CHAPTER 5

Neutrinoless double beta decay in $SO(10)$ model

The left-right symmetric gauge theories based on $G_{224D}$ [46,47] and $G_{2213D}$ [48,49,55] with $g_{2L} = g_{2R}$ may suggest the origin of parity restoration [228]. The detection of $W_R$ boson at LHC is likely to resolve the mystery of parity violation in weak interaction. With this possibility we work on a class of LR model with TeV scale $W_R, Z'$ but having parity restoration at high scale where they originate from well known Pati-Salam symmetry or $SO(10)$ GUT [52, 54]. The canonical and type-II seesaw [59–67] emerge naturally from $SO(10), G_{224D}$ and $G_{2213D}$ gauge theories provided both LH and RH neutrinos are Majorana fermions. Usually the scale of operation of conventional seesaw is far beyond the experimental reach. Therefore, we minimally extended the model under study to accommodate TeV scale inverse seesaw frame work for neutrino masses.

Currently a number of dedicated experiments on $0\nu2\beta$-decay are in progress [229–239] while a part of the Heidelberg-Moscow experiment [229, 236–239] has already claimed to have measured the effective mass parameter $|m_{ee}| \simeq (0.23 - 0.56)$ eV. This observation might be hinting towards the Majorana nature of the light neutrinos [240]. Its worthwhile to study such prospects under GUT framework. The Dirac mass matrix $M_D$ predicted using underlying quark-lepton symmetry $G_{224D}$ of $SO(10)$ would play crucial role in determining above and other results on non-unitarity and LFV.

The chapter is organized as follows: in Sec. 5.1 we briefly discuss the TeV scale left-right gauge theory with low-mass $W_R$, $Z'$ bosons, light neutrino masses and associated non-unitarity effects. In Sec. 5.2, we present various Feynman amplitudes for neutrinoless double beta decay; in Sec. 5.3, we give a detailed discussion for
standard and non-standard contributions to the effective mass parameter for 0ν2β decay rate and in Sec. 5.4, we have discussed the branching ratios for lepton flavor violating decays. In Sec. 5.5, we implement the idea in a SO(10) grand unified theory and derive Dirac neutrino mass matrix at the TeV scale.

5.1 Low scale left-right gauge theory and extended seesaw

5.1.1 The Model

Besides the standard 16-fermions per generation including the RH neutrino, we add one additional sterile fermion singlet for each generation, as in inverse seesaw. We start with parity conserving left-right symmetric gauge theory, $G_{224D}$ [46, 47] or $G_{2213D}$ [48, 49, 55], with equal gauge couplings ($g_{2L} = g_{2R}$) at high scales. In the Higgs sector we need both LH and RH triplets ($\Delta_L, \Delta_R$) as well as the LH and RH doublets ($\chi_L, \chi_R$) in addition to bi-doublet ($\Phi$) and $D$-parity odd singlet ($\sigma$) [56, 57]. Their transformation properties, can be checked in tables given in the Tab. B.10, under $G_{224D} \supset G_{2213D}$ are

\[
\begin{align*}
\sigma(1, 1, 1) & \supset (1, 1, 0, 1), & \Phi(2, 2, 1) & \supset (2, 2, 0, 1), \\
\Delta_L(3, 1, 10) & \supset (\Delta_L(3, 1, -1, 1), & \Delta_R(1, 3, \overline{10}) & \supset (\Delta_R(1, 3, -1, 1), \\
\chi_L(2, 1, 4) & \supset (\chi_L(2, 1, -1/2, 1), & \chi_R(1, 2, \overline{4}) & \supset (\chi_R(1, 2, -1/2, 1). \tag{5.1}
\end{align*}
\]

