Chapter -1

INTRODUCTION

1.1 Preliminaries:

Practically in most cases we fail to find the exact solution of a mathematical problem. This happens mainly not because we do not know the way in which the exact solution is found but usually owing to the fact that the desired solution is not expressible in elementary or other functions usually known to us. Therefore, numerical methods assume ever greater importance especially in connection with, the increasing role of mathematical methods in various fields of science and technology and owing to the appearance of high efficiency electronic computer.

By numerical analysis we mean that, the theory of constructive methods in mathematical analysis. By constructive methods we mean, a procedure that permits us to obtain the solution of a mathematical problem with an arbitrary precision in finite number of steps that can be prepared rationally. A solution obtained by a numerical method is usually an approximation, i.e. it has some error.

The following are the sources of the error of an approximate solution:

1) The lack of correspondence between a mathematical problem and real phenomenon under consideration.

2) The error of the initial data (input-parameters).

3) The error of the method of solution.

4) Round of errors in arithmetic and other types of operations on the numbers involved.

In most cases, numerical methods are approximated by themselves, i.e. even if the initial data are void of errors and all the arithmetic operations are ideally performed, they yield the solution of the original problems with some error which is called the error of method applied. This is because, a numerical method is usually applied to solve some other simpler problem, which approximate the originally given
problem. In a number of cases the chosen numerical method is constructed on the basis of an infinite process, which leads to the desired solution.

1.2 Quadrature or Numerical Integration:

Numerical integration means the numerical evaluation of the integral

\[ I(f) = \int_a^b f(x) \, dx, \quad (1.2.1) \]

where \( y = f(x) \) is a given function in the interval \([a, b]\).

Quadrature formula are widely used for approximately evaluating definite integrals and also Gaussian quadrature formula, which turn out to be exact for algebraic polynomials of higher degree.

Taylor’s Theorem:

Let \( f(x) \) has \((n+1)\) continuous derivative on \([a, b]\) for some \( n \geq 0 \) and let \( x, x_0 \in [a, b] \), then

\[ f(x) = P_n(x) + R_n(x); \quad (1.2.2) \]

\[ P_n(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0); \quad (1.2.3) \]

\[ R_n(x) = \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) \, dt \]

\[ = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad \text{for} \quad \xi \in (x_0, x). \quad (1.2.4) \]

Maclaurin’s Theorem:

In particular if \( x_0 = 0 \), then
\[ P_n(x) = f(0) + f'(0) + \frac{(x)^2}{2!} f''(0) + \ldots + \frac{(x)^n}{n!} f^n(0), \quad (1.2.5) \]

and
\[ R_n(x) = \frac{(x)^{n+1}}{(n+1)!} f^{n+1}(\xi) \text{ for } \xi \in (0, x). \quad (1.2.6) \]

**Lagrange Interpolating Polynomial:**

Let \( y = f(x) \) be a real valued function defined on an interval \([a, b]\) and let \( x_0, x_1, x_2, \ldots, x_n \) be \((n + 1)\) distinct points in that interval at which the respective values \( f(x_0), f(x_1), f(x_2), \ldots, f(x_n) \) are tabulated.

Hence the Lagrange interpolating polynomial is
\[
p_n(x) = \sum_{i=1}^{n} l_i(x)f(x_i), \quad (1.2.7)
\]

where
\[
l_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})}, \quad (1.2.8)
\]

The truncation error in Lagrange interpolating polynomial is given by
\[
E_n(f) = f(x) - p(x) = w(x) \frac{f^{n+1}(\xi)}{(n+1)!}, \quad (1.2.9)
\]

where
\[
w(x) = (x-x_0)(x-x_1)\ldots(x-x_n). \quad (1.2.10)
\]

**1.3 Basic Quadrature Formula:**

The general problem of numerical integration or quadrature formula is to find an approximate value of the integral
\[ I(f) = \int_{a}^{b} w(x)f(x)dx, \quad (1.3.1) \]

where \( w(x) > 0 \) on \([a, b]\) called weight function.

We assume that \( w(x) \) and \( w(x)f(x) \) are integrable in Riemann sense on \([a, b]\). The quadrature formula (1.3.1) can be written in the form

\[ I(f) = \int_{a}^{b} w(x)f(x)dx = \sum_{i=0}^{n} w(x_i)f(x), \quad (1.3.2) \]

where \( x_i, w_i, i = 0(1) n \) are called the nodes distributed within the limits of integration and the weights of the quadrature formula (1.3.2) respectively.

