Chapter 4

Modifying some foliated dynamical systems to guide their trajectories to specified submanifolds

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4.1 Introduction

The main purpose of this chapter is to modify some foliated dynamical systems so that their trajectories eventually reach certain pre-specified submanifold. A formal definition of foliation is given in [4]. An \( n \)-dimensional foliated manifold is composed of disjoint \( d \) dimensional \( (n > d) \) submanifolds. Each of these submanifolds is called a leaf of the foliation. Each leaf is a connected and invariant manifold. A common example of a topological foliation is a fiber bundle. Reference [39] gives an interesting example which bears some relevance to dynamical systems. If \( G \) is a Lie group, and \( H \) is a subgroup obtained by exponentiating a closed subalgebra of the Lie algebra of \( G \), then \( G \) is foliated by the cosets of \( H \) [39].

We define a dynamical system as foliated when its state phase portrait (plot of the multiple trajectories corresponding to different initial conditions) is foliated. We can illustrate this by considering the example of a harmonic oscillator with a differential equation given by \( \frac{dx_0}{dt} = x_1, \frac{dx_1}{dt} = -x_0 \). Let us consider the solutions of this equation for all possible initial conditions. Thus, for example, if we choose \( x_0 = 3 \) and \( x_1 = 4 \) as initial conditions, our solution will be a circle of radius 5. It is clear that for any sets of initial conditions, the sum of whose squares is 25 will lead to the solutions which lie on the same circle. However, if the sum of the squares is,
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say 30, we get a circle which is concentric to the first one but of a larger radius. Now if we consider the all possible solutions, it is clear that we have a set of concentric circles. These circles are one of the simplest examples of foliation.

In this chapter we look at some \( n \)-dimensional dynamical systems that are foliated by the choice of the initial conditions. In all of these cases each leaf of the foliation is \( d \)-dimensional \((n > d)\). For each leaf there is an equivalence class of the initial conditions such that if we start from any point of that equivalence class, the resulting trajectory will remain confined to a \( d \)-dimensional submanifold shared by all other initial conditions in that class. Now if we start from a point of some different equivalence class of the initial conditions, we will be confined to a different leaf. Our goal is to modify the dynamical system so that from wherever we start we always reach or converge to the same pre-specified leaf. An important constraint is that the original system and the modified system should remain identical on the target leaf. This is because the data that we wish to model in many cases, originates only on that leaf and the model must achieve stability while remaining faithful to the data.

This kind of problem arises when we embed a system into a higher dimension. For getting equations from a given time series it is very common to embed the system into a higher dimension \[41\]. One of the most promising methods in this respect is Taken’s embedding \[55\]. We have shown in reference \[61\] how embedding may create a foliated dynamical system. Foliation can also arise if the differential equation of a dynamical system is extended as, for example, in the case of applications to cryptography \[60\]. A related situation arises in reference \[52\]. In reference \[52\] some general conditions are found that lead to spatially localized periodic oscillations in networks of coupled oscillators. For conservative systems periodic solutions typically form a one-parameter family. To specify a single periodic solution, one extra equation is required which yields an overdetermined system. Reference \[44\] shows an algorithm by which this problem can be solved. In this chapter we propose a method by which for some dynamical systems we can asymptotically guide the trajectory to a pre-specified submanifold. Perhaps, the most important application of this method is to the problems of Mathematical Modeling. In this field it is often required to arrive at a set of differential equations or maps that fit a given observational data. In many problems, data points lie in a low dimensional manifold, but the model is necessarily of a higher dimension in which that low dimensional manifold is embedded. There are two problems with this. The first is that the data can validate only the behavior that corresponds to the submanifold. Therefore these models can be seen as not being faithful to the data. The second problem, which is actually a possible consequence of the first problem, is that the predicted behavior of these equations can often be unstable. Under this circumstance, we need to stabilize the system and confine it to the proper submanifold. Another important application is demonstrated at the end of this chapter. This is the problem of controlling chaos. In the next section we discuss how we can modify a foliated system so that the foliation will be collapsed. In the subsequent sections, we demonstrate few practical applications of these ideas by choosing, at first, a very simple example.
of the harmonic oscillator, followed by a case of a dynamical system generated by
the Duffing’s oscillator. In each case, we find a modification of the system such that
no matter what initial conditions we choose, the system will asymptotically reach a
specific leaf of the foliation. Our choice of the 4-dimensional form of the Duffing’s
oscillator is an unusual one. This system contains some leaves which are periodic
and some which are chaotic. We show that we can control the system to move it
from periodic to chaotic or from chaotic to periodic.