When $D$-parity odd singlet $\sigma$ acquires a VEV $\langle \sigma \rangle \sim M_P$, the LR discrete symmetry is spontaneously broken but the gauge symmetry $G_{2213}$ remains unbroken leading to $M_{\Delta_R}^2 = (M_\Delta^2 - \lambda_\Delta \langle \sigma \rangle M')$, $M_{\chi_R}^2 = (M_\chi^2 - \lambda_\chi \langle \sigma \rangle M')$, where $\lambda_\Delta, \lambda_\chi$ are trilinear couplings and $\langle \sigma \rangle, M', M_\Delta, M_\chi$ are all $\mathcal{O}(M_P)$, the RH Higgs scalar masses are made lighter depending upon the degree of fine tuning in $\lambda_\Delta$ and $\lambda_\chi$. The asymmetry in the Higgs sector causes asymmetry in the $SU(2)_L$ and $SU(2)_R$ gauge couplings with $g_{2L}(\mu) > g_{2R}(\mu)$ for $\mu < M_P$. If one wishes to have $W_R, Z'$ mass predictions at nearly the same scales and generate Majorana neutrino masses, it is customary to break $G_{2213} \rightarrow$ SM by the VEV of the right handed triplet $\langle \Delta_R^0 \rangle \sim v_R$. We rather suggest a more appealing phenomenological scenario with $M_{W_R} > M_{Z'}$, for which two step breaking of the asymmetric gauge theory to the SM is preferable: $G_{2213} \rightarrow_{M_R^+}^\rightarrow$
\( G_{2113} \xrightarrow{M_R^0} SM \), where the first step of breaking that generates massive \( W_R \) bosons is implemented through the VEV of the heavier triplet \( \sigma(1,3,0,1) \) carrying \( B-L=0 \) and the second step of breaking is carried out by \( \langle \Delta_R^0 \rangle \sim v_R \). At this stage the RH neutral gauge boson gets mass which is kept closer to the current experimental lower bound \( M_{Z'} \geq 1.162 \) TeV for its visibility by high energy accelerators. We further gauge the extended seesaw mechanism at the TeV scale for which the VEV of the RH-doublet \( \langle \chi_R^0 \rangle = v_\chi \) provides the \( N-S \) mixing. The \( G_{2113} \) symmetric low-scale Yukawa Lagrangian is

\[
\mathcal{L}_{\text{Yuk}} = Y_l \bar{\psi}_L \psi_R \Phi + f \psi_R^C \psi_R \Delta_R + F \bar{\psi}_R S \chi_R + S^T \mu S + h.c. \tag{5.2}
\]

which gives rise to the 9×9 neutral fermion mass matrix after electroweak symmetry breaking, discussed in the next subsection.

### 5.1.2 Extended inverse seesaw

The effective neutrino mass matrix in \( |\nu| = (\nu, S, N_R^C)^T \) basis is expressed as

\[
\mathcal{M}_\nu = \begin{pmatrix} 0 & 0 & M_D \\ 0 & \mu_S & M \\ M_D^T & M^T & M_N \end{pmatrix} \tag{5.3}
\]

where \( M_D = Y^l \langle \Phi \rangle \), \( M_N = f \langle \Delta_R^0 \rangle \) and \( M = F \langle \chi_R^0 \rangle \). The neutrino mass matrix \( M_D \) and the \( S-N \) mass matrix \( M \) are in general \( 3 \times 3 \) complex matrices in flavor space while \( \mu_S \) and \( M_N \) are \( 3 \times 3 \) complex symmetric. In the limit \( \mu_S, M_D << M_N \), the heaviest right handed neutrinos can be integrated out from the Lagrangian so that the effective Lagrangian becomes \( [159, 160] \)

\[
-\mathcal{L}_{\text{eff}} = (M_D M_N^{-1} M_D^T)_{\alpha \beta} \nu_{\alpha}^T \nu_{\beta} + (M_D M_N^{-1} M)_{am} (\bar{\nu}_\alpha S_m + \bar{S}_m \nu_\alpha) + (M^T M_N^{-1} M)_{mn} S_m S_n - \mu_S S_m S_n \tag{5.4}
\]

The effective mass term of the above Lagrangian

\[
\mathcal{M}_{\text{eff}} = - \begin{pmatrix} M_D M_N^{-1} M_D^T & M_D M_N^{-1} M \\ M_D M_N^{-1} M_D^T & M^T M_N^{-1} M - \mu_S \end{pmatrix} \tag{5.5}
\]
can be block diagonalized giving the light and sterile neutrino masses

\begin{align}
    m_\nu &\sim M_D M^{-1}_N \mu_S (M_D M^{-1})^T, \\
    m_S &\sim \mu_S - M M_N^{-1} M^T,
\end{align}

(5.6)

(5.7)

to the leading approximation. Note that type-I seesaw like terms \( M_D M^{-1}_N M_D^T \) cancelled out and the inverse seesaw formula \([143, 144, 241]\) emerges. This was possible only under assumption \( \mu_S \ll M M_N^{-1} \). On the other hand if \( \mu_S \gg M M_N^{-1} \), type-I seesaw dominates. The complete block diagonalization procedure of the matrix \( M_\nu \) is given in Appendix D.