The error of approximation is given by

\[ E_n(f) = I(f) - \sum_{i=0}^{n} w_i f_i. \quad (1.3.3) \]

\section*{1.4 Newton-Cotes Quadrature Formula:}

In general a quadrature formula consists of weights and nodes. There are total \((2n + 2)\) weights and nodes are involved. In Newton-Cotes quadrature formula the \((n + 1)\) nodes are given in advance. These may or may not be equispaced. Using \((n + 1)\) polynomials we determine \((n + 1)\) weights. For this purpose we assume the formula to be exact for all polynomials of degree \( \leq n \).

The general approach for the derivation of Newton-Cotes quadrature formula is based on Lagrangian interpolation technique. By this approach we first get Lagrangian interpolation polynomial of degree matching the tabular points assuming that the integrand is Riemann integrable as:

\[ f(x) \approx P_n(x); \]
so \( f(x) = P_n(x_k) \)

\[
= \sum_{k=0}^{n} I_k(x) f_k + R_n(x),
\]

where

\[
I_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}
\]

\[
= \prod_{i=0}^{n} (x-x_i) \prod_{i=0}^{n} (x_i-x_j).
\]

Hence from equation (1.3.2)

\[
I(P) = \sum w_k f_k + R_n(x),
\]

where

\[
w_k = \int_{a}^{b} w(x) I_k(x) \, dx;
\]

\[
R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \int_{a}^{b} w(x) \prod (x) \, dx,
\]

and

\[
\prod(x) = (x-x_0)(x-x_1)\ldots(x-x_n).
\]

Taking \( w(x) = 1 \) and quadrature nodes \( x_i \) are equispaced with

\[
x_0 = a; \quad x_n = b;
\]

\[
h = \frac{(b-a)}{n},
\]
we have 
\[ x_i = x_0 + ih, \quad x = x_0 + sh, \]
and 
\[ dx = hds. \]

Also 
\[ x - x_0 = sh; \]
\[ x - x_i = (s - 1)h; \]
\[ x - x_n = (s - n)h; \]
\[ x_k - x_0 = kh; \]
\[ x_k - x_i = (k - 1)h; \]
\[ x_k - x_{k+1} = \{k - (k + 1)\} h; \]
\[ x_k - x_{n-k} = \{k - (n - k)\}h; \]
\[ x_k - x_n = (k - n)h. \]

Now
\[
w_k = \int_0^s \frac{sh(s - 1) \ldots (s - n)h}{kh(k - 1)h \ldots (k - n)h} \ h \ ds \\
= \frac{(-1)^{n-k} h}{k!(n-k)!} \int_0^s \frac{s(s - 1) \ldots (s - n)}{(s - k)} ds, \tag{1.4.3} \]

and
\[
R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \int_0^s \{sh(s - 1)h \ldots (s - n)h\} h \ ds. \tag{1.4.4} \]

the formula (1.4.1) with nodes and weights given by equation (1.4.2) and equation (1.4.3) is called Newton’s quadrature formula.

**1.4.1 Trapezoidal Rule:**

The basic Trapezoidal rule for numerical integration is first order integration scheme with two points on the interval, i.e. \( n = 1 \).
Hence

\[ I(f) = \int_{a}^{b} f(x) \, dx = \sum_{i=0}^{1} w_i f(x_i); \]

\[ I(f) = w_0 f(x_0) + w_1 f(x_1), \]

where

\[ w_0 = \frac{(-1)^1 h}{1!0!} \int_{0}^{1} s(s-1) \, ds = \frac{h}{2}; \]

\[ w_1 = \frac{(-1)^0 h}{0!1!} \int_{0}^{1} s \, ds = \frac{h}{2}. \]

Now

\[ I(f) = w_0 f(x_0) + w_1 f(x_1) \]

\[ = \frac{h}{2} f(x_0) + \frac{h}{2} f(x_1) \]

\[ = \frac{h}{2} \left[ f(x_0) + f(x_1) \right] \quad (1.4.5) \]

\[ = \frac{b-a}{2} [f(a) + f(b)], \]

where \( h = x_1 - x_0 = b - a. \)

**Error in Trapezoidal Rule:**

The error in Trapezoidal rule is

\[ E_T(f) = \frac{-h^3}{12} f^{(v)}(\xi); \quad x_0 \leq \xi \leq x_n. \quad (1.4.6) \]

### 1.4.2 Simpson’s \( \frac{1}{3} \)rd Rule:

For \( n = 2, \) the Newton-Cotes quadrature formula is
\[ I(f) = \int_{a}^{b} f(x) \, dx = \sum_{i=0}^{n-2} w_k f(x_k), \]

where
\[ w_0 = \frac{(-1)^2 h^2}{0! \times 2!} \int_{0}^{2} (s-1)(s-2) \, ds = \frac{h}{3}; \]
\[ w_1 = \frac{(-1)^1 h^2}{1! \times 2!} \int_{0}^{2} s(s-2) \, ds = \frac{4h}{3}; \]
\[ w_2 = \frac{(-1)^0 h^2}{2! \times 0!} \int_{0}^{2} s(s-1) \, ds = \frac{h}{3}. \]

Now
\[ I(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) \]
\[ = \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) + \frac{h}{3} f(x_2) \]
\[ = \frac{h}{3} [f_0 + 4f_1 + f_2]; \]
\[ I(f) = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right], \tag{1.4.7} \]

where
\[ h = \frac{x_2 - x_0}{2} = \frac{b-a}{2}. \]

The Simpson’s \( \frac{1}{3} \) \( rd \) rule is obtained from the equation (1.4.7).

