Chapter 5

Other Applications

5.1 Spectrum for Charged Particle in a Class of Non-Uniform Magnetic Field

5.1.1 Introduction

The motion of charged particle in uniform magnetic field is well studied. The spectrum consists of equally spaced Landau levels which are infinitely degenerate [1, 2]. In general, it is difficult to solve the problem for non-uniform magnetic fields. However, the ground state is exactly calculable and possesses degeneracy related to the total flux [3]. The cases where the magnetic field is one-dimensional has been solved in Ref. [4]. Applying supersymmetric quantum mechanical techniques [5-11], the isospectral Hamiltonian approach has been used to obtain the partial spectrum of charged particle in a class of non-uniform magnetic fields and the spectrum obtained is again equispaced [12]. We use the same approach to obtain the energy eigenvalue spectrum of charged particle for other cases of non-uniform magnetic fields. The Pauli operator was also studied in the framework of supersymmetric quantum mechanics in many papers [13-20].

We cast the problem of charged particle in the language of supersymmetric quantum mechanics and use isospectral Hamiltonian approach to obtain the spectrum of charged particle in a class of non-uniform magnetic fields in one and two dimensions.
5.1.2 Spectrum for Charged Particle in Non-Uniform Magnetic Fields

The spectrum of charged particle is obtained for one and two dimensions separately. In the first case, we consider that the magnetic field has only a z component and depends only upon one coordinate, say y. With the asymmetric choice of the gauge, \( A_y = A_z = 0 \), one obtains [4] (we choose \( h = e = c = 1 \))

\[
\left[ P_y^2 + P_z^2 + (P_x + e A_x(y))^2 + m^2 - H_z(y) \right] \psi = E \psi \quad (5.1)
\]

The variables \( P_x \) and \( P_z \) are constants of motion and can be considered as constants. The wave function is only a function of \( y \). We choose \( A_x(y) \) such that the above equation becomes equivalent to Schrödinger equation with a solvable potential. For the choice \( A_x(y) = -H_0 y \), the equation becomes Schrödinger equation for an harmonic oscillator [21]. Let us choose the vector potential \( A_x(y) = -H_0 \tanh y \) and the corresponding magnetic field \( H_z(y) = H_0 \sech^2 y \). The Eq. 5.1 reduces to

\[
\left[ P_y^2 + \xi^2 - 2P_x \xi \tanh y - (\xi^2 + \xi) \sech^2 y \right] \psi = (E - m^2 - P_z^2 - P_x^2) \psi, \quad (5.2)
\]

here, \( \xi = eH_0 = H_0 \). Since \( P_x \) and \( P_z \) are constants, therefore, we can choose \( P_x = P_y = 0 \).

The Eq. 5.2 is reduced in the form of one-dimensional Schrödinger equation with a Rosen-Morse potential [22]. Introducing the notations

\[
\gamma = \xi^2 + \xi, \quad \epsilon = \xi^2 + m^2 + P_z^2 - E.
\]

\( \psi \) satisfies the differential equation

\[
\left[ \frac{d^2}{dy^2} - \epsilon + \gamma \sech^2 y \right] \psi = 0. \quad (5.3)
\]

The differential equation can be converted into the hypergeometric equation by change of variable \( \eta = \frac{1}{2}(1 + \tan y) \) and the transformation,

\[
\psi = \sech^\tau y \, F(y),
\]

where \( \tau = \sqrt{\epsilon} \). The differential equation satisfied by \( F(\eta) \) is

\[
\eta(1 - \eta) F''(\eta) + [(\tau + 1) - 2\eta(\tau + 1)] F'(\eta) + [\gamma - \tau(\tau + 1)] F(\eta) = 0, \quad (5.4)
\]

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The solution of the above equation corresponding to $y = -\infty$ is given by the hypergeometric function

$$F^2 \left[ \tau + \frac{1}{2} - (\gamma + \frac{1}{4})^2, \tau + \frac{1}{2} + (\gamma + \frac{1}{4})^2, \tau + 1; \eta \right].$$