4.2 Modifying dynamical system when foliation is created by embedding:

It has been shown that if we embed a dynamical system into a higher dimension it
may create a foliation [61]. Let us consider the 2D system
\[ \frac{dx_0}{dt} = x_1, \quad \frac{dx_1}{dt} = f(x_0, x_1) \] \( \text{(2.1)} \)
We want to embed it into 3D
\[ \frac{dx_0}{dt} = x_1, \quad \frac{dx_1}{dt} = f(x_0, x_1), \quad \frac{dx_2}{dt} = \frac{df}{dt} \] \( \text{(2.2)} \)
If we integrate the last equation of (2.2) we get
\[ x_2 = f(x_0, x_1) + a_2 - f(a_0, a_1) \] where \( (a_0, a_1, a_2) \) are the initial conditions. So we get a foliation as
\( a_2 \) varies. For each \( a_2 \), we get a leaf of the foliation. Now we want to modify the
system (2.2) in such a way that this foliation gets collapsed. Let us consider the set
of equations
\[ \frac{dx_0}{dt} = x_1, \quad \frac{dx_1}{dt} = f(x_0, x_1), \quad \frac{dx_2}{dt} = \frac{df}{dt} + \lambda (x_2 - f(x_0, x_1)) \] \( \text{(2.3)} \),
where \( \lambda \) is less than 0.

Let \( g(x_0, x_1, x_2) = x_2 - f(x_0, x_1). \)
So,
\[ \frac{dg}{dt} = \frac{dx_2}{dt} - \frac{df}{dt} = \lambda g \]
Hence, \( g(t) = g(0)e^{\lambda t} \). Now lambda is less than 0, so \( g \) tends to 0 as \( t \) tends
to infinity. Therefore, for this system if we obtain a numerical solution and observe
its long term behavior, we find that we eventually reach the leaf \( x_2 - f(x_0, x_1) = 0 \),whatever our initial conditions are. On the other hand, in the case of the system
(2.2) initial conditions decide in which leaf we are going to reach.

4.3 Guiding trajectories of harmonic oscillator to a specific leaf:

We have discussed in section 4.1 that harmonic oscillator
\[ \frac{dx_0}{dt} = x_1, \quad \frac{dx_1}{dt} = -x_0 \] \( \text{(3.1)} \)
is a foliated dynamical system. Choice of the initial condition gives rise to a leaf
\( S_1 \). For example, if we choose \( x_0 = 3 \) and \( x_1 = 4 \) as our initial conditions, we get
a circle of radius 5. Now, if we choose some other initial conditions we once again
begin tracing the same circle, only if the sum of the squares of the initial conditions
is 25. In general, however, we generate many different concentric circles, as we
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keep choosing a variety of the initial conditions and keep finding the trajectories generated by them. Our goal is to modify the system in such a way that we always reach a pre-specified circle, say, \( g(x_0, x_1) = x_0^2 + x_1^2 - 25 = 0 \) even if our initial condition lies outside of the circle. Let us consider the following system,

\[
\begin{align*}
\frac{dx_0}{dt} &= x_1 + \lambda_1 gx_0, \\
\frac{dx_1}{dt} &= -x_0 + \lambda_2 gx_1
\end{align*}
\] (3.2)

where \( g = x_0^2 + x_1^2 - 25 \) and \( \lambda_1 \) and \( \lambda_2 \) are negative. Multiplying the first equation of 3.2 by \( x_0 \) and the second equation of 3.2 by \( x_1 \) we get

\[
\frac{d}{dt}(x_0^2 + x_1^2 - 25) = 2g\lambda_1 x_0^3 + 2g\lambda_2 x_1^3,
\]

i.e. \( \frac{dg}{dt} = \lambda g \), where \( \lambda = 2(\lambda_1 x_0^2 + \lambda_2 x_1^2) \).

