corollary 5.3.3** Every efficient pair \((T : Z_q, p_q)\), \( q \geq 2 \), is attainable at a basic feasible solution.

**Proof**: The second efficient pair \((T : Z_2, p_2)\) is that member of \( L_1 \) for which \( Z_2 = \min\{Z \mid (T : Z, p) \in L_1\} \). Thus \((T : Z_2, p_2)\) is attained at a basic feasible solution. By definition of \( D_q \), \( L_q (q \geq 2) \), every pair \((T : Z_q, p_q)\), \( q \geq 3 \) also corresponds to a basic feasible solution.

**Remark 5.3.4** Corollary 5.3.3 justifies that \( E \) is a finite set.

**Theorem 5.3.5** For any efficient pair \((T : Z_q, p_q)\), \( p_q \) is the minimum pipeline at cost \( Z_q \) and \( Z_q \) is the minimum cost at pipeline \( p_q \).
Proof: It is sufficient to show for each $h = 1$ to $s_q$, the sets

$$S_{1h} = \left\{ (i, j) \notin B^{th} \mid \Delta_{ij} = 0, \ \Delta_{ij}' > 0, \ \bar{x}^{th}_{ij} = \bar{x}^{th}_{lm}, \ \bar{x}^{th}_{im} = 0, \ (l, m) \in B^{th} \right\}$$

and

$$S_{2h} = \left\{ (i, j) \notin B^{th} \mid \Delta_{ij} > 0, \ \Delta_{ij}' = 0, \ \bar{x}^{th}_{ij} = \bar{x}^{th}_{lm}, \ \bar{x}^{th}_{im} = 0, \ (l, m) \in B^{th} \right\}$$

are both empty. Suppose, on the contrary, that $S_{1h} \neq \emptyset$ for some $h$, $1 \leq h \leq s_q$. Then there exists a cell $(i, j)$ in $S_{1h}$ such that $\Delta_{ij} = 0, \ \Delta_{ij}' > 0, \ \bar{x}^{th}_{ij} = \bar{x}^{th}_{lm}$, $\ (l, m) \in B^{th}, \ \bar{x}^{th}_{im} = 0$. The entry of such a cell $(i, j)$ into the basis of the solution $X^{th}$ will result in a pair $(T : Z_q, p)$ with $p < p_q$. This contradicts the non-dominance of $(T : Z_q, p_q)$. Hence $S_{1h} = \emptyset$. Similarly $S_{2h} = \emptyset \ \forall \ h = 1$ to $s_q$.

Corollary 5.3.6 If $(T : Z, p)$ is a pair and $Z = Z'$, $p \neq p'$ where $(T : Z', p')$ is an efficient pair, then $(T : Z, p)$ is dominated.

Theorem 5.3.7 A non-dominated pair $(T : Z, p)$ not in $E$ satisfies the relation

$$Z = \sum_{i=1}^{M} \lambda_i Z_i, \ \ p \leq \sum_{i=1}^{M} \lambda_i p_i, \ \ \sum_{i=1}^{M} \lambda_i = 1, \ \ \lambda_i \geq 0, \ \ i = 1$ to $M.$

Proof: Since $(T : Z, p)$ is a non-dominated pair not in $E$, therefore $(T : Z, p) \notin \bigcup_{i=1}^{M} I_i$ and $(T : Z, p)$ does not correspond to a basic feasible solution. Thus $(T : Z, p)$ is yielded by a feasible solution. Clearly $Z_1 < Z < Z_M$. There exist scalars $\lambda_i$ not all zero such that $Z = \sum_{i=1}^{M} \lambda_i Z_i$, $\sum_{i=1}^{M} \lambda_i = 1, \ \ \lambda_i \geq 0, \ \ i = 1$ to $M$. Let $p^* = \sum_{i=1}^{M} \lambda_i p_i$.

Since $(T : Z, p)$ is non-dominated, $p \leq p^* = \sum_{i=1}^{M} \lambda_i p_i$.

Theorem 5.3.8 If $(T : Z, p)$ is a pair with $Z \neq Z_i$, $i = 1$ to $M$ and $Z_1 < Z < Z_M$ then there exists $p' \leq p$ such that $(T : Z, p')$ is a non-dominated pair.