This hypergeometric function diverges like $e^{2\tau}$ as $\eta$ approaches 1 which corresponds to $y = +\infty$, unless $\tau + \frac{1}{2} + (\gamma + \frac{1}{4})^2$ or $\tau + \frac{1}{2} - (\gamma + \frac{1}{4})^2$ equals a negative integer [22]. The first possibility is ruled out as $\tau$ and $(\gamma + \frac{1}{4})^2$ are positive quantities. Therefore, for $\psi$ to be finite, we must have

$$\tau = \left(\gamma + \frac{1}{4}\right) - \frac{1}{2} - n.$$

For $n = 0$, we have

$$\tau = \left(\gamma + \frac{1}{4}\right) - \frac{1}{2}$$

and the normalized ground state can be calculated as

$$\psi_0 = A_0(y) = \frac{1}{\sqrt{\beta(\frac{1}{2}, \xi)}} \text{sech}^{\xi}y.$$  \hspace{1cm} (5.5)

Using isospectral Hamiltonian formalism, one can calculate $I(y)$, $A_0$ and $H_z$ for different values of $\lambda$. All the members of the family $H_z(y, \lambda)$ give same spectrum as the undeformed magnetic field. We obtain

$$\dot{\psi}_0 = \frac{2\sqrt{\beta(\frac{1}{2}, \xi)}\sqrt{\lambda(\lambda + 1)}}{\beta(\frac{1}{2}, \xi)(2\lambda + 1) + \frac{2}{(1-2\xi)\sqrt{1-\cosh^2y}}} \text{sech}^{\xi}y,$$  \hspace{1cm} (5.6)

$$H_z(y) = \xi \text{sech}^{2}y + \frac{8(2\xi - 1) \text{sech}^{2}y \sqrt{1 - \cosh^{2}y}}{(2\xi - 1)(2\lambda + 1)} \frac{2\xi g(y) + f(y)}{\beta(\frac{1}{2}, \xi) \sqrt{1 - \cosh^{2}y + g(y)}},$$  \hspace{1cm} (5.7)

where

$$g(y) = \text{Hypergeometric}\ 2F1 \left( \frac{1}{2} - \xi, \frac{1}{2}, \frac{3}{2} - \xi; \cosh^{2}y \right) \text{sech}^{2}y \sinh^{2}y$$

and

$$f(y) = (2\xi - 1) \sqrt{1 - \cosh^{2}y} \left( \text{sech}^{2}y + \xi(2\lambda + 1) \beta \left( \frac{1}{2}, \xi \right) \tanh y \right).$$

The non-uniform magnetic field is plotted for different values of $\xi$ and $\lambda$ in figures 5.1-5.4.

The flux for deformed magnetic field is obtained as

$$\Phi = \int_{-\infty}^{\infty} H_z = \Phi.$$  \hspace{1cm} (5.8)
Figure 5.1: Magnetic field $\dot{H}_z$ for $\xi = 3$ and for $\lambda = 0.1$ (small dash), $\lambda = 0.5$ (dotted line), $\lambda = 2.0$ (large dash) and solid line represents undeformed magnetic field.

It is found that even though the undeformed and deformed magnetic fields $H_z$ and $\dot{H}_z$ are different, but the corresponding flux is same.

Another choice of $A_x(y)$ leading to exactly solvable equation is $A_x(y) = \xi(1 - e^y)$ and we get $H_z(y) = \xi e^y$. The Pauli equation reduces to a one-dimensional Schrödinger equation with Morse potential [23]. Choosing the constants $P_x = P_y = 0$ and introducing the notations

$$g = E - m^2 - P_x^2 - \xi^2,$$
$$f = 2\xi^2 + \xi,$$

the differential equation for $\psi$ is obtained as

$$\left[ \frac{d^2}{dy^2} + g + f e^y - \xi^2 e^{2y} \right] \psi(y) = 0. \quad (5.9)$$

The equation is transformed to Laguerre’s equation by the change of variable $x = 2\xi e^y$ and the transformation $\psi(y) = x^{\sqrt{g}} e^{-x/2} G(x)$. The differential equation satisfied by $G$ is

$$x G''(x) + \left( 2\sqrt{-g} + 1 - x \right) G'(x) - \left( \frac{1}{2} + \frac{f}{2\xi} + \sqrt{-g} \right) G(x) = 0. \quad (5.10)$$
Figure 5.2: Magnetic field $\tilde{H}_z$ for $\xi = 3$ and for $\lambda = -1.5$ (small dash), $\lambda = -2.0$ (dotted line), $\lambda = -5.0$ (large dash) and solid line represents undeformed magnetic field.