**Error in Simpson’s \( \frac{1}{3} \) \( rd \) Rule:**

The error in Simpson’s \( \frac{1}{3} \) \( rd \) rule is
\[ E_{\frac{1}{3}}(f) = -\frac{h^5}{90} f^{(iv)}(\xi). \tag{1.4.8} \]
1.4.3 Simpson’s $\frac{3}{8}$th Rule:

For $n = 3$ the Newton-Cotes quadrature formula is

$$I(f) = \int_a^b f(x) \, dx$$

$$= \sum_{i=0}^{n=3} w_i f(x_i)$$

$$= w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3),$$

where

$$w_0 = \frac{(-1)^3 h}{0! \times 3!} \int_0^3 (s-1)(s-2)(s-3) \, ds = \frac{3h}{8};$$

$$w_1 = \frac{(-1)^2 h}{1! \times 2!} \int_0^3 (s-2)(s-3) \, ds = \frac{9h}{8};$$

$$w_2 = \frac{(-1) h}{2! \times 1!} \int_0^3 (s-1)(s-3) \, ds = \frac{9h}{8};$$

$$w_3 = \frac{(-1)^0 h}{3! \times 0!} \int_0^3 (s-1)(s-2) \, ds = \frac{3h}{8}.$$

Now

$$I(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$= \frac{3h}{8} f(x_0) + \frac{9h}{8} f(x_1) + \frac{9h}{8} f(x_2) + \frac{3h}{8} f(x_3)$$

$$= \frac{3h}{8} [ f(a) + 3f(a + \frac{2h}{3}) + 3f\left( b - \frac{2h}{3} \right) + f(b) ],$$

(1.4.9)

where

$$h = \frac{x_3 - x_0}{3} = \frac{b - a}{3}.$$

Equation (1.4.9) is called Simpson’s $\frac{3}{8}$th rule.


Error in Simpson’s $^3_8$ th Rule:

The error associated with this method is

$$E_{^3_8}(f) = \frac{-3h^5}{80} f^{(v)}(\xi), \quad (1.4.10)$$

where $x_0 \leq \xi \leq x_4$ or $x_i \leq \xi \leq x_5$ or $a \leq \xi \leq b$.

1.4.4 Boole’s Rule:

From Newton-Cotes method for $n = 4$, we get the Boole’s formula as

$$\int_{x_0}^{x_4} f(x) \, dx = \frac{2h}{45} \left[ 7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right]. \quad (1.4.11)$$

In particular

$$\int_{-1}^{1} f(x) \, dx = \frac{1}{45} \left[ 7\{f(-1) + f(1)\} + 32\left\{ f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) \right\} + 12f(0) \right], \quad (1.4.12)$$

where the required error is

$$E_n(f) = \frac{-1}{3 \times 7!} f^{(v)}(\xi); \quad -1 < \xi < 1.$$

1.5 Lobatto Integration Method:

$$I(f) = w_0 f(-1) + w_n f(1) + \sum_{i=1}^{n-1} w_i f(x_i), \quad (1.5.1)$$

is called Lobatto integration method.

The system of equations for $f(x) = x_i$, where $i = 0,1,2,3,4,5$. 

Now

\[ w_0 + w_1 + w_2 + w_3 = 2; \]
\[- w_0 + w_1 x_1 + w_2 x_2 + w_3 = 0; \]
\[- w_0 + w_1 x_1^2 + w_2 x_2^2 + w_3 = \frac{2}{3}; \]
\[- w_0 + w_1 x_1^3 + w_2 x_2^3 + w_3 = 0; \]
\[- w_0 + w_1 x_1^4 + w_2 x_2^4 + w_3 = \frac{2}{5}; \]
\[- w_0 + w_1 x_1^5 + w_2 x_2^5 + w_3 = 0. \]

From the above results, we get

\[ x_1 = -\frac{1}{\sqrt{5}}; \quad x_2 = \frac{1}{\sqrt{5}}; \]
\[ w_0 = \frac{1}{6}; \quad w_1 = \frac{5}{6}; \quad w_2 = \frac{5}{6}; \quad w_3 = \frac{1}{6}. \]

