Chapter 1

Supersymmetric Quantum Mechanics

1.1 Introduction

The discovery of supersymmetry (SUSY) is one of the most distinguished achievements in High Energy Physics. The fundamental significance of SUSY was realized almost immediately after its discovery in 1971 by Gel’fand and Likhtman. Within a short span of time, the supersymmetric generalizations of the standard model was found, the supersymmetric theory of gravity constructed and the approaches to supersymmetric string theory developed. Physicists have long strived to obtain a unified description of all basic interactions. Now it is felt that SUSY is a necessary ingredient in any unifying approach which relates bosonic and fermionic degrees of freedom. The algebra involved in SUSY is a graded Lie algebra which closes under a combination of commutation and anticommutation relations. First of all, it was introduced in the context of the string models to unify the bosonic and fermionic sectors. A (3+1) dimensional field theory was constructed which was invariant under this symmetry and had very interesting properties such as having paired fermionic and bosonic degrees of freedom. The SUSY offered a possible way of unifying space-time and internal symmetries of S-matrix. The gravity was also generalized by incorporating SUSY to a theory called supergravity. In such theories, Einstein’s general theory of relativity turns out to be a necessary consequence of a local gauged SUSY. Thus local supersymmetric theories provide a natural framework for the unification of gravity with other fundamental interactions of nature. Despite the beauty of all these unified theories,
so far there is no experimental evidence of SUSY being realized in nature. One of the important predictions of unbroken SUSY is the existence of SUSY partners of quarks, leptons and gauge bosons which have the same masses as their SUSY counterparts. Still no particles have been seen, which implies that SUSY must be spontaneously broken. Various schemes have been invented to try to resolve the problem, including the idea of non-perturbative breaking of SUSY [1, 2].

In the early days, SUSY was studied in quantum mechanics as a testing ground for the non-perturbative methods of seeing SUSY breaking in field theory. Over the last 25 years, the idea of SUSY have stimulated new approaches to other branches of physics. For example, evidence has been found for a dynamical SUSY relating even-even and even-odd nuclei [3]. The Langevin equation and the method of stochastic quantization has a path integral formulation which embodies SUSY [4]. There have also been applications of supersymmetry in atomic, condensed matter and statistical physics [5-12]. The idea of supersymmetry have been profitably applied to many non-relativistic quantum mechanical problems. In an important paper on supersymmetry, Witten [13] considered a spin-1/2 particle moving in one-dimension as a simple example of a supersymmetric system, with no reference to a field theory at all. Witten also defined the algebra that must be satisfied by the charge operators in terms of which the supersymmetric Hamiltonian can be expressed. The algebraic relations which have been formulated firstly by Witten, have now become the defining relations of supersymmetric quantum mechanics. Just like in field theory, supersymmetry leads to specific relations between the component sectors of a supermultiplet, in supersymmetric quantum mechanics, the existence of a symmetry generating operator which commutes with the Hamiltonian leads to certain specific relations between the spectra of the component parts of the supersymmetric Hamiltonian. This is one of the main reasons for the increase of interest in supersymmetric quantum mechanics. By exploiting the degeneracy between the spectra of the bosonic and fermionic component of certain one-dimensional supersymmetric Hamiltonians, the properties of the first excited state of the bosonic component may be obtained from the knowledge of the ground state of the fermionic component. It has been shown independently by Andrianov et al. and Sukumar that all one-dimensional quantum systems have supersymmetric partners.
A simple extension of supersymmetric quantum mechanics to arbitrary dimensions is also possible [14-17].

In supersymmetric quantum mechanics, one considers a simple realization of a SUSY algebra involving the fermionic and bosonic operators. Because of the existence of fermionic operators which commute with the Hamiltonian, one obtains the relationships between the energy eigenvalues, the eigenfunctions and the S-matrices of the component parts of the full SUSY Hamiltonian. These relationships are exploited to categorize analytically solvable problems [1].

One of the key ingredients in solving exactly for spectrum of one dimensional potential problems is the connection between the bound state wave function and the potential. Once one knows the ground state wave function and its energy, then one can factorize the Hamiltonian in the following way. If the ground state energy is chosen to be zero, then from the Schrödinger equation, we have

\[ H_1 \psi_0(x) = -\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + V_1(x) \psi_0(x) = 0. \]  

Therefore, we get

\[ V_1(x) = \frac{\hbar^2}{2m} \frac{\psi_0''(x)}{\psi_0}. \]  

We can factorize the Hamiltonian as \( H_1 = A^\dagger A \) where

\[ A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \quad \text{and} \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x). \]

From eq. (1.1), we get

\[ V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x). \]  

Here \( W(x) \) is the superpotential. The solution for \( W(x) \) in terms of ground state wave function is

\[ W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi_0'(x)}{\psi_0(x)}. \]

The next step in constructing the SUSY theory related to the original Hamiltonian is to define the operator \( H_2 = AA^\dagger \), obtained by reversing the order of \( A \) and \( A^\dagger \). The operator \( H_2 \) is in fact a Hamiltonian corresponding to a new potential \( V_2(x) \).

\[ H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x), \]
\[ V_2(x) = W_2(x) + \frac{\hbar}{\sqrt{2m}} W'(x). \quad (1.6) \]