Figure 5.3: Magnetic field $\tilde{H}_z$ for $\xi = 5$ and for $\lambda = 0.1$ (small dash), $\lambda = 0.5$ (dotted line), $\lambda = 2.0$ (large dash) and solid line represents undeformed magnetic field.
Figure 5.4: Magnetic field $H_z$ for $\ell = 5$ and for $\lambda = -1.1$ (small dash), $\lambda = -1.5$ (dotted line), $\lambda = -3.0$ (large dash) and solid line represents undeformed magnetic field.

The function $G$ is the associated Laguerre polynomial $L_n^{\sqrt{\xi^2}}(x)$. So, the wave function in terms of variable $y$ is written as [4]

$$\psi_n(y) = (2\xi e^y)^{\sqrt{\xi^2}} e^{-\xi e^y} L_n^{\sqrt{\xi^2}}(2\xi e^y)$$

(5.11)

and the energy eigenvalues are calculated as $E = m^2 + P_3^2 + 2 n \xi - n^2$. The ground state wave function reads

$$\psi_0(y) = (2\xi e^y) e^{-\xi e^y}.$$  

(5.12)

Now, we can calculate the deformed wave function and the family of magnetic fields which have the same spectrum. The deformed ground state wave function (for $\ell = 2$) is

$$\hat{\psi}_0 = \sqrt{\frac{\lambda(\lambda+1)}{\lambda+1}} \frac{16e^{-2e^y + 2y}}{\left(\frac{3}{128} - \frac{1}{128} e^{-4e^y} (3 + 12 e^y + 24 e^{2y} + 32 e^{3y})\right) + \lambda}$$

(5.13)

and the deformed magnetic field reads

$$\hat{H}_z(y) = 2 e^y + \frac{1024 e^{4y} \left[3 + 9 e^y + 12 e^{2y} + 8 e^{3y} - 3(1 + \lambda) e^{4y}(1 + e^y)\right]}{\left[3 + 12 e^y + 24 e^{2y} + 32 e^{3y} - 3 e^{4y}(1 + \lambda)\right]^2}.$$  

(5.14)

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Figure 5.5: Deformed Magnetic field $\mathbf{H}_z$ for $\lambda = 0.001$ (small dash), $\lambda = 0.1$ (dotted line), $\lambda = 1.0$ (large dash) and solid line represents undeformed magnetic field.

Figure 5.6: Deformed Magnetic field $\mathbf{H}_z$ for $\lambda = -1.01$ (small dash), $\lambda = -1.1$ (dotted line), $\lambda = -2.0$ (large dash) and solid line represents undeformed magnetic field.
The deformed magnetic field is plotted for different positive and negative values of deformation parameter in figures 5.5 and 5.6. The flux for the deformed magnetic field is same as that for undeformed magnetic field.

Now we consider the problem of charged particle in two dimensions. The Pauli-Hamiltonian for the motion of charged particle in magnetic field for this case is given by

\[ 2H = (P_x + A_x)^2 + (P_y + A_y)^2 + (\nabla \times A)_z \sigma_z. \]  

(5.15)

We choose

\[ A_x = -B_y f(\rho), \quad A_y = -B_x f(\rho), \]  

(5.16)

where \( \rho = \sqrt{x^2 + y^2} \) and \( B \) is a constant then the magnetic field is given by

\[ B_z = 2B f(\rho) + B \rho f'(\rho). \]  

(5.17)

Eq. 5.15 takes the form

\[ 2H = \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) + B^2 \rho^2 f^2 + 2B f L_z + (2B f + B \rho f'(\rho)) \sigma_z. \]  

(5.18)

where \( L_z \) is the z-component of the orbital angular momentum operator. To solve the corresponding Schrödinger equation in cylindrical coordinates \( \rho \) and \( \phi \), the wave function is factorized as \( \psi(\rho, \phi) = R(\rho) e^{i m \phi} \), where \( m = 0, \pm 1, \pm 2, \ldots \) and the Schrödinger equation becomes

\[ R''(\rho) + \frac{1}{\rho} R'(\rho) - \left[ B^2 \rho^2 f^2 + \frac{m^2}{\rho^2} + 2Bm f + (2B f + B \rho f'(\rho)) \sigma_z \right] R(\rho) = -2ER(\rho), \]  

