Chapter 3

A brief overview to the world of Fractals

This chapter provides an insight into the world of fractals used for the analysis of complex irregular structures. First, the basic properties of fractals are mentioned. The existence of fractals is explained by describing the space where fractals live. The mathematical foundations of fractals like the Iterated Function Systems (IFS) and Collage Theorem are given later. A succinct overview of the mathematical modeling of images using fractals is also presented.
3.1 From Euclidean geometry to Fractals

Conventional elementary mathematics classes deals with idealized shapes like triangles, circles, spheres, squares, etc. These form the basic building blocks of Euclidean geometry. But nature never exists in idealized shapes. Shapes such as coastlines, mountains and clouds are not easily described by traditional Euclidean geometry. Thus nature exhibits a higher degree or different level of complexity. The description of these shapes by Euclidean geometry is only an estimate of how closely they resemble to the Euclidean geometry. These irregular and fragmented patterns existing in nature lead to identifying a family of shapes called "fractals".

The word fractal was coined by Mandelbrot (Man 1982) in 1975 from the Latin adjective "fractus". The corresponding Latin verb "frangere" means "to break:" to create irregular fragments. Mandelbrot defined fractals as “A rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced/size copy of the whole”. He is often considered as the father of fractal geometry.

But, the works on fractals were started long back before Mandelbrot. Their description began with classical mathematics and mathematicians like Georg Cantor (1872), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916) Gaston Julia (1918) and Felix Hausdorff (1919) were the pioneers in this field (Pei 1991).

Methods of classical geometry and calculus are unable for studying fractals and alternate techniques are required (Fal 2003). Thus, fractal geometry developed by Mandelbrot, forms a new branch of mathematics, which is appropriate for the irregular shapes in the real world.

These definitions lead to the three important properties of fractals which are given below.

3.2 Properties of Fractals

The three basic properties of fractals are
3.2 Properties of Fractals

a. Self similarity
b. Fractal dimension
c. Formation by Iteration

3.2.1 Self Similarity

Self similarity stands for "similar to itself". An example of a natural structure with self similarity is a cauliflower. The cauliflower contains branches or parts which when removed and compared with the whole, are very much the same, only smaller. These clusters again can be decomposed into smaller clusters, which again look similar to the whole as well as to the first generation branches. This self similarity carries forward, for about three or four stages. After that the structure is too small for further dissection. The self similarity of cauliflower is shown in fig 3.1.

Fig. 3.1 Four pieces of the same single cauliflower is shown in 1, 2, 3 and 4.

In a mathematical idealization, the self similarity property of a fractal may be continued through infinite number of stages (Pei 1991). But all self similar structures are not fractals. For example, a line segment or a square or a cube, can be broken into small copies which are obtained by similarity transformation. But, these structures are not fractals. There comes the second property of fractals: the fractal dimension.
3.2.2 Fractal Dimension

The fractal dimension is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales. Before coming to fractal geometry, the dimensions in the Euclidean geometry are considered. A point has no dimension as it does not have length, width or height. A straight line has dimension 1 since it has finite length, but no width or height. But a plane has two dimensions length and width but no height. For a cube, the dimension is three, as it has three dimensions length, width and depth extending to infinity in all three directions.

For further investigations on dimension, take a self-similar figure like a line segment, and double its length. Doubling the length gives two copies of the original segment. Considering another self similar figure like a square, when the length and width are scaled by two, it gives four copies of the original cube. Similarly eight copies are obtained when a cube is doubled in its length, width, and height. The table

<table>
<thead>
<tr>
<th>Figure</th>
<th>Dimension (D)</th>
<th>Scaling Factor (r)</th>
<th>Number of copies (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Line</td>
<td>1</td>
<td>2</td>
<td>2=2^1</td>
</tr>
<tr>
<td>Plane</td>
<td>2</td>
<td>2</td>
<td>4=2^2</td>
</tr>
<tr>
<td>Cube</td>
<td>3</td>
<td>2</td>
<td>8=2^3</td>
</tr>
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</table>
3.1 shows the relation between the number of copies (N), scaling factor (r) and the
dimension (D). Thus the number of copies N is given by:

\[ N = r^D \]  \hspace{1cm} (3.1)

Taking logarithm on both sides,

\[ \log(N) = D \log(r) \]

or

\[ D = \frac{\log(N)}{\log(r)} \]  \hspace{1cm} (3.2)

Thus, the dimension is an integer in Euclidean geometry. Now, consider the most
common example of fractal, the Sierpinski triangle, introduced by the Polish
construction of the Sierpinski triangle (gasket) is as follows.