The potentials \( V_1(x) \) and \( V_2(x) \) are known as supersymmetric partner potentials. The energy eigenvalues, the wave functions and the S-matrices of \( H_1 \) and \( H_2 \) are related to each other. We begin with the Schrödinger equation

\[ H_1 \psi_n = \epsilon_n \psi_n, \quad (1.7) \]

\[ A \psi_n = \lambda_n \psi_n, \]

Multiplying both sides with \( A \), we get,

\[ A A^\dagger (A \psi_n) = \epsilon_n (A \psi_n), \]

\[ H_2 (A \psi_n) = \epsilon_n (A \psi_n), \]

\[ H_2 \chi_n = \epsilon_n \chi_n. \quad (1.8) \]

Here \( H_2 \) is the supersymmetric partner Hamiltonian of \( H_1 \), with eigenfunctions \( \chi_n = A \psi_n \).

It is obvious that \( H_2 \) has the same eigenvalue spectrum as that of \( H_1 \), but for the case \( A \psi_0 = 0 \), which is the case of supersymmetry broken. Explicitly, the relation between the eigenvalues and the eigenfunctions of two Hamiltonians reads [18] \((n = 0, 1, 2, 3, \ldots)\)

\[ E_n^{(2)} = E_n^{(1)}; \quad E_0^{(1)} = 0, \]

\[ \psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A \psi_n^{(1)}, \]

\[ \psi_n^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^\dagger \psi_n^{(2)}. \]

The superscript \( E_n^{(2)}, E_n^{(1)} \) refer to the spectrum of the respective Hamiltonians \( H_2, H_1 \). If \( \psi_n^{(1)} \) and \( \psi_n^{(2)} \) are normalized then the wave functions \( \psi_n^{(2)} \) and \( \psi_n^{(1)} \) are also normalized.

The operator \( A(A^\dagger) \) not only converts an eigenfunction of \( H_1 (H_2) \) into an eigenfunction of \( H_2 (H_1) \) with the same energy, but it also destroys (creates) an extra node in the eigenfunction. Since the ground state wave function of \( H_1 \) is annihilated by the operator \( A \), this state has no SUSY partner. Thus, knowing all the eigenfunctions of \( H_1 \), we can determine the eigenfunctions of \( H_2 \) using the operator \( A \) and using \( A^\dagger \) we can construct all the eigenfunctions of \( H_1 \) from those of \( H_2 \) except for the ground state.
The reason for the degeneracy of the spectra of the two Hamiltonians $H_1$ and $H_2$ can be understood from the properties of the SUSY algebra. Consider a matrix SUSY Hamiltonian of the form,

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \equiv \begin{bmatrix} A^1 A & 0 \\ 0 & A A^1 \end{bmatrix}.$$ 

The matrix Hamiltonian contains both bosonic and fermionic operators with commutation and anticommutation relations. Consider the charge operators

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$$

and

$$Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}$$

in conjugation with $H$. The following commutation and anticommutation relations then satisfy the closed superalgebra.

$$[H, Q] = [H, Q^\dagger] = 0,$$

$$\{Q, Q^\dagger\} = H; \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0.$$ 

Since supercharge $Q$ and $Q^\dagger$ commute with $H$, which is responsible for degeneracy. The operators $Q$ and $Q^\dagger$ can be taken as operators, which change bosonic degrees of freedom into fermionic ones and vice-versa.

### 1.2 Factorization of the Hamiltonian

The Hamiltonian $H_1$ can be factorized in terms of the operators $A_1$ and $A_1^\dagger$. The ground state wave function of the partner Hamiltonian $H_2$ is obtained from the first excited state of $H_1$ using operator $A_1$. The Hamiltonian $H_2$ can again be factorized in terms of $W_2$ and the partner of this Hamiltonian is another Hamiltonian $H_3$ which has one bound state less than $H_2$ and two bound states less than $H_1$. Therefore, for an exactly solvable potential for $H_1$, we can solve for the energy eigenvalues and wave functions for the Hamiltonian created with repeated refactorizations. If $E_0^{(1)}$ is the ground state energy of the Hamiltonian $H_1$ with eigenfunction $\psi_0^{(1)}$, then we can write (using $\hbar = 2m = 1$),

$$H_1 = A_1^\dagger A_1 + E_0^{(1)} = -\frac{d^2}{dx^2} + V_2(x),$$

(1.9)
where
\[ A_1 = \frac{d}{dx} + W_1(x), \]
\[ A_1^\dagger = -\frac{d}{dx} + W_1(x) \]
and
\[ W_1(x) = \frac{d}{dx} \ln \psi_0^{(1)}(x). \]

The partner Hamiltonian of \( H_1 \) is given by
\[ H_2 = A_1 A_1^\dagger + E_0^{(1)} = -\frac{d^2}{dx^2} + V_2(x), \] (1.10)
where
\[ V_2(x) = W_1^2 + W_1' + E_0^{(1)} \]
\[ = V_1(x) + 2W_1' \]
\[ = V_1(x) - 2 \frac{d}{dx} \ln \psi_0^{(1)}(x). \] (1.11)