Hence the method becomes

\[ \int_{-1}^{1} f(x) \, dx = \frac{1}{6} \left[ f(-1) + 5f\left(\frac{-1}{\sqrt{5}}\right) + 5f\left(\frac{1}{\sqrt{5}}\right) + f(1) \right]. \]  \hspace{1cm} (1.5.2)

The error associated with this method is given by

\[ E_{L4}(f) = \frac{-32}{526 \times 6!} f^{(4)}(\xi); \quad -1 < \xi < 1. \]  \hspace{1cm} (1.5.3)

1.6 Birkhoff-Young Quadrature Formula for Analytic Functions:

The Birkhoff-Young (1950) [9] have derived the five point formula

\[ R_{BY}(f) \] as follows:
\[ \int_{-1}^{1} f(z) \, dz = \frac{8}{5} f(0) + \frac{4}{15} \{ f(1) + f(-1) \} 
- \frac{1}{15} \{ f(i) + f(-i) \} + E_{by}(f); \]  
\[ R_{by}(f) = \int_{z_0-h}^{z_0+h} f(z) \, dz \]
\[ = \frac{h}{5} \left[ \frac{4}{3} \{ f(z_0 + h) + f(z_0 - h) \} + 8 f(z_0) \right] 
- \frac{1}{3} \{ f(z_0 + ih) + f(z_0 - ih) \}, \]
\[ |E_{by}(f)| \leq \frac{1}{1890} \max_{z \in S} |f^{vi}(z)|. \]

Where \( S \) is the square with vertices defined as arguments in equation (1.6.3).

### 1.6.1 The Modified Birkhoff-Young Formula:

Let us consider an integral \( \int_{-1}^{1} f(z) \, dz \) where \( z \rightarrow f(z) \) is an analytical function in the square whose vertices are \( 1,-1,i,-i \). If we integrate Taylor expansion of \( f \) around \( z = 0 \) we obtain

\[ \int_{-1}^{1} f(z) \, dz = \sum_{n=0}^{\infty} \frac{f^{2n}(0)}{(2n+1)!}, \]  
where

\[ f^2(0) = \frac{1}{2k^2} \{ f(k) + f(-k) - f(ik) - f(-ik) \} 
- 2 \left\{ \frac{k^4}{6!} f^{vi}(0) + \frac{k^8}{10!} f^v(0) + \frac{k^{12}}{14!} f^{iv}(0), \ldots \right\}; \]
\[ f^4(0) = 6k^4 \left \{ f(k) + f(-k) + f(ik) + f(-ik) - 4f(0) \right \} \]
\[ -24 \left( \frac{k^4}{8!} f^{\text{viii}}(0) + \frac{k^8}{12!} f^{\text{xii}}(0) + \ldots \right), \] 

(1.6.6)

where \( k, -k, ik, -ik \ (k \leq 1) \) are the vertices of the square in which we calculate the function \( f \) for an approximate computation of derivatives.

Applying equation (1.6.5) and equation (1.6.6) in equation (1.6.4), we obtain an integration rule

\[ \int f(z) \, dz = 2 \left( 1 - \frac{1}{5k^4} \right) f(0) + \left( \frac{1}{6k^2} + \frac{1}{10k^4} \right) \left \{ f(k) + f(-k) \right \} + \left( \frac{-1}{6k^2} + \frac{1}{10k^4} \right) \left \{ f(ik) + f(-ik) \right \} + E, \] 

(1.6.7)

with the error term

\[ E = \left( -\frac{2}{3} \frac{1}{6!} \frac{2}{7!} k^4 + \frac{2}{9!} - \frac{2}{5 \times 8!} k^4 \right) f^{\text{xii}}(0) + \ldots. \]

If we put \( k = 1 \), in equation (1.6.7), we derive the \( R_{BY}(f) \) formula (1.6.1) with a slightly changed error term as follows

\[ \int f(z) \, dz = \frac{8}{5} f(0) + \frac{4}{15} \left \{ f(1) + f(-1) \right \} - \frac{1}{15} \left \{ f(i) + f(-i) \right \} - \frac{1}{1890} f^{\text{xii}}(0) + \frac{1}{226800} f^{\text{xii}}(0) + \ldots. \] 

(1.6.8)

It is to be noted that for

\[ \frac{-1}{6k^2} + \frac{1}{10k^4} = 0, \text{ i.e., for } k = \sqrt{\frac{3}{5}}, \]

when the coefficient \( \left \{ f(ik) + f(-ik) \right \} \) vanishes, we obtain Gauss-Legendre-3 point formula \( R_{GL3}(f) \) as
\[
\int_{-1}^{1} f(z) \, dz = \frac{8}{9} f(0) + \frac{5}{9} \left[ f \left( \frac{3}{\sqrt{5}} \right) + f \left( -\frac{3}{\sqrt{5}} \right) \right] + \frac{1}{15750} f^{(3)}(0) - \frac{1}{226800} f^{(5)}(0) + \ldots
\]  

(1.6.9)

1.7 Gaussian Quadrature:

In contrast to Newton-Cotes formula in the integration method (1.4.1) the nodes \( x_i \)'s and the weights \( w_i \)'s are also obtained by making the formula exact for polynomial of degree up to \( n \).