Draw an equilateral triangle in a plane. Pick the midpoints of its three sides
(fig 3.2a). When the three midpoints are joined four triangles are formed as in fig.
3.2(b), out of which the central one is dropped. This is the basic construction step.

Thus, after the first step, three congruent triangles, which are exactly half the size of
the original triangle, which touch at three points i.e. common vertices of the two
contiguous triangles. The same procedure is followed with the three remaining
triangles and repeat the basic step as often as desired. Therefore, starting with one
triangle, succeeding stages produces, 3, 9, 27, 81, 243......triangles, each of which is
an exact scaled down version of the triangles in the preceding step.

Now, defining the dimension of Sierpinski triangle, when the length of the
sides is doubled (since the black triangles are holes, they are not counted), three
copies of the original triangles are obtained. Following the convention in equation (3.1), number of copies \(3 = 2^D\), where \(D\) is the dimension. From equation (3.2)

\[
D = \frac{\log 3}{\log 2} = 1.5849
\]

Thus the fractal dimension is not an integer. So fractals are geometrical objects that have a non integer or fractional dimension.

### 3.2.3 Formation by Iteration

Fractals are often formed by an iterative procedure. To make a fractal, take a familiar geometric figure, like a triangle, a line segment, etc and operate on it so that the new figure is more "complicated" in a special way. Then, in the same way, operate on that resulting figure, and get an even more complicated figure. Repeat the process again and again...and again. It should be repeated many times. Consider the previous example of Sierpinski triangle, the final figure obtained is very complex compared to the starting figure of a triangle.

Thus, a fractal is a design of infinite details. It is created using a mathematical formula. No matter how closely they are looked at, a fractal never loses its detail. It is infinitely detailed, yet it can be contained in a finite space. Thus, fractals are generally self similar and are independent of scale.

Some of the popular fractal sets defined by mathematicians, other than the Sierpinski triangle, are Cantor set, Koch curve, Julia set, Bandelbrot set etc. and are shown in the figs.3.3(a)-3.3(d).
3.2 Properties of Fractals

**Fig 3.3** (a) Basic steps for the construction of Cantor set

Step 0

Step 1

Step 2

Step 3

Step 4

**Fig 3.3** (b) Basic steps for the construction of Koch Curve

**Fig 3.3** (c) Julia Set
The mathematical foundations of fractals can be explained by considering the analogy of a simple Xerox machine with feedback incorporated in it and is explained in the next section.

### 3.3 Multiple Reduction Copy Machine

Consider a simple Xerox (copy) machine that takes an image as the input. It has several independent lens systems, each of which reduces the input image and places it somewhere in the output image. The assembly of all reduced copies in some pattern is finally produced as output (Pei 1991). Schematic of such a copy machine is given in fig. 3.4.
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The crucial point in the design is that the machine runs in a feedback loop; its own output is fed back as its new input again and again. When there are more than one reduction lens, powerful and exciting results are obtained. Since the output image is the reduced copy of the original image, that copy is similar to the original. The process to generate a copy is called similarity transformation or similitude. After ten or more cycles, any initial image will reduce to just a point. Such a machine is referred to as the Multiple Reduction Copy Machine (MRCM).

Now; consider the MRCM with three lens systems, each of which is set to reduce the input image by a factor of $\frac{1}{2}$. The resulting copies are assembled in the configuration of an equilateral triangle. Fig. 3.5 shows an MRCM with a square as the input image and the reduced copies of the square are placed in the form of an equilateral triangle.

