Chapter 7

General discussion and recommendations

In this chapter, we summarize the thesis findings and write conclusions. We will also make some recommendations and suggest some areas for future research work.

7.1 Conclusions

This thesis is in the area of numerical analysis of transport equations motivated by neuroscience. My research interests are in applied and computational mathematics driven by applications to biology. We are interested in the challenging issues in modern neuronal sciences stemming from the modelling of neuronal variability. We investigate the mathematical and numerical analysis of partial differential models for neuroscience. This area of research remained relatively unexplored but during the end of the last century, considerable progress has been made in understanding the basis of the initiation and transmission of neuronal impulses in quantitative terms. The resulting mathematical model is a transport equation involving point-wise delay which models the distribution of time intervals between successive neuronal firings. We have studied the transport equations, developing some algorithms for the solution, finding
mathematical estimates for stability and convergence, and running it on a computer to obtain the results needed in practice. Numerical methods have been derived in the research papers [79, 80, 81, 82] for a model problem in neuronal transport which leads to a linear advection equation with multiple point-wise delay. This specificity is common to many models in biology and neuroscience (to take into account impulses or signal propagation time). We considered several solution strategies, depending on the concrete situation.

### 7.2 Scope for future work

We will consider the non-linear version of neuronal model, i.e., transport equation having point-wise delay as well as advance. For more details about the model we refer to the Chapter 2. Non-linearity comes from intensity of incoming current which can be nonlinear. Let intensity of the incoming current is \( I(t) \). The change in the probability \( F(v, t) \) with time is given by

\[
\frac{\partial F}{\partial t}(v, t) + \left[ -\frac{v}{\tau} + I(t) \right] \frac{\partial F}{\partial v}(v, t) = p_e[F(v - 1, t) - F(v, t)] + p_i[F(v + v_0, t) - F(v, t)],
\]

(7.1)

\[
F(v, 0) = F_0(v)
\]

where \( 0 < v < r, \ t > 0, \ F_0 \) is some initial data, \( F(v, t) \) is the probability that the depolarization \( V_i \leq v \) at time \( t \), \( p_e \) and \( p_i \) are the frequencies of excitatory and inhibitory impulses, respectively. After a refractory period of duration \( t_0 \), an excitatory impulse produces unit depolarization, while an inhibitory impulse produces \( v_0 \) unit repolarization, and if the depolarization reaches a threshold of \( r \) units, the neuron fires. For sub-threshold levels, the depolarization decays exponentially between impulses with time constant \( \tau \).
7.2.1 Motivation towards the recent model

In deterministic leaky integrate-and-fire models for the excitatory networks the probability of a neuron in potential \( v \) at any time \( t \) is given by a transport equation which we have studied in the Chapter 6. We include jumps by excitatory and inhibitory impulses. We consider the following nonlinear version of neuronal model in which probability \( p(t,v) \) of a neuron in potential \( v \) at any time \( t \) is given by the following equation

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial v} \left[ (-v + I(t))p(t,v) \right] + A \int_{v > v_F} p(t,v) \, dv = \int [p(t,v + \alpha) - p(t,v)]M^+(\alpha) \, d\alpha \\
+ \int [p(t,v - \beta) - p(t,v)]M^-(\beta) \, d\beta \\
+ N(t)\delta(v = V_R), \ v \in \mathbb{R}, \ t > 0
\]

\[
p(0,v) = p^0(v) \geq 0, \ \int p^0(v) dv = 1, \quad (7.2)
\]

where \( \alpha \) is the addition in the voltage due to the excitatory impulses and \( \beta \) is the reduction due to the inhibitory impulses. The intensities of these impulses are measured by the functions \( M^+(\alpha) \) and \( M^-(\beta) \) which are given external inputs due to the noise in the network. Firing rate is given by

\[
N(t) := A \int_{v > v_F} p(t,v) dv. \quad (7.3)
\]

This is the nonlinear version of the leaky integrate-and-fire model with Lévy noise. Possible non-linearity comes from \( I(t) \) depending on the total activity of the network, viz.

\[
I(t) = I_0 + aN(t).
\]

We are working on mathematical theory of finite difference and finite volume methods for hyperbolic models from neuroscience. We anticipate that the experience in dealing with the numerical analysis of hyperbolic phenomena would be of great use in dealing with modeling, analysis and optimization of complex dynamical systems. Dealing
with integrate-and-fire model with Lévy noise is the motivation to work on non-linear version of integrate-and-fire model. Further studies are therefore necessary to consider the non-linear version of neuronal model.

7.3 Scope for future work in another area

At this stage we would like to extend this work in another directions. There are various time-dependent processes in natural and engineering sciences that hinge essentially on the complete process history. One of the challenging issues in the modern material sciences is the modelling of memory materials. We propose to study the longitudinal motion of materials where the destabilizing influence of nonlinear elastic response competes with the damping effect of the fading memory. The resulting mathematical model is a system of conservation laws involving a time convolution integral. On the basis of a recently developed well-posedness theory; we construct, implement, and analyze a new numerical method. The implementation relies on numerical methods for advection equations with (discrete) delay which have been previously developed by us for the advection equation. Finally we apply the complete algorithm to the elastic memory problem to make quantitatively predictions for the behavior of the material.