So \( \lambda \) is a function of \( x_0 \) and \( x_1 \) and hence it is a function of time \( t \). Integrating the above equation we get

\[
g = g(0) \exp(\int_0^t \lambda dt).
\]

The Jacobian matrix of this system at the origin is

\[
\begin{pmatrix}
-25\lambda_1 & 1 \\
-1 & -25\lambda_2
\end{pmatrix}
\]

The eigenvalues are the solutions of the following equation \( x^2 + 25(\lambda_1 + \lambda_2)x + (625\lambda_1\lambda_2 + 1) = 0 \). The sum of the two eigen values is \( -25(\lambda_1 + \lambda_2) \) which is positive and the product of the two eigen values is \( (625\lambda_1\lambda_2 + 1) \) which is also positive. Hence both the eigenvalues of the Jacobian matrix of this system at the origin are greater than 0. So the origin is a repeller of this system. Since \( \lambda_1 \) and \( \lambda_2 \) are negative, \( x_0 \) and \( x_1 \) are not simultaneously 0 and the origin is a repeller, the value of \( \lambda \) will be strictly less than 0. Hence \( \int_0^t \lambda dt \) tends to \( -\infty \) as \( t \) tends to \( \infty \). So \( g \) tends to 0 as \( t \) tends to infinity. As a consequence we eventually reach the leaf \( g = 0 \), even when we start outside of the leaf.

Figure 4.1: left: trajectory of harmonic oscillator when initial condition is (2,3), right: trajectory of modified harmonic oscillator when initial condition is (2,3)

Empirically in the Runge Kutta method, we choose \( x_0 = 2 \) and \( x_1 = 3 \) as our initial conditions and the values of \( \lambda_1 \) and \( \lambda_2 \) are - 0.2 and - 0.3 respectively. In the case of the harmonic oscillator (3.1) we reach a circle of radius 3.6, but in system (3.2) when we start from the same initial conditions we reach a circle of radius 5, i.e. \( g = 0 \) (fig 4.1). In fact in system (3.2) we always converge to \( g = 0 \) whatever
4.4 Guiding trajectories of Duffing system to a specific leaf:

Duffing is a 2-dimensional non-autonomous system given by the following equation
\[ \frac{dy_0}{dt} = y_1, \quad \frac{dy_1}{dt} = -cy_1 - ky_0 - \delta y_0^3 + F \cos(\omega t + \alpha). \]...(4.1).

It is known to us that this equation shows chaos for the following values of the parameters: \( c = 0.044964, k = 0, \delta = 1, \omega = 0.44964, \alpha = 0 \) and \( F = 1.02 \). It is well known that keeping the other parameters fixed, if we change \( F \) we see that the system is periodic when \( F \) is 0.2. If \( F \) is 1.02, this system becomes chaotic (as empirically found in figure 4.2). As \( F \) varies we get different trajectories in the state space picture of Duffing. So we can treat \( F \) as a parameter of the foliation.

We show that we can modify this system in such a way that we eventually reach a specific periodic orbit even if we start from an orbit which displays chaotic behavior. It has been shown that the 2-dimensional non-autonomous Duffing equation can be extended to a 4-dimensional autonomous equation [60]. The equation is given by
\[ \frac{dy_0}{dt} = y_1, \quad \frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = y_3, \quad \frac{dy_3}{dt} = -cy_3 - ky_2 - 3\delta y_0^2 y_2 - 6\delta y_0 y_1^2 - \omega^2 y_2 - \omega^2 cy_1 - \omega^2 ky_0 - \omega^2 \delta y_0^3 \]...(4.2).

Since \( \frac{dy_1}{dt} = y_2 \), from (4.1) we get
\[ F \cos(\omega t + \alpha) = y_2 + cy_1 + ky_0 + \delta y_0^3 \]...(4.3).

By differentiating (4.3) we get
\[ F \omega \sin(\omega t + \alpha) = -cy_2 - ky_1 - 3\delta y_0 y_1 - y_3 \]...(4.4).