![Fig. 3.5 Multiple Reduction Copy Machine with three reduction lenses and places the input figure in the form of an equilateral triangle](image)

Fig. 3.5 Multiple Reduction Copy Machine with three reduction lenses and places the input figure in the form of an equilateral triangle

equilateral triangle. Fig. 3.6 shows the effect of the machine run three times, beginning with different initial images. Looking into the output images obtained in fig. 3.6, it is observed that the same final structure is approximated as the machine is run infinite number of times. It doesn’t matter in the least, whether the input images to the machine are rectangles, triangles, or any other shape, the same final image is approached in each case - the Sierpinski gasket. This final image is totally independent of the initial image with which the operation was started. Thus, in mathematical terms, it is a process which produces a sequence of results tending towards one final object which is independent of how the process began. This property is called stability. The final image to which the algorithm converges is called the attractor. Moreover, when the machine is started with the attractor, then nothing
happens, or the attractor is left invariant or fixed. The arrangement of the lens and its reduction factor determines the final image it produces. Thus it is equivalent to applying certain transformations on the input image, which are called the *affine transformations*.

Fig. 3.6 The input images to the MRCM and their corresponding output images obtained after three iterations

The mathematical foundations of the fractals can be approximated to the working of the above mentioned Multiple Reduction Copy Machine (MRCM) algorithm.

### 3.4 Mathematical Foundations

Irregular sets such as those seen in section 3.2 provide a much better representation of many natural phenomena than the figures of classical geometry. Fractal geometry provides a general frame work for the study of such irregular sets. A few basic definitions in the fractal space are explained below (Fal 2003).
3.4 Mathematical Foundations

3.4.1 The Space of Fractals

Fractal geometry is concerned with the structure of sub-sets of various very simple “geometrical” spaces (Bar 1988). Such a space is denoted by \( X \). Fractal is just a sub-set of a space. Though the space is simple, the fractal sub-set is geometrically complicated.

The space \( X \) is a set. The points of the space are the elements of the set. The nomenclature “space” implies that there is some structure to the set, some sense of which points are close to which. Example: when \( X = \mathbb{R} \), \( \mathbb{R} \) denotes the set of real numbers. Each "point" \( x \in X \) is a real number. When \( X = \mathbb{R}^2 \), it is the Euclidean plane or the coordinate plane of calculus. Any pair of real numbers \( x_1, x_2 \in \mathbb{R} \) determine a single point in \( \mathbb{R}^2 \). That is, in \( \mathbb{R}^2 \) space, any point is a function of two co-ordinates \((x, y)\). Any image can be considered as a set in the \( \mathbb{R}^2 \) space or in other words images are all possible subsets of \( \mathbb{R}^2 \) space.

The next section gives certain definitions which are necessary for building the mathematical foundations of fractals.

3.4.1.1 Metric spaces

A metric space \((X,d)\) is a space \( X \) together with a real valued function \( d : X \times X \to \mathbb{R} \), which measures the distance between pairs of points \( x \) and \( y \) in \( X \). It is required that \( d \) should obey the following axioms.

1. \( d(x, y) = d(y, x) \quad \forall x, y \in X \)
2. \( 0 < d(x, y) < \infty \quad \forall x, y \in X, x \neq y \)
3. \( d(x, x) = 0 \quad \forall x \in X \)
4. \( d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X \)

The fourth axiom is called the \textit{triangular inequality}.

Such a function is called a \textit{metric}. The concept of shortest paths between points in a space, \textit{geodesics}, is dependent on the metric. The metric may determine a geodesic
structure of the space. Geodesics on a sphere are great circles and in the plane with the Euclidean metric they are straight lines.

Fractal geometry is concerned with the description, classification, analysis and observation of subsets of metric spaces \((X, d)\). The metric spaces are usually of an inherently simple geometric character and the subsets are typically geometrically complicated. There are a number of general properties for the subsets of metric spaces, which occur over and over again, which are very basic, and which form part of the vocabulary for describing fractal sets and other subsets of metric spaces.

### 3.4.1.2 Cauchy Sequence

A sequence \(\{x_n\}_{n=1}^{\infty}\) of points in a metric space \((X, d)\) is called a Cauchy sequence if, for any given number \(\varepsilon > 0\), there is an integer \(N > 0\), so that

\[
d(x_n, x_m) < \varepsilon, \forall m, n > N
\]

The above equation implies that, further along the sequence if one goes, the closer together become the points in the sequence as in fig.3.7. Just as the points move close together as one move along the sequence, it will finally converge to a point.