In this direction we focus on elastic memory materials with fading memory. We refer to [77] for the mathematical modelling background. If we consider the longitudinal motion of an infinitely long elastic bar, the problem can be described by the following partial integro-differential equation for the unknown strain \( w = w(x, t) > -1 \) and velocity \( v = v(x, t) \in \mathbb{R} \).

\[
\begin{align*}
    w_t - v_x &= 0 \\
    v_t - \sigma(w)_x &= \int_0^t k(t - \tau) \sigma(w(\cdot, \tau))_x \, d\tau \quad \text{in } \mathbb{R} \times (0, \infty). \\
\end{align*}
\]

Here \( \sigma \in C^1(-1, \infty) \) is the given nonlinear stress strain relation with \( \sigma' \geq 0 \). We denote the memory kernel by \( k \) and \( k \in C^1(0, \infty) \). For \( k \geq 0 \) and \( k' \leq 0 \) the
system (ELAS) is a problem with **fading** memory. Appropriate initial data have to be added.

For \( k = 0 \) the system (ELAS) is a hyperbolic system of nonlinear conservation laws that can develop singularities, i.e., discontinuous solutions, for arbitrarily small initial data. It is the striking result of Hrusa [42] that (ELAS) for nonvanishing \( k \) can have global smooth solutions if the initial data are small enough. In this case the damping effect of the memory dominates. On the other hand Dafermos shows that for large enough data singularity blow up of the solution happens [19]. Up to our knowledge no quantitative information of this striking threshold phenomenon, which we will call the Hrusa-Dafermos hypothesis, are known.

Recently a well-posedness theory for a model problem for (ELAS) with scalar unknown \( u := u(x, t) \in \mathbb{R} \), namely

\[
\begin{align*}
&u_t + f(u)_x = -\int_0^t k(t - \tau)f(u(\cdot, \tau))_x\,d\tau \quad \text{in } \mathbb{R} \times (0, \infty) \\
&\text{(MODEL)$_0$}
\end{align*}
\]

has been derived [14]. In (MODEL)$_0$ the given function \( f \in C^1(\mathbb{R}) \) is the (in general nonlinear) flux. Also for this model the above-mentioned threshold behaviour can be observed, such that it is necessary to consider weak solutions. These weak solutions can be realized as the \( L^1 \)-limit of a sequence of smooth functions \( \{u^\varepsilon\} \) which solve the regularized problem

\[
\begin{align*}
&u^\varepsilon_t + f(u^\varepsilon)_x = -\int_0^t k(t - \tau)f(u^\varepsilon(\cdot, \tau))_x\,d\tau + D^\varepsilon[u], \\
&D^\varepsilon[u] = -\varepsilon\left(u_{xx} + \int_0^t k(t - \tau)u_{x\tau}(\cdot, \tau)d\tau\right) \quad \text{in } \mathbb{R} \times (0, \infty) \quad \text{(MODEL)$_\varepsilon$}
\end{align*}
\]

The interesting point of the analysis is that the viscous regularization with \( D^\varepsilon[u] = \varepsilon u_{xx} \) does not suffice to establish the limit. The latter is the case for homogeneous conservation laws ((MODEL)$_0$ with \( k = 0 \)). However, successful numerical schemes for homogeneous conservation laws, as e.g. monotone finite volume schemes, rely exactly on the viscous regularization. Concerning previous numerical work for (MODEL)$_0$ we refer representatively to [58] and references therein. These procedures transfer mono-
tone numerical methods for homogeneous conservation laws directly to conservation laws with memory. The approach leads in some examples to unstable numerical behaviour. We conjecture that a finer tuning of the numerical dissipation in the sense of $(\text{MODEL})_\varepsilon$ can stabilize the computations. Up to our knowledge no numerical method for $(\text{MODEL})_0$ with $k \neq 0$ which mimics the dissipative effect of the new term $D^\varepsilon[u]$ in $(\text{MODEL})_\varepsilon$ has been suggested.

Let us finally note that the problem $(\text{MODEL})_0$ becomes an ordinary Volterra-type integro-differential equation if the spatial derivatives are skipped. In this case many efficient and reliable numerical methods have been suggested, so that we just cite the recent work [10] which gives a certain overview on the existing methods.

If the discretization for $(\text{MODEL})_0$ first uses some quadrature for the time integral it can be understood as a differential equation with pointwise delay. For this situation, we already made contributions. For a model problem in neuronal transport which leads to a linear advection equation with pointwise delay numerical methods have been derived in [79, 80, 81, 82]. Note that the complete scheme of $(\text{MODEL})_0$ will require the discretization of point-wise delay terms. This work includes finite volume, finite difference and discontinuous Galerkin methods. A-priori and a-posteriori error estimates should have been derived. We refer the work of Rohde et al. (e.g. [45, 46]. This covers also the coupling with (spatial) integral terms.

### 7.4 Recommendations about future work

It would be interesting to consider the non-linear version of neuronal model and assess the effects of delay on solution behavior. A future study would be investigating the model from material science also would be very interesting. Let us summarize the future plan of research work.

1. Numerical analysis of nonlinear neuronal model (7.1) and (7.2); and application of the finite volume method and analysis of the resulting scheme for hyperbolic
equation.

2. Extension of the model (7.1) to higher space dimensions and application of the discretization schemes developed in the preceding item 1.

3. Development of a mesh-adapted scheme such that there is no restriction on the size of delay and advanced arguments.

4. Investigation of implicit discretization schemes with emphasis to finite volume and discontinuous Galerkin methods.

5. Development and analysis of numerical scheme for the model problem $\text{(MODEL)}_\varepsilon$. 