We can have arbitrary choice of the initial conditions for \( y_0 \) and \( y_1 \) in system (4.2). But we have to select IC for \( y_2 \) and \( y_3 \) by using the relations (4.3) and (4.4). If \( Y_0, Y_1, Y_2 \) and \( Y_3 \) are the values of \( y_0, y_1, y_2 \) and \( y_3 \) at \( t = 0 \), from these relations we get
\[ Y_2 = F \cos(\alpha) - (cY_1 + kY_0 + \delta Y_0^3), \quad Y_3 = -F \omega \sin(\alpha) + (-cY_2 - kY_1 - 3\delta Y_0^2 Y_1). \]

In this chapter we want to propose an alternative form of the 4-dimensional Duffing equation. From (4.1) we get...
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\[ F \cos(\omega t + \alpha) = \frac{dw_1}{dt} + cw_1 + kw_0 + \delta w_0^3. \]

We define \( w_2 \) as \( w_2 = F \cos(\omega t + \alpha) \).

Then \( \frac{dw_1}{dt} = w_2 - (cw_1 + kw_0 + \delta w_0^3) \), and further let \( w_3 = \frac{dw_2}{dt} = -\omega F \sin(\omega t + \alpha) \).

Then \( \frac{dw_3}{dt} = -\omega^2 w_2 \).

So we get a new system

\[ \frac{dw_1}{dt} = w_1, \quad \frac{dw_2}{dt} = w_2 - (cw_1 + kw_0 + \delta w_0^3), \quad \frac{dw_3}{dt} = w_3, \quad \frac{dw_3}{dt} = -\omega^2 w_2. \quad \text{...(4.5).} \]

It will be equivalent to (4.1) if at \( t = 0 \), we take \( w_2 = F \cos(\alpha) \), \( w_3 = -\omega F \sin(\alpha) \). Hence we can conclude that system (4.5) with the initial condition \((W_0, W_1, W_2, W_3)\) is equivalent to the system (4.1) with the initial condition \((Y_0, Y_1)\), where \( W_0 = Y_0, W_1 = Y_1, W_2 = F \cos(\alpha), W_3 = -\omega F \sin(\alpha) \), \( F \) is the same as in (4.1). From the definition of the initial conditions of \( W_2 \) and \( W_3 \) we get \( W_2^2 + \frac{W_2^2}{\omega^2} = F^2 \). System 4.5 will show chaotic behavior if we choose \( F \), and hence \((W_2^2 + \frac{W_2^2}{\omega^2})\) as 1.02 and it will show periodic behavior when \( F \) is 0.2. So the choice of our initial conditions \((W_2 \) and \( W_3)\) decides in which leaf we will reach. Now let us consider the following system

\[ \frac{dw_1}{dt} = w_1, \quad \frac{dw_2}{dt} = w_2 - (cw_1 + kw_0 + \delta w_0^3), \quad \frac{dw_3}{dt} = w_3 + \lambda_1 w_2 (w_2^2 + \frac{w_2^3}{\omega^2} - G^2), \quad \frac{dw_3}{dt} = -\omega^2 w_2 + \lambda_2 w_3 (w_2^2 + \frac{w_2^3}{\omega^2} - G^2). \quad \text{...(4.6),} \]

where \( G = 0.2 \) and \( \lambda_1 \) and \( \lambda_2 \) are negative.

Let \( g = w_2^2 + \frac{w_2^3}{\omega^2} - G^2 \). So from 4.6 we can write

\[ \frac{dg}{dt} = \frac{w_3}{\omega^2} - \frac{\lambda w_2^3}{\omega^2} = -\omega^2 w_2 + \lambda_2 w_3 g. \]

Multiplying the first equation by \( w_2 \) and the second equation by \( w_3/\omega^2 \) we get

\[ \frac{d}{dt}(w_2^2 + \frac{w_2^3}{\omega^2} - G^2) = 2(\lambda_1 w_2^2 + \frac{\lambda_2 w_2^3}{\omega^2}) g. \]

i.e. \( \frac{dg}{dt} = 2\lambda g \). So \( \lambda = \lambda_1 w_2^2 + \frac{\lambda_2 w_2^3}{\omega^2} \). So \( \lambda \) is a function of \( w_2 \) and \( w_3 \) and hence it is a function of time \( t \). Integrating the above equation we get

\[ g = g(0) \exp(\int_0^t \lambda \, dt). \]

The Jacobian matrix of this system at the origin is

\[ \begin{pmatrix} -G^2 \lambda_1 & 1 \\ -1 & -G^2 \lambda_2 \end{pmatrix} \]