A sequence \(\{x_n\}_{n=1}^{\infty}\) of points in a metric space \((X, d)\) is said to converge to a point \(x \in X\) if, for any given number \(\varepsilon > 0\), there is an integer \(N > 0\) so that

\[
d(x_n, x) < \varepsilon, \forall n > N
\]
In this case the point \( x \in X \), to which the sequence converges, is called the limit point of the sequence:

\[
x = \lim_{n \to \infty} x_n
\]  

(3.6)

If a sequence of points \( \{x_n\} \) in a metric space \((X, d)\) converges to a point \( x \in X \), then \( \{x_n\} \) is a Cauchy sequence.

A metric space \((X, d)\) is complete if every Cauchy sequence \( \{x_n\} \) in \( X \) has a limit point \( x \in X \).

### 3.4.2 Affine Transformations

Consider mappings from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) such that the new points formed are members of the space \( \mathbb{R}^2 \). Let the mapping be given by, \((x, y) \rightarrow (x, y)\) or in other words, the new \((x, y)\) formed are functions of previous points \( x \) and \( y \), i.e

\[
\begin{pmatrix}
  x_{n+1} \\
  y_{n+1}
\end{pmatrix} = f
\begin{pmatrix}
  x_n \\
  y_n
\end{pmatrix}
\]  

(3.7)

A transformation \( \omega: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), of the form

\[
\omega (x_{n+1}, y_{n+1}) = (ax_n + by_n + e, cx_n + dy_n + f)
\]  

(3.8)

where \( a, b, c, d, e \) and \( f \) are real numbers is called affine transformation.

or

\[
\begin{pmatrix}
  x_{n+1} \\
  y_{n+1}
\end{pmatrix} =
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  x_n \\
  y_n
\end{pmatrix} +
\begin{pmatrix}
  e \\
  f
\end{pmatrix}
\]  

(3.9)

\[
= Ax + t
\]

Here \( A=\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \) is two dimensional, \( 2 \times 2 \) real matrix and \( t \) is the column vector \( \begin{pmatrix}
  e \\
  f
\end{pmatrix} \), which cannot be distinguished from the co-ordinate pair \((e, f)\)\( \in \mathbb{R}^2 \). The matrix \( A \) can always be written in the form:
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\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} =
\begin{pmatrix}
  r_1 \cos \theta_1 & -r_2 \sin \theta_2 \\
  r_1 \sin \theta_1 & r_2 \cos \theta_2
\end{pmatrix}
\]

(3.10)

where \((r_1, \theta_1)\) are the polar coordinates of the point \((a, c)\) and \((r_2, (\theta_2 + \pi/2))\) the polar coordinates of the point \((b, d)\). The linear transformation in \(\mathbb{R}^2\) given by equation 3.11, maps any parallelogram with a vertex at the origin to another parallelogram with a vertex at the origin. The parallelogram is turned over by the transformation. A general affine transformation \(\omega(x) = Ax + t\) in \(\mathbb{R}^2\) consists of a linear transformation, \(A\), which deforms space relative to the origin, i.e., flipping, rotation etc, followed by a translation or shift specified by the vector \(t\) as shown in fig. 3.8. Let \(\omega : X \rightarrow X\) be a transformation on a metric space. A point \(x_f \in X\) such that \(f(x_f) = x_f\) is called a fixed point of the transformation. The fixed points of a transformation are very important. They are pinned into space i.e. they are not changed by the transformation. They restrict the motion of the space under non violent, nonripping transformation of bounded deformation.

![Fig 3.8 Deformation obtained for the parallelogram](image)

### 3.4.3 The Contraction Mapping Theorem

Let \(x\) and \(y\) be two points in space with a distance \(d(x, y)\) between them. Define two functions \(\omega(x)\) and \(\omega(y)\). They will produce two new different points with a distance
of \(d(\omega(x), \omega(y))\) between them. A transformation \(\omega: X \rightarrow X\) on a metric space is called contractive or contraction mapping if there is a constant \(0 \leq s < 1\) such that

\[
d(\omega(x), \omega(y)) \leq s \cdot d(x, y) \quad \forall x, y \in X
\]

where \(s\) is called the contractivity factor. For the contractivity factor to be less than one the determinant

\[
\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix} < 1.
\]

A contraction mapping has at most one fixed point. Moreover, the Banach fixed point theorem states that every contraction mapping on a nonempty complete metric space has a unique fixed point, and that for any \(x\) in \(X\) the iterated function sequence converges to the fixed point.