(5.19)

and upon substituting \( R(\rho) = \rho^{-1/2} A(\rho) \), we get,

\[ \left\{ -\frac{d^2}{d\rho^2} + \left[ B^2 \rho^2 f^2 - 2B f + 2Bm f - B \rho f'(\rho) + \frac{m^2}{\rho^2} - \frac{1}{\rho} \right] \right\} A(\rho) = 2EA(\rho). \]  

(5.20)

For \( m \leq 0 \), the left hand side can be written as \( a^2 \) where,

\[ a = \frac{d}{d\rho} + B f - \left| m \right| + \frac{1}{\rho}. \]  

(5.21)

The ground state wave function is obtained by solving the equation

\[ \left[ \frac{d}{d\rho} + B f - \left| m \right| + \frac{1}{\rho} \right] A_0(\rho) = 0 \]  

(5.22)
and we get

\[ A_0(\rho) = N \rho^{\frac{m+1}{2}} e^{-\int B_\rho dp}. \]  

(5.23)

One can choose the different forms of \( f \) and obtain the ground state wave function. Using isospectral Hamiltonian formalism, the ground state wave function for the one parameter family of isospectral potentials is given by

\[ \hat{A}_0(\rho, \lambda) = \frac{\sqrt{\lambda(1 + \lambda)A_0(\rho)}}{I(\rho) + \lambda}. \]  

(5.24)

The corresponding \( \dot{f} \) can be calculated as

\[ \dot{f} = f + \frac{1}{B\rho} \frac{d}{d\rho} \ln(I + \lambda). \]  

(5.25)

The isospectral magnetic field is given by

\[ \hat{B}_z = B_z + \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} \ln(I + \lambda) \right). \]  

(5.26)

The flux for \( \hat{B}_z \) is calculated as

\[ \Phi = \int B_z d^2 \rho + 2\pi \int_0^\infty \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} \ln(I + \lambda) \right) d\rho = \Phi. \]  

(5.27)

Now, we can choose the different forms of \( f \), i.e. different magnetic fields and calculate the isospectral family of non-uniform magnetic fields which give the same spectra as that of the undeformed magnetic field. For \( f = 1 \), one obtain uniform magnetic field consisting of equally spaced Landau levels. Using isospectral formalism, the family of non-uniform magnetic fields consisting of equispaced but non-degenerate spectrum is obtained [12]. For the choice, \( f = \tanh \rho/\rho \), the magnetic field is

\[ B_z = B \left[ \frac{\tanh \rho}{\rho} + \text{sech}^2 \rho \right]. \]  

(5.28)

and the corresponding ground state wave function reads

\[ A_0(\rho) = N \rho^{\frac{m+1}{2}} \left( \text{sech}\rho \right)^{B}. \]  

(5.29)

Though the spectrum is not exactly known, but once the ground state is obtained, this approach can be applied to obtain the deformed magnetic field which will give same
Figure 5.7: Magnetic field $B_z$ for $m = 0$ and for $\lambda = 0.5$ (small dash), $\lambda = 1.5$ (dotted line), $\lambda = 3.0$ (large dash) and solid line represents undeformed magnetic field.

spectrum. Now we can calculate $I(\rho), \hat{A}_0, \hat{f}$ and $\hat{B}_z$ for each value of $m$ so that one has a family of magnetic fields giving same spectrum. $\hat{B}_z$ is calculated as,

$$\hat{B}_z = B_x + \frac{N^2 \rho^{2m+1}(\text{sech}^2 \rho)^B}{I(\rho) + \lambda} \left[ 2 + 2 m - 2 B \rho \tanh \rho - \frac{N^2 \rho^{2m+2}(\text{sech}^2 \rho)^B}{I(\rho) + \lambda} \right],$$

(5.30)

where $N$ is the normalization and

$$I(\rho) = N^2 \int \rho^{2|m|+1}(\text{sech}^2 \rho)^B \, d\rho.$$  

Thus we are getting the family of magnetic fields which have same spectrum. The magnetic fields are plotted for $m = 0$ and $m = -1$ and for different values of the deformation parameter $\lambda$ in figures 5.7-5.10. As $\lambda \to \pm \infty$, $\hat{B}_z \to B_z$ i.e. for these values of $\lambda$, we get back the undeformed magnetic field.