When the \( n \) nodes are known, the corresponding method is called Newton-Cotes method. When the \( (2n+1) \) nodes are to be determined, the method is called Gaussian quadrature method.

An important thing in this method is to transform the interval \([a,b]\), to the interval \([-1,1]\), using the transformation \( 2x = (b-a)t + (b+a) \).

\[
I(f) = \int_{-1}^{1} w(x)f(x) \, dx = \sum_{i=0}^{n} w_i f(x_i). 
\]

(1.7.1)

For \( w(x) = 1 \), the method (1.7.1) reduces to

\[
I(f) = \int_{-1}^{1} f(x) \, dx = \sum_{i=0}^{n} w_i f(x_i). 
\]

(1.7.2)

Rules for Basic Problems:

The associated polynomials are Legendre polynomial \( P_n(x) \) with the \( n^{th} \) degree polynomial normalized to give \( P_n(1) = 1 \). The \( i^{th} \) Gauss nodes \( x_i \) is the \( i^{th} \) root of \( P_n(x) \); its weight is given by Abramwoitz and Stegum [43].
Table-1.1: Nodes and Weights in Gauss-Legendre Rule

<table>
<thead>
<tr>
<th>Number of points</th>
<th>Nodes</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0, $\pm \frac{3}{\sqrt{5}}$</td>
<td>$\frac{8}{9}, \frac{5}{9}$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm \frac{1}{7} \left( \sqrt{3 - 2 \frac{6}{\sqrt{5}}} \right)$, $\pm \frac{1}{7} \left( \sqrt{3 + 2 \frac{6}{\sqrt{5}}} \right)$</td>
<td>$\frac{18 + \sqrt{30}}{36}, \frac{18 - \sqrt{30}}{36}$</td>
</tr>
<tr>
<td>5</td>
<td>0, $\pm \frac{1}{3} \sqrt{5 - 2 \frac{10}{7}}$, $\pm \frac{1}{3} \sqrt{5 + 2 \frac{10}{7}}$</td>
<td>$\frac{128}{225}, \frac{322 + 13\sqrt{70}}{900}, \frac{322 - 13\sqrt{70}}{900}$</td>
</tr>
</tbody>
</table>

1.7.1 Gauss-Legendre-2 Point Formula:

In this case all nodes and weights are unknown.

For $n = 1$, the method becomes

$$\int_{-1}^{1} f(x) \, dx = \sum_{i=0}^{n=1} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1). \tag{1.7.3}$$

There are four unknowns in equation (1.7.3) and to determine the unknowns, it is assumed to be exact for polynomials of degree up to 3.
The system of equations for \( f(x) = x^i \) where \( i = 0,1,2,3 \), is

\[
\begin{align*}
w_0 + w_1 &= 2; \\
w_0 x_0 + w_1 x_1 &= 0; \\
w_0 x_0^2 + w_1 x_1^2 &= \frac{2}{3}; \\
w_0 x_0^3 + w_1 x_1^3 &= 0.
\end{align*}
\]

From the above equations

\[
\begin{align*}x_0 &= \frac{1}{\sqrt{3}}; \quad x_1 = \frac{-1}{\sqrt{3}}; \quad w_0 = w_1 = 1.\end{align*}
\]

The method (1.7.3) becomes

\[
\int_{-1}^{1} f(x) \, dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).
\]

The error term associated with this method can be written as

\[
E_{GL2}(f) = \frac{f^{iv}(\xi)}{135}; \quad -1 < \xi < 1.
\]

1.7.2 Gauss-Legendre-3 Point Formula:

In this case all nodes and weights are unknown. For \( n = 2 \), the method becomes

\[
\int_{-1}^{1} f(x) \, dx = \sum_{i=0}^{2} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2).
\]

The system of equations for \( f(x) = x^i \) where \( i = 0,1,2,3,4,5, \)

\[
w_0 + w_1 + w_2 = 2;
\]
\[ w_0 x_0 + w_1 x_1 + w_2 x_2 = 0; \]
\[ w_0 x_0^2 + w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}; \]
\[ w_0 x_0^3 + w_1 x_1^3 + w_2 x_2^3 = 0; \]
\[ w_0 x_0^4 + w_1 x_1^4 + w_2 x_2^4 = \frac{2}{5}; \]
\[ w_0 x_0^5 + w_1 x_1^5 + w_2 x_2^5 = 0. \]