147
Proof: Since \( Z \neq Z_i \), \( i = 1 \) to \( M \), therefore \((T : Z, p) \notin E\).

Let \( Z_k < Z < Z_{k+1} \), \( k \in \{1, 2, \ldots, M\} \). Then there exists \( \lambda \) such that \( Z = \lambda Z_k + (1 - \lambda)Z_{k+1} \), \( 0 < \lambda < 1 \). Consider \( p^* = \lambda p_k + (1 - \lambda)p_{k+1} \). Clearly \( p_{k+1} < p^* < p_k \).

Also \( p^* \leq p \), because if \( p^* > p \) then \((T : Z, p^*)\) is dominated by \((T : Z, p)\) which is a contradiction as \((T : Z, p^*)\) being a convex combination of adjacent efficient pairs, must be non-dominated. Setting \( p^* = p' \), the desired result is obtained.

**Theorem 5.3.9** \( E \) is the exhaustive set of efficient pairs.

Proof: The proof is divided into two pairs.

**Case I.** No efficient pair other than the ones in \( E \) can be derived from a dominated pair.

Suppose on the contrary that \((T : Z', p')\) is an efficient pair derived from a dominated pair \((T : Z, p)\), such that \( (T : Z', p') \neq (T : Z_i, p_i) \), \( i = 1 \) to \( M \). Now \((T : Z', p')\) is a non-dominated pair not in \( E \) and so from Corollary 5.3.3 it follows that it does not correspond to a basic feasible solution. This violates the very character of an efficient pair.

**Case II.** Any pair \((T : Z, p)\) derived from an efficient pair \((T : Z_q, p_q)\) with \( Z < Z_q \), \( p > p_q \) is either identical with one of the efficient pairs \((T : Z_i, p_i)\), \( i = 1 \) to \( q - 1 \) or is a dominated pair or is a convex combination of efficient pairs \((T : Z_i, p_i)\), \( i = 1 \) to \( M \). Now \( Z_1 \) being the global minimum cost at time \( T \), \( Z_1 \leq Z < Z_q \). If \( p > p_1 \), then \((T : Z, p)\) is dominated by \((T : Z_1, p_1)\) and the conclusion follows. Let now \( p \leq p_1 \). Thus \( p_q < p \leq p_1 \). Two exhaustive cases arise:

(i) \( Z = Z_i \) for some \( i \in \{1, 2, \ldots, q - 1\} \) In this case \( p = p_1 \) because if \( p_1 > p \) then \((T : Z_i, p_1)\) is dominated by \((T : Z_i, p)\) which contradicts the efficient character of \((T : Z_i, p_1)\). Thus in this case \((T : Z, p)\) is identical with \((T : Z_i, p_i)\)

(ii) \( Z \neq Z_i \). Let \( Z_r < Z < Z_{r+1} \) where \( r \geq 1 \), \( q \geq r + 1 \). Let \( Z = \lambda Z_r + (1 - \lambda)Z_{r+1} \) where \( 0 < \lambda < 1 \) and \( p^* = \lambda p_r + (1 - \lambda)p_{r+1} \). Thus \( p_{r+1} < p^* < p_r \).

If \( p^* > p \), then \((T : Z, p^*)\) is dominated by \((T : Z, p)\) which is a contradiction, because \((T : Z, p^*)\) is a convex combination of two adjacent efficient pairs and therefore must be a non-dominated pair. Thus \( p^* \leq p \), if \( p^* < p \) then clearly \((T : Z, p)\) is a dominated pair.
and if $p^* = p$, then $(T : Z. p)$ is a convex combination of efficient pairs in $E$.

**Theorem 5.3.10** The last pair in $E$ gives the global minimum pipeline at pivotal time $T$.

**Proof** : Since $(T : Z_M, p_M)$ is the last pair, $L_{M-1}$ is a singleton, viz. $L_{M-1} = \{(T : Z_M, p_M)\}$ and $D_M = \emptyset$. Thus $X^{Mh} = \emptyset$. Therefore there does not exist any cell $(i, j) \notin B^{Mh}$ with $\Delta_i < 0$, $\Delta'_j > 0$, the entry of which into the basis of solution $X^{Mh}$ results in a solution with time of transportation equal to $T$. Thus the only possibility for cell $(i, j) \notin B^{Mh}$ with $\Delta' > 0$ are $\Delta_{ij} = 0$, $\Delta'_i > 0$ or $\Delta_{ij} > 0$, $\Delta'_j > 0$. In both cases, the fact that $(T : Z_M, p_M)$ is efficient, is contradicted. Hence $\Delta'_{ij} \neq 0$. Thus the set $P$ given by