The energy eigenvalues and eigenfunctions of the two Hamiltonians \( H_1 \) and \( H_2 \) are related by
\[ E_{n+1}^{(1)} = E_n^{(2)}, \]
\[ \psi_n^{(2)} = \left( \frac{E_n^{(1)} - E_0^{(1)}}{E_n^{(1)}} \right)^{-\frac{1}{2}} A_1 \psi_n^{(1)}. \]

Now, factorizing the Hamiltonian \( H_2 \) with its ground state energy \( E_0^{(2)} = E_1^{(1)} \), we get the supersymmetric partner \( H_3 \) for \( H_2 \). Therefore, we have
\[ H_2 = A_1 A_1^\dagger + E_0^{(1)} = A_2^\dagger A_2 + E_1^{(1)} \] (1.12)
where
\[ A_2 = \frac{d}{dx} + W_2(x), \]
\[ A_2^\dagger = -\frac{d}{dx} + W_2(x) \]
and
\[ W_2(x) = -\frac{d}{dx} \ln \psi_0^{(2)}(x). \]
If we start from \( H_3 \), we obtain

\[
H_3 = A_3 A_2^1 + E_1^{(1)} = -\frac{d^2}{dx^2} + V_3(x),
\]

where

\[
V_3(x) = W_3^2 + W_2' + E_1^{(1)}
= V_2(x) - 2\frac{d^2}{dx^2} \ln \psi_0(x)
= V_1(x) - 2\frac{d^2}{dx^2} \ln(\psi_0^{(1)}(x)\psi_0^{(2)}(x)).
\]

The relation between energy eigenvalues and eigenfunctions for the two Hamiltonians \( H_2 \) and \( H_3 \) is given as

\[
E_n^{(3)} = E_n^{(2)} = E_n^{(1)},
\]

\[
\psi_n^{(3)} = \left[\left(E_n^{(2)} - E_0^{(1)}\right)\right]^{-\frac{1}{2}} A_2 \psi_n^{(2)}
= \left[\left(E_n^{(1)} - E_0^{(1)}\right)\right]^{-\frac{1}{2}} \left[\left(E_n^{(2)} - E_0^{(1)}\right)\right]^{-\frac{1}{2}} A_2 \psi_n^{(1)}.
\]

Thus, starting from the original Hamiltonian \( H_1 \), with \( m \) bound states, we can generate \((m - 1)\) Hamiltonians \( H_2, \ldots, H_m \), such that \( H_m \) has same eigenvalue spectrum as that of \( H_1 \), except that the \((m - 1)\) eigenvalues of \( H_1 \) are missing. Therefore, in general, we can write,

\[
H_m = A_m A_m^1 + E_1^{(1)} = -\frac{d^2}{dx^2} + V_m(x),
\]

where

\[
A_m = \frac{d}{dx} + W_m(x),
\]

\[
W_m(x) = -\frac{d}{dx} \ln \psi_0^{(m)}(x).
\]

The energy eigenvalues and eigenfunctions are related as,

\[
E_n^{(m)} = E_n^{(m-1)} = \ldots = E_{n+m-1},
\]

\[
\psi_n^{(m)} = \left[\left(E_n^{(m-1)} - E_{n-2}^{(m-2)}\right)\right]^{-\frac{1}{2}} \ldots \left[\left(E_n^{(1)} - E_0^{(1)}\right)\right]^{-\frac{1}{2}} A_{m-1} \ldots A_1 \psi_n^{(m-1)}.
\]
and the corresponding potential reads,

\[ V_m(x) = V_1(x) - 2 \frac{d^2}{dx^2} \ln \left( \psi_0^{(1)}(x) \cdots \psi_0^{(m-1)}(x) \right). \]  

(1.18)

Thus, knowing all the eigenvalues and eigenfunctions of \( H_1 \), we can obtain the energy eigenvalues and eigenfunctions of all the \((m-1)\) Hamiltonians.

Using the ideas of supersymmetric quantum mechanics and an integrability condition called the shape invariance, it can be shown that the operator method for the harmonic oscillator can be generalized to the whole class of shape invariant potential (SIP) which include all the popular, analytically solvable potentials [19]. For such potentials, the generalized operator method quickly yields all the bound state energy eigenvalues, eigenfunctions as well as the scattering matrix [20]. Here, shape invariance means that the supersymmetric partner potentials are similar in shape and differ only in the parameters appearing in them. If the partner potential \( V_{1,2}(x, a_1) \) satisfies the condition

\[ V_2(x, a_1) = V_1(x, a_2) + R(a_1), \]

where \( a_1 \) is a set of parameters, \( a_2 \) is a function of \( a_1 \) and remainder \( R(a_1) \) is independent of \( x \), then the potentials \( V_1(x, a_1) \) and \( V_2(x, a_2) \) are said to be shape invariant. Starting from the supersymmetric partner Hamiltonians \( H_1 \) and \( H_2 \) whose eigenvalues and eigenfunctions are related by supersymmetry, the entire spectrum of \( H_1 \) can be easily obtained algebraically by using the shape invariance condition. If the Hamiltonian \( H_1 \) has \( p \) bound states, we can construct \( p \) such Hamiltonians, \( H_1, H_2, \ldots, H_p \), with same spectrum but less number of bound states. Using the shape invariance condition, we obtain,

\[ H_p = -\frac{d^2}{dx^2} + V_1(x; a_p) + \sum_{k=1}^{p-1} R(a_k), \]  