From the above equations we get
\[ x_0 = -\sqrt{\frac{3}{5}}; \quad x_1 = 0; \quad x_2 = \sqrt{\frac{3}{5}}; \]
\[ w_0 = \frac{5}{9}; \quad w_1 = \frac{8}{9}; \quad w_2 = \frac{5}{9}. \]

Hence the method (1.7.6) becomes
\[
\int_{-1}^{1} f(x) \, dx = \frac{1}{9} \left[ 5 f\left(-\sqrt{\frac{3}{5}}\right) + 8 f(0) + 5 f\left(\sqrt{\frac{3}{5}}\right) \right].
\] (1.7.7)

The error term associated with the method (1.7.7) can be written as
\[
E_{Gl3}(f) = \frac{8}{175 \times 6!} f^{(iv)}(\xi); \quad -1 < \xi < 1.
\] (1.7.8)

Similarly the nodes and weights of Gauss-Legendre 4-point rule as well as 5-point rule are given in table-1.1.

1.8 Extrapolation Methods:

The technique of combining two computed values obtained by using the same method with two different step sizes, to obtain a higher order method is called extrapolation.
Richardson Extrapolation:

We can write the error term for the Integrand \( f(x) \) as

\[
E_n(f) = \int_a^b f(x) \, dx - h \sum_{j=0}^n f^{(j)}(x_j)
\]

\[
= -\sum_{i=1}^m \frac{\beta_{2i}}{(2i)!} h^{2i} \left[ f^{(2i-1)}(b) - f^{(2i-1)}(a) \right]
\]

\[
+ \frac{h^{2m+2}}{(2m+2)!} \int_a^b \beta_{2m+2} \left( \frac{x-a}{h} \right)^{(2m+2)} f(x) \, dx,
\]

as

\[
I - I_n = \frac{d_2^{(0)}}{n^2} + \frac{d_4^{(0)}}{n^4} + \frac{d_{2m}^{(0)}}{n^{2m}} + F_{n,m}, \tag{1.8.1}
\]

where \( I_n \) denotes Trapezoidal rule and

\[
F_{n,m} = \frac{(b-a)^{2m+2}}{(2m+2)!} \frac{h^{2m+2}}{n^{2m+2}} \int_a^b \beta_{2m+2} \left( \frac{x-a}{n} \right)^{(2m+2)} f(x) \, dx,
\]

\[
d_{2j}^{(0)} = -\frac{\beta_{2j}}{(2j)!} (b-a)^{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right]. \tag{1.8.2}
\]

For \( n \) even,

\[
I - I_{n/2} = \frac{4d_2^{(0)}}{n^2} + \frac{16d_4^{(0)}}{n^4} + \frac{64d_6^{(0)}}{n^6} + \ldots. \tag{1.8.3}
\]

Multiplying equation (1.8.1) by \( (4)^j \) and subtracting equation (1.8.3) from it, we obtain

\[
4(I - I_n) - (I - I_{n/2}) = -\frac{12d_4^{(0)}}{n^4} - \frac{60d_6^{(0)}}{n^6} \ldots;
\]

\[
I = \frac{4I_n - I_{n/2}}{3} - \frac{4d_4^{(0)}}{n^4} - \frac{20d_6^{(0)}}{n^6} \ldots.
\]
Now
\[ I_n^{(1)} = \frac{1}{3} \left[ 4f_n^{(0)} - I_n^{(0)} \right]; \quad n \text{ is even and } n \geq 2 \quad (1.8.4) \]

and
\[ I_n^{(1)} = I_n. \]

We call \( \{I_n^{(1)}\} \) the Richardson extrapolation of \( \{I_n^{(0)}\} \).

The sequence \( I_2^{(1)}, I_4^{(1)}, I_6^{(1)} \ldots \) is a numerical integration rule.

We have the error as
\[ I - I_n^{(1)} = \frac{d_4^{(1)}}{n^4} + \frac{d_6^{(1)}}{n^6} + \ldots; \quad (1.8.5) \]
\[ d_4^{(1)} = -4d_4^{(0)}, \quad d_6^{(1)} = -20d_6^{(0)}. \quad (1.8.6) \]

Now to get the explicit formula for \( I_n^{(1)} \),

let \( h = \frac{b - a}{n} \) and \( x_j = a + jh \) for \( j = 0, 1, \ldots, n \). Then using equation (1.8.4) and

Trapezoidal rule we get
\[ I_n^{(1)} = \frac{4h}{3} \left[ \frac{f_0}{2} + f_1 + f_2 + f_3 + \ldots + f_{n-1} + \frac{f_n}{2} \right] \]
\[ -\frac{2h}{3} \left[ \frac{f_0}{2} + f_2 + f_4 + f_6 + \ldots + f_{n-2} + \frac{f_n}{2} \right], \]
\[ I_n^{(1)} = \frac{h}{3} \left[ f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n \right]. \quad (1.8.7) \]

Equation (1.8.7) is a Simpson’s rule with \( n \) subdivisions.