The eigenvalues are the solutions of the following equation

\[ x^2 + G^2(\lambda_1 + \lambda_2)x + (G^4 \lambda_1 \lambda_2 + 1) = 0. \]

The sum of the two eigen values is \(-G^2(\lambda_1 + \lambda_2)\) which is positive and the product of the two eigen values is \((G^4 \lambda_1 \lambda_2 + 1)\) which is also positive. Hence both the eigenvalues of the Jacobian matrix of this system at the origin are greater than 0. So the origin is a repeller of this system. Since \( \lambda_1 \) and \( \lambda_2 \) are negative, \( x_0 \) and \( x_1 \) are not simultaneously 0 and the origin is a repeller, the value of \( \lambda \) will be strictly less than 0. Hence \( \int_0^t \lambda \, dt \) tends to \(-\infty\) as \( t \) tends to \( \infty \).

So \( g \) tends to 0 as \( t \) tends to infinity. As a consequence we eventually reach \( g = 0 \), i.e. \( w_2^2 + \frac{w_2^3}{\omega^2} - G^2 = 0 \), even if the initial conditions of \( W_2 \) and \( W_3 \) were chosen in such a way that \( w_2^2 + \frac{w_2^3}{\omega^2} = F^2 \), where \( F \) is not equal to \( G \).

We run the Runge Kutta method for both system (4.5) and (4.6). We take \( F = 1.02 \) for both (4.5) and (4.6) for choosing initial conditions \( W_2 \) and \( W_3 \). We choose rest of the parameters the same as before for both the systems. For system
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(4.6) we take the value of $\lambda_1$ as -0.002 and the value of $\lambda_2$ as -0.003. We take $10^7$ points in the method. If we plot the first 2800 points we get almost similar pictures for both the system (4.5) and system (4.6) as figure 4.3 reveals. Then we plot the last 60000 points. As we see in figure 4.4, in the case of (4.5) we get a chaotic orbit, whereas in the case of (4.6) we get a periodic orbit. In system (4.5) we take $F$ as 1.02 and it shows chaos. In system (4.6) too, we take $F = 1.02$, so initially it gives a similar trajectory as that of system (4.5). But as we have taken both $\lambda_1$ and $\lambda_2$ negative, $(w_2^2 + w_3^2 - G^2)$ tends to 0 eventually. As a consequence, the long term behavior of the system is governed not by the value of $F$, but by the value of $G$. Since, in this example, we have chosen $G = 0.2$, the long term behavior is similar to that generated by the system (4.5) when $F$ for that system is 0.2. At $F = 0.2$ the conventional Duffing system (given by the equation 4.5) is periodic, so the system (4.6) ends up showing periodic behavior. Thus, the main interesting characteristic of the system (4.6) is that it initially resembles a system which is chaotic, but at the end, it converges to a periodic orbit.

Figure 4.3: left: initial behavior of the trajectory of system (4.5), right: initial behavior of the trajectory of system (4.6)

Figure 4.4: left: At the end, system (4.6) shows periodic behavior, right: At the end, system (4.5) still acts chaotically
4.5 Conclusion:

In this chapter we have discussed foliation and how it appears in some practical problems. We have shown how we can guide the trajectory of a foliated dynamical system to a specific leaf of the foliation. We have demonstrated this by taking the examples of the harmonic oscillator and the Duffing’s oscillator. In each of these cases we find an alternative set of equations which represents a collapsed foliation, such that no matter what initial conditions we choose, the system will asymptotically reach the same desired submanifold. We have shown that this can be achieved while making sure that the original system and the modified system remain identical on the target leaf. This is because the data that we wish to model in many cases originates only from that leaf and the model must achieve stability while remaining faithful to the data.

In the case of the Duffing system, as parameter $F$ varies we get different trajectories. These different trajectories exhibit different qualitative behavior. Some of them are periodic whereas some of them are chaotic. We have shown how we can create a trajectory which acts chaotically in the beginning but eventually exhibits periodic behavior. So one of the important aspects of this method is that it shows the control of chaos. Thus it is very easy to synthesize the cases in which a system begins in a chaotic region, but is guided to a periodic region and vice versa. It could also happen that we move from an orbit of one period to an orbit of another period.