(1.19)

where \( a_p = f^{p-1}(a_1) \). If we compare the spectrum of \( H_p \) and \( H_{p+1} \), we have,

\[ H_{p+1} = -\frac{d^2}{dx^2} + V_1(x; a_{p+1}) + \sum_{k=1}^{p} R(a_k) \]

\[ = -\frac{d^2}{dx^2} + V_2(x; a_p) + \sum_{k=1}^{p-1} R(a_k). \]  

(1.20)

Here, \( H_p \) and \( H_{p+1} \) are supersymmetric partner Hamiltonians which have identical bound state spectra except for the ground state with energy \( E_0 = \sum_{k=1}^{p-1} R(a_k) \). The bound state
wave function for any shape invariant potential can be easily obtained from its ground state wave function, because the operators $A$ and $A^\dagger$ relates the eigenfunctions of the same energy for the supersymmetric partner Hamiltonians. The shape invariance is a sufficient and necessary condition for exact solvability [21]. The Coulomb potential and harmonic potential in $n$ dimensions are shape invariant potentials. For the SIPs, the reflection and transmission amplitudes can also be calculated by operator method. Using the idea of SUSY and shape invariance, the number of potential problems can be solved algebraically. Most of these potentials are either one dimensional or central potentials. Some non-central but separable potential problems can also be solved using these techniques.

The semiclassical WKB method is one of the most useful approximation for computing the energy eigenvalue of the Schrödinger equation. Combining the ideas of SUSY and with lowest order WKB method, one can obtain the lowest order SUSY WKB quantization in case supersymmetry is broken and can be shown that it yields energy eigenvalues which are not only exact for large quantum number but are also exact for the ground state [22].

1.3 Isospectral Hamiltonians

Two Hamiltonians are said to be strictly isospectral, if they have exactly same energy eigenvalue spectrum and S-matrix [23, 24]. The wave functions and hence the wave function dependent quantities like moments, transition amplitudes could be different. Though the idea of generating isospectral Hamiltonians using the Gelfand-Levitan approach [25], Darboux procedure [26] were known for some time, but the supersymmetric quantum mechanical techniques (which deals with first order operators) make the procedure look simpler [27, 28]. When one deletes a bound state of a given potential $V(x)$ and re-introduce the state, it involves solving a first order differential equation, which admits a free parameter. Thus, a one dimensional family of potentials $\hat{V}(x, \lambda)$ can be constructed which have the same energy spectrum as that of $V(x)$. In general, for any one dimensional potential (full line or half-line) with $n$ bound states, one can construct an $n$-parameter family of strictly isospectral potentials, i.e. potentials with eigenvalues, reflection and transmission coefficients identical to those for original potential. This aspect has been utilized prof-
itably in many physical situations. In soliton physics, the stability of the soliton/kink is ensured by the occurrence of a zero energy ground state of the stability equation when small oscillations around the soliton/kink are considered [29]. The stability equation can be considered as an one dimensional Schrödinger equation with potential $V(x)$ and one can construct an isospectral partner for it. The partner stability equation will have the same energy spectrum as that of the original equation. Then, one can reconstruct the soliton solution and hence the potential, $V(\phi)$, which admits the soliton solution from the partner stability equation. This generalizes the class of Hamiltonians which admits soliton/kink solutions that share the same stability data [30, 31]. The spectrum of a charged particle in uniform magnetic field consists of equally spaced Landau levels which are infinitely degenerate. Using isospectral deformation, it has been shown that equispaced spectrum can also be obtained for a wide class of non-uniform magnetic fields [32]. Another application of this procedure is in the context of thermic denaturation of the DNA macromolecules [33], in which the stability of these molecules depends on the H-bonds between the complementary base-pairs as well as on the stacking interactions between adjacent bases along the strands. This interaction can be modelled by Morse potential or a harmonic potential. The generalized isospectral harmonic potential does not change the ground state eigenvalue, but the eigenfunction is sensitive to the deformation. Using the superalgebra formalism, the isospectral potential from harmonic oscillator is obtained, in which the deformation parameter can be chosen considering the experimental data and one obtain the melting point which is very close to experimental value.

1.3.1 One-parameter family of Isospectral Hamiltonians

In this family of Hamiltonians, we delete the lowest bound state and re-introduce it, solving a first order differential equation, which introduces one free parameter. The plan for the construction of partner Hamiltonians and the Strictly isospectral family of potentials is shown in figure 1.1. The superpotential relates the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW}{dx}. \tag{1.21}$$
### Partner Hamiltonians:

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### Strictly Isospectral:

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Figure 1.1: General plan to construct the family of partner Hamiltonians and strictly isospectral potentials.
It is well known that for the potential \( V_2(x) \), the original potential \( V_1(x) \) is not unique \([27, 28]\). The argument is as follows. Suppose \( H_2 \) has another factorization \( B B^\dagger \), where \( B = \frac{d}{dx} + \hat{W}(x) \), then, \( H_2 = A A^\dagger = B B^\dagger \) but \( H_1 = B^\dagger B \) is not \( A^\dagger A \) rather it defines a certain new Hamiltonian. For superpotential \( \hat{W}(x) \), the partner potential \( V_2(x) \) is

\[
V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{1.22}
\]