$$P = \left\{(i, j) \notin B^{Mh}, \ b = 1 \in S_M \left| \Delta'_i < 0, \ \Delta'_j < 0, \ x_{im}^{Mh}, \ \bar{x}_{im}^{Mh} = 0, \ (i, m) \in B^{Mh} \right| \ \text{and} \ \max_{(r, w) \in \{r | x_{rj}^{Mh} > 0\}} t_{rw} = T \right\}$$

is empty. Thus the pipeline cannot be reduced further at pivotal time $T$ of transportation. Hence $p_M$ is the global minimum pipeline at pivotal time $T$.

### 5.4 Algorithm for finding cost and pipeline trade-off pairs

Initially set $r = 0$ and $l = 0$. Let the partition of various time routes be given by $t_0 > t_1 > t_2 \cdots > t_s$, where $t_i = \min\{t_{ij} \mid i \in I, j \in J\}$

**Step 0.** Solve the problem $(P_{3,3})$ at time $T = t_i$ and go to next step.

**Step 1.** If every optimal basic feasible solution of problem $(P_{3,3})$ yield time $T$, then declare $T$ as pivotal time and go to next step, otherwise set $l = l + 1$ and go to Step 0.

**Step 2** (Finding 1st efficient pair; $(T : Z_1, p_1)$). Read optimal basic feasible solution of problem $(P_{3,3})$ corresponding to time $T$ with minimum cost as $Z_1$ and pipeline $p_1$ and corresponding basis as $B_r$. 

149
Step (2.a) Construct \( N_r = \{(i,j) \notin B_r \mid \Delta_{ij} = 0, \Delta'_{ij} > 0\} \). If \( N_r = \emptyset \), then go to Step (2.c), otherwise go to Step (2.b).

Step (2.b) Choose \((s,t) \in N_r\) such that \( \Delta_{st} = \max\{\Delta'_{ij} \mid (i,j) \in N_r\}\) and enter cell \((s,t)\) into the basis \(B_r\), set \( r = r+1 \) and obtain new basic feasible solution as \(X_r\) with basis \(B_r\) and go to Step (2.a).

Step (2.c) Record \((T : Z_1, p_1)\) as the first efficient pair and the corresponding basic feasible solution as \(X_1\) with basis \(B_1\). Construct the set \(H = \{(i,j) \notin B_1^T \mid \Delta_{ij} = 0, \Delta'_{ij} = 0\}\) (as mentioned in Remark 5.3.2), compute \(X_{1h}, h = 2,3\ldots s_1\) and set \(X_1 = \{X_{1h}, h = 1,2\ldots s_1\}\), go to next step.

Step 3. Initially record \(E = \{(T : Z_1, p_1)\}\), set \( q=1 \) and go to next step.

Step 4. Construct the set \(N^h, h = 1,2\ldots s_q\). If \( N^h = \emptyset \) for all \( h = 1,2\ldots s_q\), go to terminal step, otherwise go to Step 5.

Step 5. Construct \(D_q, L_q^q\) and \(L_q\). Then note the \((q+1)\)th efficient pair given as \((T : Z_{q+1}, p_{q+1})\), where \( z_{q+1} = \min\{Z \mid (T : Z, p) \in L_q\}\) and set \( E = E \cup \{(T : Z_{q+1}, p_{q+1})\}\), set \( q = q + 1 \) and go to Step 4.

Step 6 (Terminal step). Set \( E \) is the exhaustive set of efficient pairs corresponding to the pivotal time \( T \) (as proved in Theorem 5.3.9).

5.5 Numerical illustration

Consider a 4 \times 6 transportation problem given by Table 1. The upper left corner in each cell gives the time of transportation on the corresponding route and the lower right corner gives the per unit cost on that route. The partition of various time routes is given by \( t^0(=45) > t^1(=40) > t^2(=36) > t^3(=34) > t^4(=21) > t^5(=20) > t^6(=18) > t^7(=13) > t^8(=12) > t^9(=11) > t^{10}(=7)\).