Now to obtain error, using equation (1.8.6) and equation (1.8.2) we have
\[ I - I_n^{(1)} = -\frac{h^4}{180} \left[ (f^3(b) - f^3(a)) + \frac{h^6}{1512} [f^5(b) - f^5(a)] \right] + \ldots; \quad (1.8.8) \]
\[ I - I_n^{(1)} = \frac{16d_4^{(1)}}{n^4} + \frac{64d_6^{(1)}}{n^6} + \ldots \; ; \]

\[ 16(I - I_n^{(1)}) - \left( I - I_n^{(1)} \right) = -\frac{48d_6^{(1)}}{n^6} + \ldots \; , \]

then

\[ I = \frac{16I_n^{(1)} - I_n^{(1)}}{15} - \frac{48d_6^{(1)}}{15n^6} + \ldots \; ; \quad (1.8.9) \]

\[ I_n^{(2)} = \frac{16I_n^{(1)} - I_n^{(1)}}{15} ; \quad n \geq 4. \quad (1.8.10) \]

Where \( n \) is divisible by 4. We call \( \{ I_n^{(2)} \} \) the Richardson extrapolation of \( \{ I_n^{(1)} \} \). Now using equation (1.8.6) we get

\[ I - I_n^{(1)} = \frac{16I_n^{(1)} - I_n^{(1)}}{15} - I_n^{(1)} + \frac{d_6^{(2)}}{n^6} + \ldots = \frac{I_n^{(1)} - I_n^{(1)}}{15} + \frac{d_6^{(2)}}{n^6} + \ldots \; , \]

using

\[ h = \frac{(b - a)}{n} ; \]

\[ I - I_n^{(1)} = \frac{1}{15} \left[ I_n^{(1)} - I_n^{(1)} \right] + 0(h^6), \quad (1.8.11) \]

and

\[ I - I_n^{(1)} = \frac{1}{15} \left[ I_n^{(1)} - I_n^{(1)} \right] , \quad (1.8.12) \]

Since both terms are \( O(h^4) \) and the remainder term is \( O(h^6) \), this is called Richardson error estimate for Simpson’s rule.

The extrapolation can be defined by

\[ I_n^{(k)} = \frac{4^k I_n^{(k-1)} - I_n^{(k-1)}}{4^k - 1} , \quad n \geq 2^k , \quad (1.8.13) \]
with $n$ a multiple of $2^k, k \geq 1$. It can be shown that the error has the form

$$I - I_n^{(k)} = \frac{d_{2k+2}^{(k)}}{n^{2k+2}} + \ldots$$

$$= A_k (b - a) h^{2k+2} f^{2k+2} (\xi_n); \quad a < \xi_n < b,$$

with $A_k$, a constant independent of $f$ & $h$ and

$$d_{2k+2}^{(k)} = A_k (b - a)^{2k+2} \left[ f^{(2k+1)} (b) - f^{(2k+1)} (a) \right].$$

Finally it can be shown that for any $f \in C [a, b]$,

$$\lim_{n \to \infty} I_n^{(k)} (f) = I(f).$$

The rules $I_n^{(k)} (f)$ for $k > 2$ bear no direct relation to the composite Newton-Cotes rule.

1.9 Integral Equation:

For $a \leq x \leq b, a \leq t \leq b$ we have taken the equations

$$f(x) = \int_a^x K(x,t) U(t) dt;$$

$$U(x) = f(x) + \lambda \int_a^x K(x,t) U(t) dt;$$

$$U(x) = \int_a^b K(x,t) [U(t)]^2 dt.$$
An integral equation is called linear, if only linear operations are performed in it upon the unknown function. On the other hand, an integral equation which is not linear is known as non-linear integral equation. In the above definition, equation (1.9.1) and equation (1.9.2) are linear integral equation, while equation (1.9.3) is a non-linear integral equation.

**Hammerstein Type:**

The special non linear Fredholm integral equation of the type

\[ x(t) = \frac{1}{\lambda} \int_a^b k(t, s)F(x(s))ds + y(t) \quad ; \lambda \neq 0, \ a \leq t \leq b, \quad (1.9.4) \]

is called Hammerstein type equation.

### 1.10 Adomain Decomposition Method:

The main principle in adomain decomposition method (ADM) is to convert non linear integral equation to linear with the help of adomain polynomials.

\[ A_n = \left[ \frac{1}{n!} \frac{d^n}{d\mu^n} f(u(\mu)) \right]_{\mu=0}, \quad (1.10.1) \]

where

\[ \left[ \frac{d^n}{d\mu^n} U(\mu) \right]_{\mu=0} = n! U_n. \]

The main application of the method is utilizing adomain polynomials which permit for the solution of convergence, of the non linear portion of the equation without simply linearizing the systems. The principle based on decomposition of a solution of a non linear operator equation in series of functions.