Consider the most general solution as \( \hat{W}(x) = W(x) + \phi(x) \), which demands that,

\[
\phi^2(x) + 2W(x)\phi(x) + \phi'(x) = 0. \tag{1.23}
\]

We find that \( y \equiv 1/\phi(x) \) satisfies the Bernoulli’s equation

\[
y' = 1 + 2Wy \tag{1.24}
\]

The solution of the above equation is

\[
\frac{1}{y(x)} = \phi(x) = \frac{d}{dx} \ln[I(x) + \lambda],
\]

where \( I(x) = \int_{-\infty}^{x} \psi_0^2(x')dx' \) and \( \lambda \) is a constant. Therefore, we obtain,

\[
\hat{W}'(x) = W(x) + \frac{d}{dx} \ln[I(x) + \lambda]. \tag{1.25}
\]

The corresponding one parameter family of potentials \( \hat{V}_1(x, \lambda) \) is given as

\[
\hat{V}_1(x, \lambda) = V_1(x) - 2\frac{d^2}{dx^2} \ln[I(x) + \lambda]). \tag{1.26}
\]

\( V_1(x) \) is also a member of this family, since as \( \lambda \to \pm \infty, \hat{V}_1(x, \lambda) \to V_1(x) \). The normalized ground state wave function corresponding to the potential \( \hat{V}_1(x, \lambda) \) reads

\[
\hat{\psi}_0(x, \lambda) = \sqrt{\frac{\lambda(1 + \lambda)}{I(x) + \lambda}} \psi_0(x), \tag{1.27}
\]

which will be acceptable wave function only if it is square integrable. Since \( I(x) \) lies between 0 and 1 depending upon the value of \( x \), so we can have different cases for the values of \( \lambda \).

**Case I. When \( \lambda > 1 \) or \( \lambda < -1 \):**

In this case, the wave function is square integrable, so it is an acceptable wave function. Therefore, for any potential, there is a one-parameter family of isospectral
potentials which is strictly isospectral to it. Each member of the family has same number of bound states and scattering matrix. If the potential $V(x)$ is exactly solvable, then $\hat{V}(x)$ is also exactly solvable with $E_n = \hat{E}_n$. $\psi_0(x)$ is obtained in terms of $\psi_0$ and $\psi_{n+1}(x)$ can also be easily obtained in terms of $\psi_{n+1}(x)$. The excited state eigenfunctions for the potential $\hat{V}(x,\lambda)$ are given by

$$\psi_{n+1}(x, \lambda) = \psi_{n+1}(x) + \frac{1}{E_{n+1}} \left( \frac{f(x)}{I(x) + \lambda} \right) \left( \frac{d}{dx} + W(x) \right) \psi_{n+1}(x). \quad (1.28)$$

**Case II.** When $\lambda = 0$ or $\lambda = -1$:

In these cases, the wave function will not be square integrable, as $(I(x) + \lambda)$ will vanish at $x = -\infty$ or $x = \infty$. In both the cases, SUSY is broken and the isospectral potential loses a bound state. For $\lambda = 0$, this was pointed out by Pursey [23] and for $\lambda = -1$, this work was done by Abraham and Moses [24]. We discuss the two cases separately:

(i) **Pursey Potential ($\lambda = 0$):** For any potential, the corresponding Pursey potential is phase equivalent but has one bound state less. One obtain the potential

$$V^P_I(x) = V_I(x) - 2 \frac{d^2}{dx^2} \ln(I(x)). \quad (1.29)$$

The normalized excited state eigenfunctions corresponding to $V^P_I(x)$ are obtained as

$$\psi^P_n(x) = \psi_{n+1}(x) + \frac{1}{E_{n+1}} \left( \frac{f(x)}{I(x)} \right) \left( \frac{d}{dx} + W(x) \right) \psi_{n+1}(x). \quad (1.30)$$

(ii) **Abraham-Moses potential ($\lambda = -1$):** The Abraham-Moses potential is obtained by setting $\lambda = -1$,

$$V^{AM}_I(x) = V_I(x) - 2 \frac{d^2}{dx^2} \ln(I(x) - 1). \quad (1.31)$$

The normalized excited state eigenfunctions corresponding to $V^{AM}_I(x)$ are obtained as

$$\psi^{AM}_n(x) = \psi_{n+1}(x) + \frac{1}{E_{n+1}} \left( \frac{f(x)}{(I(x) - 1)} \right) \left( \frac{d}{dx} + W(x) \right) \psi_{n+1}(x). \quad (1.32)$$

In both the cases supersymmetry between $V(x)$ and $V^{AM}_I(x)$ or $V^P_I(x)$ is broken.

**Case III.** When $-1 < \lambda < 0$: In this case, the potential will be singular for finite values of $x$, so the potential is not of much use for these values of $\lambda$. 

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The one-parameter family of potentials obtained is unique. If we start with the potential $V_1(x)$ and try to find out its one-parameter family $\hat{V}_1(x)$, then we do not get anything new. The one-parameter family of potentials for $V_1(x)$ is given by,

$$\hat{V}_1(x) = V_1(x) - 2 \frac{d^2}{dx^2} \ln((f(x) + \lambda))$$

(1.33)

Solving the above equation, we get,

$$\hat{V}_1(x) = V_1(x) - 2 \frac{d^2}{dx^2} \ln[f(x) + q],$$

(1.34)

where $q = \frac{\lambda}{1+\lambda}$. Therefore, in the potential only $\lambda$ is changed with $q$, which has same range as that of $\lambda$ [18].