### 3.4.3.1. Banach’s Contraction Mapping Theorem.

Let \((X, d)\) be a complete metric space, and \(\omega(x)\) be the transformation applied on \(x\), which is contractive mapping. Then

\[
d(\omega(x), \omega(y)) \leq \lambda d(x, y), \quad 0 < \lambda < 1, \quad \forall x, y \in X
\]

Then \(\omega\) has a unique fixed point. When there are a set of transformations, \(\omega_1, \omega_2, \omega_3\) with contractivity factors \(s_1, s_2\) and \(s_3\) respectively, then the contractivity of \(\omega = \omega_1 \cup \omega_2 \cup \omega_3\) is \(s = \max\{s_1, s_2, s_3\}\).

### 3.4.4. Iterated Function System (IFS)

An Iterated Function System (IFS) consists of a complete metric space \((X, d)\) together with a finite set of contraction mappings \(\omega_n: X \rightarrow X\), with respective contractively factors, \(s_n\), for \(n=1,2,N\). Let a transformation defined as \(\Omega: H(X) \rightarrow H(X)\) by the transformation \(\Omega\) be applied on the set \(B\) in the complete metric space.

\[
\Omega(B) = \omega_1(B) \cup \omega_2(B) \cup \ldots \cup \omega_n(B)
\]

then \(\forall B \in H(X)\)
Ω is a contraction mapping with contractivity factor. Then 
\( h(\Omega(B),\Omega(C)) \leq s.h(B,C), \forall B,C \in H(X) \). Its unique fixed point \( A \in H(X) \) obeys
\[
A = \Omega(A) = \bigcup_{n=1}^{\infty} \omega_n(A)
\]
and is given by \( A = \lim_{n \to \infty} \Omega^n(B) \) for any \( B \in H(X) \). The fixed point \( A \in H(X) \) is called the attractor of the IFS. This implies that when starting with any set or image, on the repeated application of the IFS, it will ultimately converge to the fixed point or another fixed image. This is similar to the property of MRCM discussed in section 3.3.

Till this point, the problem of how to generate the image when a set of IFS is given is considered. But, an inverse problem exists, which is, given a fractal object how to generate the IFS. This is based on the collage theorem which is given in the next section.

### 3.4.5. Collage Theorem

This theorem is the corollary of the Banach’s fixed point theorem and the central theorem with which the inverse theorem is based. The explanation given here is based on the derivation by Barnsley (Bar 1988). According to the Banach’s fixed point theorem the distance between the two point’s \( x_m \) and \( x_n \) in the sequence is given by:
\[
d(x_m, x_n) \leq d(x_0, x_1) \frac{\lambda^m}{1-\lambda}
\]
Setting \( m=0 \) and \( n=\infty \) the above equation becomes:
\[
d(x_0, x_\infty) \leq d(x_0, x_1) \frac{1}{1-\lambda}
\]
Let \( (X,d) \) be a complete metric space. Let \( L \), a fractal object, \( L \in H(X) \), be given and let \( \varepsilon \geq 0 \) be also given. Choose an IFS such that \( \{X; (\omega_0), \omega_1, \omega_2, \ldots, \omega_n\} \) with contractivity factor \( 0 \leq s < 1 \), so that
where \( h(d) \) is the Hausdorff metric. For a complete metric space, the Hausdorff distance between points \( A \) and \( B \) is defined by

\[
h(A, B) = d(A, B) \lor d(B, A)
\]

The notation \( \lor \) is used to mean the maximum of the two real numbers. Then

\[
h(L, A) \leq \frac{\varepsilon}{1 - s}
\]

where \( A \) is the attractor of the IFS. Or Equivalently:

\[
h(L, A) \leq \frac{1}{1 - s} h \left( L, \bigcup_{n=1}^{\infty} \omega_n(L) \right)
\]

The theorem tells that in order to find an IFS whose attractor is “close to” a given set, find a set of transformations, that is, contraction mappings on a suitable space within which the given set lies, such that the union , or collage of images of the given set under the transformations is near to the given set. Nearness is measured using Hausdorff metric.

Thus while fractal image coding, find out which transformation when applied to the whole image, will give a part of the whole image. Then make collage, and this collage should be as close as the whole image.