### 5.1.3 Conclusion

We have obtained the spectrum of charged particle in a wide class of non-uniform magnetic fields in one and two dimensions. Though the undeformed and deformed magnetic fields are different but the corresponding flux is found to be same. One can also obtain the
Figure 5.8: Magnetic field $B_z$ for $m = 0$ and for $\lambda = -1.1$ (small dash), $\lambda = -1.5$ (dotted line), $\lambda = -5.0$ (large dash) and solid line represents undeformed magnetic field.

Figure 5.9: Magnetic field $B_z$ for $m = -1$ and for $\lambda = 0.1$ (small dash), $\lambda = 0.5$ (dotted line), $\lambda = 3.0$ (large dash) and solid line represents undeformed magnetic field.
Figure 5.10: Magnetic field $\hat{B}_z$ for $m = -1$ and for $\lambda = -1.01$ (small dash), $\lambda = -1.1$ (dotted line), $\lambda = -2.0$ (large dash) and solid line represents undeformed magnetic field.

A multi-parameter family of magnetic fields $\hat{B}_z(\lambda_1, \lambda_2, ...)$, all giving the same spectrum following the work of Keung et al. [24].
5.2 Quarkonium Model

5.2.1 Introduction

The particles containing heavy quarks can be divided into two large groups: particles containing a heavy quark and a heavy antiquark and the particles containing only one heavy quark or antiquark. The quarkonia i.e. mesons consisting of a heavy quark and a heavy antiquark have been described successfully in terms of simple potential models. But the particles consisting of one heavy quark and some light stuff are more difficult to describe. In the valence approximation, the light stuff reduces to an antiquark in the case of two mesons and to two quarks or a diquark in the case of baryons. The non-relativistic potential models have some success in describing the spectra of such particles [25]. However, it is not considered very respectable because the resulting velocities of the light quarks come out relativistic. The first quarkonia, the bound $c\bar{c}$ systems were discovered about thirty years ago [26, 27] and it was realized that such systems can be described using the Schrödinger equation with simple, more or less QCD motivated potentials. The $b\bar{b}$ quarkonia discovered three years later [28, 29] consist of heavier partons, which consequently have smaller velocities.

Non-relativistic potential models have been very successful in explaining the spectroscopy of the heavy quarkonia. Some potentials like Martin, Cornell, Logarithmic, Indiana, Buckmuller-Tye etc. are also able to reproduce the spectra adequately [30-33]. But about the level spacings of the higher $n$ energy levels, there remain some discrepancies which suggest some modification for the assumption of concavity of these potentials [34]. The agreement with experiment on the decays of the quarkonia leaves a lot to be desired [35]. Leptonic decays are the physical quantities which are very sensitive to the form of the potential. Due to the dependence of decay widths upon $\partial V/\partial R$, these serve as a good check for the appropriate potential [36]. Many authors have predicted the leptonic widths in various models of quarkonia and after analysis, they reached the conclusion that in most potential models, it is difficult to resolve the discrepancy between the values obtained and the experimental values for the spectra and leptonic widths. However, parton fragmentation into $J/\psi$ and $\psi'$ at $p\bar{p}$ collision has been examined in experiments at the
CDF collaboration at Fermi lab and the discrepancy between the experimental production rates and the production rates calculated from various models have been noted. It has been established that the production rates are dependent on the quarkonium wave function [37-42]. Thus, an accurate calculation of the wave function in quarkonium model is necessary to give correct leptonic widths as well as production rates. This motivates us to take potentials which modify the bound state eigenfunctions and hence the eigenfunction dependent quantities without destroying the agreement already achieved between the predicted and observed spectra. For achieving this, we propose to use the isospectral Hamiltonian approach in supersymmetric quantum mechanics which provides a simple procedure for generating the partner potential for any one-dimensional potential with the same energy eigenvalues. However, the bound state eigenfunctions changes and hence this may be the precise technique required to get agreement of decay widths for quarkonium potentials. We shall deal with this problem at formulation level only. The expertise in the particle physics phenomenology and numerical computation are yet to be acquired.