### 1.3.2 The $n$-parameter family of Isospectral Hamiltonians

Generalizing the idea of constructing one-parameter family of Isospectral Hamiltonians, we can construct a $n$-parameter family [34] by repeated SUSY operations, where $n$ is the number of bound states in the original potential. The $n$-parameter family contains strictly isospectral potentials, which has same eigenvalues, reflection and transmission amplitudes. They are also connected with solutions of non-linear partial differential equations such as KdV equation.

Deleting the lowest bound state $\psi_0$ from potential $V_0$, we get SUSY partner $V_1$ which has the same eigenvalues as $V_0$ except the bound state at $E_0$. The partner potential $V_1$ is

$$V_1(x) = V_0(x) - 2 \frac{d^2}{dx^2} \ln \psi_0(x).$$

The ground state wave function $\psi_0$ for the potential $V_1$ is located at energy $E_1$. The procedure can be repeated by deleting the lowest bound state from potential $V_1$ to get $V_2$ which has two bound states less than $V_0$. Repeating the procedure, we produce the partner potentials $V_3$, $V_4$, $\ldots$, $V_n$ with ground states $\psi_3$, $\psi_4$, $\ldots$, $\psi_n$ at energies $E_3$, $E_4$, $\ldots$, $E_n$.

In order to produce a two-parameter family of isospectral potentials, we go from $V_0$ to $V_1$ to $V_2$ by deleting two states from $V_0$ and then we re-add two states at energies $E_1$ and $E_0$ by supersymmetric transformations. The general solution of the Schrödinger
equation for the potential $V_2$ is $\phi_1(\lambda_1) = \frac{I_1 + \lambda_1}{\psi_i}$ at energy $E_1$ and $A_1\phi_0(\lambda_0)$ at energy $E_0$.

The supersymmetric operator relates solutions of $V_i$ and $V_{i+1}$,

$$A_i = \frac{d}{dx} - (\ln \psi_i)' .$$

(1.35)

The isospectral one-parameter family is obtained as

$$\hat{V}_1(x, \lambda_1) = V_i(x) - 2\frac{d^2}{dx^2} \ln[I_1 + \lambda_1].$$

(1.36)

The solutions of Schrödinger equation for potential $V_i(x)$ and $\hat{V}_1(x, \lambda_1)$ are related by supersymmetric operator,

$$\hat{A}_i^T(\lambda_1) = -\frac{d}{dx} - (\ln \phi_i(\lambda_1))'.$$

(1.37)

The solution $\phi_0(\lambda_0, \lambda_1)$ at $E_0$ for $\hat{V}_1(x, \lambda_1)$ is given by,

$$\phi_0(\lambda_0, \lambda_1) = \hat{A}_i^T(\lambda_1)A_i\phi_0(\lambda_0),$$

(1.38)

and the two parameter family of isospectral potential is obtained as,

$$\hat{V}_0(x, \lambda_0, \lambda_1) = V_0(x) - 2\frac{d^2}{dx^2} \ln (\phi_0(I_1 + \lambda_1)\phi_0(\lambda_0, \lambda_1)).$$

(1.39)

The procedure is generalized to a $n$-parameter family of isospectral potentials for an initial potential with $n$ bound states. The supersymmetric operators relating the potentials are,

$$A_i = \frac{d}{dx} - (\ln \psi_i)'$$

and

$$\hat{A}_i^T(\lambda_1, ..., \lambda_{n-1}) = -\frac{d}{dx} + (\ln \phi_i(\lambda_1, ..., \lambda_{n-1}))'.$$

(1.40)

The generalization for an $n$-parameter family are

$$\phi_i(\lambda_i) = \frac{I_i + \lambda_i}{\psi_i}, \quad i = 0, 1, ..., (n-1).$$

(1.41)

$$\phi_i(\lambda_i, \lambda_{i+1}, ..., \lambda_{n-1}) = \hat{A}_{i+1}^T(\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_{n-1})\hat{A}_{i+2}^T(\lambda_{i+2}, \lambda_{i+3}, ..., \lambda_{n-1}) \times \hat{A}_{n-1}^T(\lambda_{n-1})A_{n-1}A_{n-2}...A_{n+1}\phi_i(\lambda_i).$$

(1.42)

$$\hat{V}_0(x, \lambda_0, ..., \lambda_{n-1}) = V_0(x) - 2\frac{d^2}{dx^2} \ln (\phi_0(\chi_0, \chi_1, ..., \chi_{n-1})\phi_0(\lambda_0, ..., \lambda_{n-1})).$$

(1.43)

Using the above equations, we can construct $n$-parameter family of isospectral potentials.
1.4 Applications

This approach has been applied to many physical situations by various authors. As the eigenfunctions of the isospectral Hamiltonians are different, therefore the eigenfunction dependent quantities can be changed suitably without destroying the agreement already achieved with the energy eigenvalues. Some of the applications are listed below.