### 3.5 Fractal Image Coding

The theorems mentioned above in section 3.4, which employs the self similarity property of the fractal images are exploited for fractal image coding. Since deterministic fractal objects are redundant objects, in the sense that, they are made up of transformed copies of either themselves or parts of themselves. This property is mainly used in compression of images based on fractal coding. For a general image, there is no need to store all the parts of the image, since affine redundancies will be present in parts of the image. Image redundancy can be exploited by modeling it, as is present in fractal objects.
In the basic method of fractal image coding proposed by Barnsley, the image to be coded is divided into blocks, called range blocks as shown in fig.3.9. Then the entire image is searched for a corresponding domain block such that, the best block when coded gives the range block. The size of the domain is generally chosen to be greater than range, to ensure contractive mapping between domain and range. Usually the size of the domain is chosen to be twice that of the range block. The domains are scaled and rotated in different directions and compared for best mapping between the domain and the range and thus finding the affine transformation, mapping the domain and the range block. This is computationally intensive as the entire image has to be searched to find the best mapping from a domain to the range. Then the domain and range locations along with the six parameters (a, b, c, d, e, and f) computed using equation 3.9, are stored. Finally, collage theorem is used to get the coded image from these parameters. Thus instead of storing the image block its corresponding fractal codes are stored. This tremendously reduces the storage space leading to a large compression ratio.

![Fig 3.9 Image divided into non overlapping range blocks and the most suitable domain block is found by searching the entire image](image)

The above method introduced by Barnsley has been modified by many mathematicians to reduce the time required to find the matching domain for a range
block. This fractal block coding forms the basis for modeling of mammograms used to identify the presence of microcalcifications described in chapter 5.

3.6 Literature Survey

Fractal objects which are generated from the mathematical theory of Iterated Sequences, were first “tagged” as mathematical “curiosities” or “monsters” by mathematicians in the beginning of the twentieth century.

Mathematicians Gaston Julia, Karl Weierstrass and Waclaw Sierpinski were among the first to explain the geometric properties of fractal (Fal 2003). They lacked the tools to properly analyze and understand them. They remained in nearly complete oblivion till they were rediscovered in the 1970’s. It began by the pioneering work of Benoit Mandelbrot who also coined the name fractals (Man 1982).

Benoit Mandelbrot, a French mathematician, is known as the father of fractal geometry. He was the first to confer the term fractal, and described these structures and his ideas in his book ‘The Fractal Geometry of Nature’ (Man 1982). His most famous contribution was a set of points in the complex plain, now known as the Mandelbrot set, which form an amazing fractal. One of the most impressive things about the development of this fractal, as with all early fractals, is that it was drawn by hand rather than generated with computing technology. Since this pioneering in fractal history, the Mandelbrot set has been drawn and redrawn time and time again, not for the sake of complex dynamics, but for the creation of art.

Due to availability of computers and automatic graphic tools, it was possible to render and visualize them as complex, beautiful, often realistic looking objects or scenes. Later on fractals have been a part of a set of tools in a variety of fields in physics, where they are closely related to chaos theory.

Fractals have blossomed tremendously in the past few years and have helped to reconnect pure mathematics research with both the natural sciences and computing. Fractal geometry and its concepts has become the central tool in most of the natural sciences: physics, chemistry, biology, geology, meteorology and materials science. There are lots of books available in literature describing the basics of fractals (Pei
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1988, Hel 2007, Tri 1997). Selection of the classical mathematical papers by masters like, G.Cantor, Sierpinski, B. Mandelbrot, Felix Hausdorff related to fractal geometry is compiled (Edg 2004). Wavelets and fractals were analyzed based on probability by Jorgensen (Jor 2006).

Barnsley (Bar 1988) explains the method of fractal image compression. The original approach of fractal image coding devised by Barnsley, became a practical reality when Jacquin (Jaq 1992, Jaq 1993) introduced the partitioned iterated function system in that each of the individual mappings operates on a subset of the image, rather than the entire image. Franceschetti and Riccio (Fra 2007) mention the fundamentals of stochastic fractals. The chaos theory was detailed by Stupnicka in (Stu 2003) and Schroeder (Sch 1991).

Iftekharuddin et. al (Ift 2003) implemented an improved Piecewise Modified Box Counting (PMBC) and Piecewise Triangular Prism Surface Area (PTPSA) methods, to find the differences in intensity histogram and fractal dimension between normal and tumor images to detect and locate the tumor in the brain MR images.