Step 0. Solve the problem \((P_{5,3})\) at time \( T = t^0(=45) \) and go to Step 1.
Step 1. An optimal feasible solution is depicted in the following table.

**Table 1**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>13</td>
<td>34</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>32</td>
<td>8</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>36</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>30</td>
<td>40</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>45</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>8</td>
<td>20</td>
<td>22</td>
</tr>
</tbody>
</table>

$\begin{align*}
    b_j \rightarrow & \quad 10 \quad 18 \quad 25 \quad 19
\end{align*}$

Since the (OFS) of this problem does not yield time $T(= 45)$, therefore $T = 45$ is not pivotal time. Set $l = 1$ and go to Step 0.

**Step 0.** Solve the problem $(P_{i,4})$ at time $T = t^l(= 40)$ and go to Step 1.

**Step 1.** This problem has two alternate optimal feasible solutions and are depicted

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>10</td>
<td>34</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>32</td>
<td>7</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>9</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>9</td>
<td>30</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>9</td>
<td>45</td>
<td>21</td>
</tr>
<tr>
<td>40</td>
<td>8</td>
<td>20</td>
<td>19</td>
<td>22</td>
</tr>
</tbody>
</table>

$\begin{align*}
    b_j \rightarrow & \quad 10 \quad 18 \quad 25 \quad 19
\end{align*}$

151
From tables 3 and 4, it is observed that (OFS) of problem \((P_{5,3})\) at time \(T = 40\) yield the time of transportation as 36 and 40 respectively. Therefore \(T = 40\) is not a pivotal time. Set \(l = 2\) and go to Step 0.
Step 0. Solve the problem \((P_{5,3})\) at time \(T = t^2(= 36)\) and go to Step 1.

Step 1. An optimal feasible solution of this problem is depicted in Table 5.

Table 5

<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th>13</th>
<th>34</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>19</td>
<td>32</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>36</td>
<td>7</td>
<td>27</td>
</tr>
<tr>
<td>70</td>
<td>9</td>
<td>40</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>M</td>
<td>21</td>
<td>19</td>
</tr>
<tr>
<td>40</td>
<td>8</td>
<td>20</td>
<td>22</td>
<td></td>
</tr>
</tbody>
</table>

Since no alternate optimal solution to one depicted in Table 5 exists, so \(T = 36\) is a pivotal time. Go to Step 2.

Step 2. Note the pair \((T : Z, p) = (36 : 1726, 18)\) and the corresponding basis as \(B_0\) and go to Step (2.a).

Step (2.a). Corresponding to the basis \(B_0\), depicted in Table 5, the value of \(\Delta'_{ij}\) and \(\Delta_{ij}\) for \((i, j) \notin B_0\) is calculated using the formula \(\Delta'_{ij} = u'_i + v'_j - c'_{ij}\) and \(\Delta_{ij} = u_i + v_j - c_{ij}\) respectively, and is shown in Table 6. Initially \(u_1 = u'_1 = 0\).

Table 6

<table>
<thead>
<tr>
<th>(i,j)</th>
<th>(1.2)</th>
<th>(1.4)</th>
<th>(2.1)</th>
<th>(2.4)</th>
<th>(3.1)</th>
<th>(3.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta'_{ij})</td>
<td>-1</td>
<td>-M-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1-m</td>
</tr>
<tr>
<td>(\Delta_{ij})</td>
<td>-34</td>
<td>0</td>
<td>-19</td>
<td>-16</td>
<td>-11</td>
<td>-2</td>
</tr>
</tbody>
</table>

Since the set \(N_0 = \{(i, j) \notin B_0 \mid \Delta_{ij} = 0, \Delta'_{ij} > 0\} = \emptyset\), the current recorded pair \((36 : 1726, 18)\) is efficient, go to Step (2.c).

Step (2.c). Record the first efficient pair as \((T : Z, p) = (36 : 1726, 18)\) with the
corresponding solution as \( X^1 = X^{11} = \{x_{11}^1 = 10, x_{12}^1 = 7, x_{21}^1 = 9, x_{22}^1 = 18, x_{32}^1 = 9, x_{33}^1 = 19\} \) and go to Step 3.