1.4.1 Differential Equations

Most of the differential equations appearing in physics can be factorized by means of ladder operators. The one-parameter isospectral special functions were constructed, using a combination of raising and lowering operators. The second order differential operators associated with the special functions of the mathematical physics are factorized and used to construct isospectral special functions. This formalism is applied to Hermite polynomials, Laguerre polynomials, Legendre polynomials, Chebyshev polynomials, Jacobi functions, Jacobi polynomials and Bessel functions extensively by Reyes et al. [35]. The connection of unbroken supersymmetric quantum mechanics in its strictly isospectral form with the nonlinear superposition principle is pointed out by Rosu [36]. The strictly non-relativistic isospectral scheme based on the general Riccati solution and Darboux transformation function corresponding to excited states is applied to harmonic oscillator, the square well potential and the hydrogen atom [37].

The $\delta(x)$-potential is a well known zero-range potential with applications in solid state physics and many other areas [38]. It has been used as a textbook example for many mathematical procedures in quantum mechanics. Rosu applied the supersymmetric approach to the attractive $\delta$-potential problem based on the general superpotential [39]. The strictly isospectral extension of the attractive $\delta(x)$-potential introduced may be relevant for many applications, as one allows a physical significance for the deformation parameter. The parameter may give the effect of static or moving distant boundaries and also as sample-size dependence [40, 41]. One may think of the isospectral method as allowing to introduce singularities in both wave functions and potentials which are required to explain the extra losses of ultracold neutrons at the walls [42]. The parameter dependence of the zero energy sector of the unbroken supersymmetric quantum mechanics, which in
1.4.2 Thermic Denaturation of DNA macromolecule

The isospectral harmonic oscillator potential is used to study the thermic denaturation of DNA macromolecules [33]. The stability of these molecules depends on the hydrogen bonds between the complimentary base pairs and the stacking interactions between adjacent bases along the strands. Harmonic coupling is assumed for the stacking interactions between neighboring bases along the strands with a common value for both the coupling constant and mass for each base [44, 45]. The isospectral potential does not change the eigenvalue spectrum, but the eigenfunction is sensitive to the deformation. The potential obtained consist of a free parameter, which can be fixed considering the experimental data and the melting temperature obtained is very close to experimental data.

1.4.3 Calculation of Resonances

Calculation of resonance energies in potentials presenting a surface barrier is a difficult numerical work, especially when the barrier is shallow and the corresponding width of the resonance is large. The isospectral potential with an appropriately chosen deformation parameter is very useful in the calculation of energy and other observable for a resonance. Thus, this formalism is a powerful tool for very broad and weakly resonant states, which are of great interest in the physics of unstable and weakly bound systems. The deformation parameter can be chosen as to optimize the stability of the resonant state. Extremely small values of deformation parameter would make the resonant state like δ-function, which is not desirable from computational point of view [46, 47].

1.4.4 Spectrum of charged particle in a class of non-uniform magnetic field

The spectrum of a charged particle in uniform magnetic field consists of equally spaced Landau levels which are infinitely degenerate. Using isospectral deformation, it has been shown that equispaced spectrum can also be obtained for a wide class of non-uniform
1.4.5 Generalization of soliton solutions

In Soliton physics, the stability of the soliton/kink is ensured by the occurrence of a zero energy ground state of the stability equation when small oscillations around the soliton/kink are considered [29]. The stability equation can be considered as an one dimensional Schrödinger equation with potential $V(x)$ and one can construct an isospectral partner for it. The partner stability equation will have the same energy spectrum as that of the original equation. Then, one can reconstruct the soliton solution and hence the potential $V(\phi)$ which admits the soliton solution from the partner stability equation. This generalizes the class of Hamiltonians which admits kink/soliton solutions and share the same stability data [30, 31].

1.5 Plan of work

Using supersymmetric quantum mechanics techniques, isospectral Hamiltonian approach has been profitably applied to study many applications. This approach can further be applied to different physical situations, which are of interest to various fields.

In Chapter 2, we apply the isospectral Hamiltonian approach to information entropy. Boltzmann-Shannon information entropy is a fundamental quantity, closely related to thermodynamical entropy, which measures the spread or extent of the single particle density in the context of Density Functional Formalism used in many particle system [48]. Information entropy plays a crucial role in a stronger formulation of the uncertainty relations. An interesting uncertainty relation was discovered by Bialynicki-Birula and Mycielski [49]. It was proved that for wave functions normalized to unity, $S_{\text{pos}} + S_{\text{mom}} \geq d(1 + \ln \pi)$ where $d$ is the dimension and $S_{\text{pos}}, S_{\text{mom}}$ are entropies in position space and momentum space respectively. Though the $S_{\text{pos}}$ and $S_{\text{mom}}$ are individually unbounded, their sum is bounded from below. The analytical determination of the position and momentum space entropies have been carried out only for a few quantum mechanical systems [50-53]. The entropies of hydrogen atom and simple harmonic oscillator were exactly calculated for the ground state in both position and momentum space.
The information entropies in various contexts, e.g., information theory, mathematics, mathematical physics, chemical physics and other areas of physics have been extensively analyzed in recent times [54-57]. We consider the hyperbolic Pöschl-Teller potential and calculate the position and momentum space information entropy exactly for ground state and first excited state. Using isospectral Hamiltonian approach, the deformed potential and their wave functions are constructed and used to calculate the information entropy for the isospectral potential.