A method incorporating gray relational pattern analysis into the self-organizing feature maps (SOFM) network to develop a GSOFM network to speed up the encoding time to about 500 was proposed by Jianwei and Jinguang (Jnw 2008).

Berizzi et. al (Ber 1997) proposed a two-dimensional fractal model of the sea surface, by means of the solution of the sea hydrodynamic differential equations and based on the band-limited Weierstrass-Mandelbrot (WM) fractal functions. Fractal geometry is used to take into account of the multi scale nature of the sea and to give a better description of the fine structure of the sea surface and they have also determined a directional wave spectrum of the sea.

Kinsner et. al. (Kin 2009) describes a novel approach of fractal modeling and coding of residuals for excitation in the linear predictive coding of speech. The new speech coder implemented using the piecewise self-affine fractal model gave a signal-to-noise ratio of 10.9 dB.

A new type of turbulence model based on fractals, applicable both in a Reynolds averaged Navier–Stokes (RANS) and a large-eddy simulation (LES) formulation which assumes an isotropic behavior for the turbulent viscosity was
developed by Giacomazzi et.al (Gia 1999). This can be applied for simulating
turbulent combustion irrelevant of its mode (premixed or non-premixed) and was able
to turn itself off in the laminar zones of the flow, and in particular near walls.

Partial discharge (PD) occurring in XLPE power cables is a cryptic
phenomenon with detectable features differing in a thousand ways. The authors have
investigated the use of fractal features for recognition of 3-d PD patterns as a fractal
surface and it was found that two fractal features, fractal dimension and self-
similar characteristic, possess reasonably good pattern recognition (Luo 2002). The
study of the nature of the fractal features of 3-d PD patterns provided an
efficient method for recognizing and picking-up the faint PD pulse from noise
based on fractal theory.

Zho et.al (Zho 2010) proposed a multirange fractal model to calculate
transition curves of multirange fractals by utilizing relevant fractal dimensions of Sn
melt at different temperatures.

Most natural images, such as geographical images, are all textural in nature.
In remote sensing images, different regions possess different texture and have
different multifractal exponents. These properties were utilized for image
segmentation by Du and Yeo (Du 2002).

In astrophysics, structures with fractal characteristics are important. A novel
approach to characterize prefactors of cover functions, like lacunarity, based on the
formalism of regular variation (in the sense of Karamata) is proposed by Stern (Ste
1997). This approach allowed in deriving bounds on convergence rates for scaling
exponent algorithms and provided a more precise characterization for fractal-like
objects of interest for astrophysics.

Deering and West (Dee 1992) related the complexity of physiological
structures like cochlea and lungs to fractal geometry.

A watermarking method which utilizes a special type of orthogonalization
fractal coding method was proposed by Pi et. al (PiH 2006). Here, the watermark
embedding procedure inserts a permuted pseudo-random binary sequence into the
quantized range block means and was found to be robust against common signal and
geometric distortion such as JPEG compression, low-pass filtering, rescaling, and clipping.

Winding currents in transformers vary to different extent depending upon the type of impulse fault. Fractal analyses using features like fractal dimension and lacunarity, of such complex current waveforms have been reported by Purkait and Chakravorti (Pur 2003) for classification of impulse faults in transformers. Experimental results obtained for a 3 MVA transformer and simulation results obtained for 3 MVA, 5 MVA and 7 MVA transformers are presented.

The application of fractals in stock markets is explained by Mandelbrot (Man 1997). Losa et. al (Los 2005) presented the biological and medical applications of fractals.

The fractal nature of the imaging modalities that measure flow vector fields (flow-sensitive MRI and Doppler ultrasound) were modeled by Tafti et. al (Taf 2010) using FBM along with vector models then used them to analyze 3D flow measurements obtained using phase-contrast MRI.

Thus, it can be seen that fractals find applications in a various fields. In the present research, the fractal dimension is used for the classification of mammograms while the self similarity property is used for fractal modeling of mammograms.

3.7 Chapter Summary

The new word fractal was introduced by Mandelbrot to encompass all the complex geometric shapes. Unlike Euclidean shapes, fractals have dimension which is non integer or fractional. This chapter provides a brief introduction to fractals. The basic concept and the theoretical background of fractals are presented. Literature survey of the application of fractals in diverse fields is also incorporated here.