The functional form of the potentials [31, 37, 43, 44] which give the reasonable accounts of the $c\bar{c}$ and $b\bar{b}$ spectra is as given:

The QCD motivated potential given by Buchmuller and Tye with the following parameters,

\[ m_c = 1.48 \text{ GeV}/c^2, \quad m_b = 4.88 \text{ GeV}/c^2, \]

and a power-law potential

\[ V(r) = -8.064 \text{ GeV} + (6.898 \text{ GeV})(r \times 1 \text{ GeV})^{0.1}, \]

with

\[ m_c = 1.8 \text{ GeV}/c^2, \quad m_b = 5.174 \text{ GeV}/c^2. \]

The logarithmic potential is

\[ V(r) = 0.6635 \text{ GeV} + (0.733 \text{ GeV}) \ln(r \times 1 \text{ GeV}), \]

with

\[ m_c = 1.5 \text{ GeV}/c^2, \quad m_b = 4.906 \text{ GeV}/c^2 \]
and the Cornell potential (Coulomb plus linear potential) is,

$$V(r) = -\frac{k}{r} + \frac{r}{a^2}$$  \hspace{1cm} (5.33)

with

$$m_c = 1.84 \text{ GeV}/c^2, \quad m_b = 5.18 \text{ GeV}/c^2,$$

$$k = 0.52 \quad a = 2.34 \text{ GeV}^{-1}.$$  

At short distances, this potential is columbiaic and at large distances it is linear. These potentials are generally used to explain the quarkonium spectra. The values of the radial wave functions at origin for these potentials are tabulated in Ref. [37].

### 5.2.2 Isospectral Partner for Quarkonium Potential

As an illustration of the applicability of isospectral Hamiltonian approach to quarkonium physics, we consider the commonly used potential and solve the Schrödinger equation for the isospectral family of the potentials. One generally used potential is Cornell potential which gives quarkonium levels quite accurately, but does not give the required leptonic widths. In case of isospectral potential, we get a free parameter which can be adjusted such as to reduce the discrepancy between calculated and experimental values. The wave functions at the origin for various potentials have been calculated by Eichten and Quigg and is given in table 5.1 and 5.2 for charmonium and bottomium respectively. We solve for the eigenfunctions of the potential and adjust the free parameter to give the correct wave function to fit data.

The Schrödinger wave equation in 3-dimensions is written as

$$\frac{-\hbar^2}{2\mu^2} \nabla \psi(r) + [V(r) - E] \psi(r) = 0,$$  \hspace{1cm} (5.34)

where $2\mu = m_q$. Writing

$$\psi(r) = R(r)Y(\theta, \phi).$$  \hspace{1cm} (5.35)

The equivalent radial equation can be written as

$$-\frac{\hbar^2}{m_q} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R(r) - \left[ E - V - \frac{l(l+1)}{m r^2} \right] R(r) = 0.$$  \hspace{1cm} (5.36)
Table 5.1: Charmonium radial wave functions in various potential models. Column 1 is for Buckmuller and Tye; 2 for power law potential; 3 for logarithmic potential and 4 is for Cornell potential. The last column gives the known experimental data.

<table>
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Table 5.2: Bottomium radial wave functions in various potential models. Column 1 is for Buckmuller and Tye; 2 for power law potential; 3 for logarithmic potential and 4 is for Cornell potential. The last column gives the known experimental data.