### 1.11 Gauss-Divergence Theorem:

The Gauss-divergence theorem is the relation between triple integral and surface integral. In analytic let \( F(x, y, z) \) be a vector function which is continuous and continuous partial derivative in some domain containing \( R \), where \( R \) be a closed
bounded region in space whose boundary is piecewise smooth orientable surface $S$, then

$$\iint_S F \cdot n \, dA = \oiint_R \text{div} F \, dv,$$  \hspace{1cm} (1.11.1)

where $F = [F_1, F_2, F_3]$ and $n$ is outer unit normal vector.

### 1.12 Cleanshaw-Curtis Rule:

The Chebyshev's polynomials

$$T_j(x) \text{ by } T_j(\cos \theta) = \cos(j\theta);$$

$$T_n(x) = \cos n\theta,$$

where $x = \cos \theta$.

Here the zeros of $T_n(x)$ are

$$x_j = \cos \frac{2j - 1}{2n} \pi, \ j = 1, 2, 3, \ldots, n.$$  \hspace{1cm} (1.12.1)

#### 1.12.1 Cleanshaw-Curtis-5 Point Rule ($R_{cc5}(f)$):

The Cleanshaw-Curtis quadrature method proposed by Cleanshaw and Curtis [24].

(a) In real integral form we have

$$R_{cc5}(f) = \frac{1}{15} \left[ f(-1) + 8f \left( \frac{-1}{\sqrt{2}} \right) + 12f(0) + 8f \left( \frac{1}{\sqrt{2}} \right) + f(1) \right].$$  \hspace{1cm} (1.12.2)

(b) In complex integral form we have
\[ R_{cc5}(f) = \frac{h}{15} \left[ f(z_0 + h) + f(z_0 - h) + 8f \left( z_0 + \frac{h}{\sqrt{2}} \right) + 8f \left( z_0 - \frac{h}{\sqrt{2}} \right) + 12f(z_0) \right] \] . \hspace{1cm} (1.12.3)

1.12.2 Cleanshaw-Curtis-7 Point Rule \((R_{cc7}(f))\):

The Cleanshaw-Curtis-7 point rule \(R_{cc7}(f)\) is defined as

\[ R_{cc7}(f) = \frac{1}{315} \left[ 9\left\{f(-1) + f(1)\right\} + 80\left\{f\left(-\frac{\sqrt{3}}{2}\right) + f\left(\frac{\sqrt{3}}{2}\right)\right\} + 144\left\{f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right\} + 164f(0) \right] \] . \hspace{1cm} (1.12.4)

1.13 Mixed Quadrature Rule:

The mixed quadrature rule of higher degree of precision is obtained by taking linear combination of two or more rules of lower degree of same precision in real and analytic sense.

1.13.1 Mixed Quadrature Rule Over Analytic Functions:

(a) We mixed up Birkhoff-Young rule and Cleanshaw-Curtis-5 point quadrature rule each of precision five. A new rule of precision 7 “Numerical treatment of analytic functions via mixed quadrature rule” is formed to integrate numerically analytic functions which are included in chapter-2.

(b) The chapter-3 “Triple mixed quadrature rule for analytic functions through extrapolation” contains a rule of degree of precision nine, which is obtained by the application of Birkhoff-Young modified rule through extrapolation and Gauss-Legendre-4 point rule of precision seven. Again combining the above rule with Gauss-Legendre-5-point rule to form a rule of degree of precision eleven.
1.13.2 Mixed Quadrature Rule For Real Integrals:

In this thesis chapters 4, 5 and 6 contain some mixed quadrature rules for real definite integrals. The chapter-4 contains, “Approximation of real definite integrals via hybrid quadrature domain”. Again the chapter-5 bears “Mixed quadrature over sphere” and “An efficient quadrature rule for approximate solution of non linear integral equation of Hammerstein type” has been taken in chapter-6.

1.14 Degree of Precision (DP):

The degree of precision of a quadrature formula is the maximum degree of arbitrary polynomial for which the formula gives exact value of integrals over any interval.

In general the degree of precision for Gaussian quadrature is \((2n-1)\) whereas the degree of precision for Newton-Cotes type rule is

\[
\begin{align*}
\text{n; if } n \text{ is odd} \\
\text{n-1; if } n \text{ is even}
\end{align*}
\]

For example for Trapezoidal rule DP =1, For Simpson’s rule DP = 3, for Cleanshaw-Curtis, Booles and Lobatto DP=5, etc.