In Chapter 3, this approach is used in the generalization of the soliton solutions. Soliton is a nonlinear wave which propagates without change in its properties like shape, velocity etc. and these are stable against mutual collisions and retain their identities [58]. As the term 'soliton' suggests, these solitary waves behave like particles which are non-dispersive, localized packets of energy moving uniformly. When they are located mutually far apart, each of them is approximately a travelling wave with constant shape and velocity. As two such solitary waves get closer, they gradually deform and finally merge into a single wave packet, soon splits into two solitary waves with the same shape and velocity as before collision. Soliton/kink solutions are obtained by different models like KdV equation, Sine-Gordon equation, $\phi^4$ theory, $\phi^6$ theory which describe physical phenomena in various fields [hydrodynamics, polymer systems, particle physics]. The generalization of a class of Hamiltonians which admits kink/soliton bearing solutions, was done for $\phi^4$ theory partially and Sine-Gordon equation, using the isospectral Hamiltonian approach. Now, the generalization of the soliton solutions is done for $\phi^4$ theory in detail and also for $\phi^6$ theory and logarithmic potential [59, 60].

In Chapter 4, we calculate the Franck-Condon factors using isospectral Morse potential. The Morse potential is a convenient model for the description of potential energy curves of a diatomic molecule. It is much used in spectroscopic applications as it is possible to solve Schrödinger equation for this system. The functional form of the potential is $V = D_{eq}(1 - e^{-a(r - r_{eq})})^2$, where $r_{eq}$ is the equilibrium bond length and $D_{eq}$ is the potential energy for bond formation and "a" a parameter controlling the width of the potential well. It is an empirical potential that describe the stretching of a chemical bond. It allows the calculation of a wide range of energy levels. The different types of
intensity distributions are explained by Franck-Condon principle whose basic idea is that an electronic transition in a molecule takes place so rapidly compared to the vibrational motion of the nuclei that the instantaneous internuclear distance and the velocity of the nuclei can be considered remaining unchanged during the electronic transition. The relative intensities of the various bands of the system which corresponds to various vibrational transitions, mainly depend on the square of overlap integral \( \int \psi_i^* \psi_f^* dr \), which is the integral over the product of the vibrational wave functions of the two combining electronic states. It is known as Franck-Condon factor \( q^{i',f'} \) that is

\[
q^{i',f'} = \left[ \int \psi_i^* \psi_f^* dr \right]^2. \tag{1.44}
\]

The amount of overlap and hence the value of Franck-Condon factor, depends on the relative position of the maxima and the nodes of the vibrational wave function \( \psi_i \) in each of the two combining electronic states and also upon the relative shapes and positions of the corresponding potential energy curves [61, 62].

First, we calculate the moments for the different levels of Poschl-Teller potential. Using isospectral Hamiltonian approach, we deformed the potential and hence the wave functions. Then we rearrange the moments for different levels using the deformation parameter. After these calculations, this approach is used to calculate Franck-Condon factors using Morse potential. We consider the Morse potential, deform the potential and the wave function using the same approach and then use these for calculations of the Franck-Condon factors which contain a free parameter that can be adjusted considering the experimental data.

In Chapter 5, we study the application of this approach to find the spectrum of charged particle in a class of non-uniform magnetic field [63, 64] and to Quarkonium Model [65]. The behavior of a charged particle in an external uniform magnetic field is well known and the spectrum consists of equally spaced levels with level being infinitely degenerate. The problem is not in general solvable for non-uniform magnetic fields, however the ground state is exactly known and the degeneracy of the ground state can be related to the total flux. Using the isospectral Hamiltonian approach, it has been shown that the spectrum is equispaced but non-degenerate for a wide class of non-uniform magnetic fields. Now
we have considered the problem for another non-uniform magnetic fields and studied its implications.

Non-relativistic potential models have been very successful in explaining the spectroscopy of the heavy quarkonia. Some potentials like Martin, Cornell, Logarithmic, Indiana, Buckmuller-Tye etc. are also able to reproduce the spectra adequately. But about the level spacings of the higher $n$ energy levels, there remain some discrepancies which suggest some modification for the assumption of concavity of these potentials. Leptonic decays are the physical quantities which are very sensitive to the form of the potential. Due to the dependence of decay widths upon $\partial V/\partial R$, these serve as a good check for the appropriate potential. Many authors have predicted the leptonic widths in various models of quarkonia and after analysis, they reached on the conclusion that in most potential models, it is difficult to resolve the discrepancy between the values obtained and the experimental values for the spectra and leptonic widths. It is found that the production rates are dependent on the quarkonium wave function. Thus, an accurate calculation of the wave function in quarkonium model is necessary to give correct leptonic widths as well as the production rates. This motivates us to take potentials which modify the bound state eigenfunctions and hence the eigenfunction dependent quantities without destroying the agreement already achieved between the predicted and observed spectra. For achieving this, we are using the isospectral Hamiltonian approach, which provides a simple procedure for generating the partner potential for any 1-dimensional potential with the same energy eigenvalues. However, the bound state eigenfunctions changes and hence this may be the precise technique required to get agreement of decay widths for quarkonium potentials.

In Chapter 6, we conclude with the brief discussion of the work done and discuss some open problems.
Bibliography