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<td>1.324</td>
<td>3.663</td>
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To make the equation amenable to the application of our technique, we introduce
\[ U(r) = r R(r) \]  
(5.37)
and \( \rho = r m \). Then, the equation for \( m U(r) \) is the one-dimensional Schrödinger wave equation
\[ \frac{d^2}{dr^2} [m U(r)] = \left[ -\frac{E}{m} + \frac{V(l+1)}{\rho^2} \right] m U(r), \]
(5.38)
which can be further simplified by \( m U(\rho) = \phi(\rho) \) to the familiar
\[ \phi'' = \left[ -\frac{E}{m} + \frac{V(l+1)}{\rho^2} \right] \phi(\rho). \]
(5.39)
The radial wave function is given by
\[ R(r) = \frac{\phi(\rho)}{\rho} = \frac{m U(r)}{m r} = \frac{U(r)}{r}. \]
(5.40)
In terms of the potential,
\[ \frac{dU(r)}{dr} = R(r) + r R''(r) \]
(5.41)
and
\[ R(0) = \frac{dU(r)}{dr}|_{r=0}. \]
(5.42)
We have
\[ -\frac{U''}{2} + V(r) U + \frac{l(l+1)}{r^2} U = E U. \]
(5.43)
\[ \left| \frac{dU(r)}{dr} \right|^2 = 2 \int_0^\infty U^2 \frac{dV}{dr} \, dr. \]
(5.44)
By calculating \( \phi_0(\rho) \), we calculate the isospectral potential of \( V_1(\rho) \),
\[ V_1(\rho) = \frac{V(\rho)}{m} - \frac{l(l+1)}{\rho^2}, \]
(5.45)
which are given by
\[ V_2(\rho) = V_1(\rho) \pm 2 \frac{d^2 \ln (l(\rho) + \lambda)}{d\rho^2} \]
(5.46)
\( V_2 \) contains a free parameter which can be chosen in accordance with the experimental data. The corresponding deformed wave functions at origin \( \hat{R}(0) \) can be calculated.

The leptonic decay rate of the quarkonium system is given by the Van-Royen Weisskopf formula, which in terms of the radial wave function is given as
\[ \Gamma(Q\bar{Q} \rightarrow e^+ e^-) = \frac{4 \alpha_s^2 e_Q^2 |R(0)|^2}{3 M_{Q\bar{Q}}} \left( 1 - \frac{16 \alpha_s}{3\pi} \right), \]
(5.47)
where $N_c = 3$; $e_Q =$ heavy quark charge. There is a note of caution about the QCD correction, which reduces the magnitude of the leptonic width, but the amount of reduction is uncertain and this fact is used by many authors to account for the discrepancy obtained between the experimental and theoretical values of leptonic widths in almost all the models. We will assume that this formula is valid and from values of the radial wave function at origin, we get the values of the leptonic widths. For the S-levels of charmonium the experimental values of the leptonic widths are:

$$
\Gamma[\psi(1 S)] = 5.36 \pm .29 \text{KeV}, \Gamma[\psi(2 S)] = 2.39 \pm .21 \text{KeV}, \Gamma[\psi(3 S)] = 0.75 \pm .15 \text{KeV}.
$$ (5.48)

Taking the running coupling constant $\alpha_s(m_c) = 0.29$, gives us the following values of $|R_{ns}|^2$:

$$
|R_{n0}(\psi(1 S))|^2 = 1.068 \text{ GeV}^3, |R_{n0}(\psi(2 S))|^2 = 0.604 \text{ GeV}^3, |R_{n0}(\psi(3 S))|^2 = 0.254 \text{ GeV}^3.
$$ (5.49)

For the S wave levels of the bottomium,

$$
\Gamma[\psi(1 S)] = 1.34 \pm .04 \text{KeV}, \Gamma[\psi(2 S)] = 0.56 \pm .14 \text{KeV}, \Gamma[\psi(3 S)] = 0.44 \pm .07 \text{KeV}.
$$ (5.50)

Taking the running coupling constant $\alpha_s(m_b) = 0.19$, we get

$$
|R_{n0}(\psi(1 S))|^2 = 7.49 \text{ GeV}^3, |R_{n0}(\psi(2 S))|^2 = 3.516 \text{ GeV}^3, |R_{n0}(\psi(3 S))|^2 = 2.948 \text{ GeV}^3.
$$ (5.51)

From table 5.1 and 5.2, we see that the Cornell potential overestimate the radial wave function at the origin and hence the leptonic widths. Since for the isospectral potential, we introduce a free parameter $\lambda$ which enables us to adjust the radial wave function at the origin without spoiling the agreement achieved with the spectra. We propose that a suitable choice of $\lambda$ will give a better fit as far as the widths are concerned.
Bibliography