**Step 3.** Record \( E = \{(36 : 1726, 18)\} \), set \( q = 1 \) and go to Step 4.

**Step 4.** Construct the set \( N^{11} = \{(2, 1), (3, 1)\} \) from Table 6 and go to Step 5.

**Step 5.** Construct the set \( D_1 = D_{11} = \{(36 : 1825, 9), (36 : 1916, 8)\} \), by entering nonbasic cells from the set \( N^{11} \) successively in single pivot operation. since there are no dominated pairs in \( D_1 \), we have \( L_1 = D_1 \) and

\[
(T : Z_2, p_2) = \min\{Z \mid (T : Z, p) \in L_1\}.
\]

we get \( (T : Z_2, p_2) = (36 : 1825, 9) \). Update the set \( E = \{(36 : 1726, 18), (36 : 1825, 9)\} \), set \( q = 2 \) and go to Step 4.

**Step 4.** The table depicting second efficient pair \( (T : Z_2, p_2) = (36 : 1825, 9) \) is shown below

<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th>13</th>
<th>34</th>
<th>16</th>
<th>32</th>
<th>19</th>
<th>8</th>
<th>12</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>18</td>
<td></td>
<td>36</td>
<td></td>
<td>7</td>
<td></td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td></td>
<td></td>
<td>30</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>18</th>
<th>25</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Construct the set \( N^{21} = \{(2, 1), (2, 4)\} \) on the same lines at we constructed above the set \( N_0 \). Since \( N^{21} \neq \emptyset \), go to Step 5.
Step 5. Construct the set \( D_2 = \{(36 : 1844, 8), (36 : 1852, 8)\} \), we have

\[
L'_2 = L_1 \setminus \{(36 : 1825, 9)\} = (36 : 1916, 8)
\]

and \( L_2 = \{(36 : 1844, 8)\} \). Therefore the third efficient pair is given as \((36 : 1844, 8)\), update the set \( E = \{(36 : 1726, 18), (36 : 1825, 9), (36 : 1844, 8)\} \). Set \( q = 3 \) and go to Step 4.

Step 4. Table depicting the third efficient pair is given below

\[
\begin{array}{cccccc}
12 & 13 & 34 & 17 & M & 17 \\
19 & 32 & & 8 & 12 & \\
7 & 18 & 36 & 8 & 7 & 27 \\
70 & 30 & 40 & 60 & & \\
11 & 20 & M & 21 & 19 & 28 \\
40 & 8 & 20 & & & \\
\end{array}
\]

Since the set \( N^{th} = \emptyset \), go to Step 6.

Step 6. The exhaustive set of efficient pairs corresponding to the pivotal time \( T = 36 \) is given by \( E = \{(36 : 1726, 18), (36 : 1825, 9), (36 : 1844, 8)\} \)

5.6 Concluding Remarks

1. In this paper we have presented an algorithm, which enumerates all the independent, non-dominated cost-pipeline pairs called efficient pairs, which correspond to basic feasible solutions (BFS) starting from the minimum cost solution at pivotal time \( T \) chosen. When the time taken for transportation is
not pivotal, there always exists an alternate solution of problem \((P_{5.3})\), which yields zero unit of pipeline at time \(T\), thereby reducing the total time of transportation from \(T\) to some other time \(T' < T\) and yielding a unique cost pipeline pair with zero pipeline, and hence there is no need to provide nondominated solution.

2. Each efficient pair at pivotal time \(T\) corresponds to an extreme point of the set of feasible solutions of \((P_{5.3})\). As there are finite number of extreme points of the feasible set and the choice of sets \(L'_e\) and \(L_q\) is such that none of the extreme points is repeatedly examined, the proposed algorithm terminates in finite number of steps.

3. It may be noted that in order to find all efficient pairs, cost minimization transportation problems \((P_{5.3})\) and RP-T are being solved repeatedly and by Corollary 5.3.3, each such pair corresponds to an extreme point of \(S\). Therefore only finite number of such problems are to be solved. The best polynomial running time for cost minimization transportation problem is \(o(n\log(n(m + n\log n)))\), where \(|I| = m\), \(|J| = n\) (Orlin [67]). Hence the proposed algorithm is a polynomial time algorithm.

4. This problem can further be explored to the case when decision variable are taken as bounded.

5. The problem of finding cost pipeline efficient pairs with positive pipelines at a given particular time which may not be a pivotal time, can be explored further.