In the third step, \( m_\nu, m_S \) and \( m_N \sim M_N \) are further diagonalized by the respective unitary matrices to give their corresponding eigenvalues

\begin{align}
    U_\nu^\dagger m_\nu U_\nu^* &= \hat{m}_\nu = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}), \\
    U_S^\dagger m_S U_S^* &= \hat{m}_S = \text{diag}(m_{S_1}, m_{S_2}, m_{S_3}), \\
    U_N^\dagger m_N U_N^* &= \hat{m}_N = \text{diag}(m_{N_1}, m_{N_2}, m_{N_3}).
\end{align}

(5.8)

The complete mixing matrix \([242–244]\) diagonalizing the matrix \((M_\nu)_{9\times9}\), given in eq. (5.3), can be expressed as

\begin{equation}
    \mathcal{V} \equiv \left( \begin{array}{ccc}
        \nu^{\nu\nu}_{\alpha i} & \nu^{\nu\nu}_{\alpha j} & \nu^{\nu\nu}_{\alpha k} \\
        \nu^{\nu\nu}_{\beta i} & \nu^{\nu\nu}_{\beta j} & \nu^{\nu\nu}_{\beta k} \\
        \nu^{\nu\nu}_{\gamma i} & \nu^{\nu\nu}_{\gamma j} & \nu^{\nu\nu}_{\gamma k}
    \end{array} \right)
\end{equation}

(5.9)

\begin{equation}
    \begin{pmatrix}
        (1 - \frac{1}{2} X X^\dagger) U_\nu \\
        -X^\dagger U_\nu \\
        y^* X^\dagger U_\nu
    \end{pmatrix}
    \begin{pmatrix}
        (X - \frac{1}{2} Z Y^\dagger) U_S \\
        (1 - \frac{1}{2} (X^\dagger X + YY^\dagger)) U_S \\
        -Y^\dagger U_S
    \end{pmatrix}
    \begin{pmatrix}
        Z U_N \\
        (Y - \frac{1}{2} X^\dagger Z) U_N \\
        (1 - \frac{1}{2} Y^\dagger Y) U_N
    \end{pmatrix}
\end{equation}

(5.10)

where \( X = M_D M^{-1}, Y = M M_N^{-1}, Z = M_D M_N^{-1} \) and \( y = M^{-1} \mu_S \).

5.1.3 Neutrino parameters and non-unitarity constraints on \( M \)

Using the constrained diagonal form of \( M \) as mentioned above, the mass matrix \( \mu_S \) is determined using the gauged inverse seesaw formula and neutrino oscillation data provided that the Dirac neutrino mass matrix \( M_D \) is also known. The determination of \( M_D \) at the TeV scale, basically originating from high-scale quark-lepton symmetry \( G_{224D} \) or \( SO(10) \) GUT, is carried out by predicting its value at the high scale from
Table 5.1: Experimental bounds of the non-unitarity matrix elements $|\eta_{\alpha\beta}|$ (column C0) and their predicted values for degenerate (column C1), partially-degenerate (column C2), and non-degenerate (column C3) values of $M = \text{diag} (M_1, M_2, M_3)$ as described in cases (a), (b) and (c), respectively, in the text.

<table>
<thead>
<tr>
<th>measure of non-unitarity</th>
<th>Expt. bound $[187-190]$</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\eta_{ee}</td>
<td>$</td>
<td>$2.0 \times 10^{-3}$</td>
<td>$3.5 \times 10^{-8}$</td>
</tr>
<tr>
<td>$</td>
<td>\eta_{e\mu}</td>
<td>$</td>
<td>$3.5 \times 10^{-5}$</td>
<td>$3.9 \times 10^{-7}$</td>
</tr>
<tr>
<td>$</td>
<td>\eta_{e\tau}</td>
<td>$</td>
<td>$8.0 \times 10^{-3}$</td>
<td>$9.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>$</td>
<td>\eta_{\mu\mu}</td>
<td>$</td>
<td>$8.0 \times 10^{-4}$</td>
<td>$4.7 \times 10^{-6}$</td>
</tr>
<tr>
<td>$</td>
<td>\eta_{\mu\tau}</td>
<td>$</td>
<td>$5.1 \times 10^{-3}$</td>
<td>$1.1 \times 10^{-4}$</td>
</tr>
<tr>
<td>$</td>
<td>\eta_{\tau\tau}</td>
<td>$</td>
<td>$2.7 \times 10^{-3}$</td>
<td>$2.7 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

This value of $M_D$ will be utilized for all applications discussed subsequently in this work including the fit to the neutrino oscillation data through the inverse seesaw formula, predictions of effective mass parameters in $0\nu2\beta$, computation of non-unitarity and CP-violating effects, and lepton flavor violating decay branching ratios.

The light active Majorana neutrino mass matrix is diagonalized by the PMNS mixing matrix $U_\nu$ such that $U_\nu^\dagger m_\nu U_\nu^* = \hat{m}_\nu = \text{diag} (m_{\nu_1}, m_{\nu_2}, m_{\nu_3})$. The non-unitarity matrix at the leading order is still $N \approx (1 - \eta) U_\nu$, see Appendix D for details.

Thus $\eta$ is a measure of deviation from unitarity in the lepton sector on which there has been extensive investigations in different models $[45, 187-197, 245, 246]$. Assuming $M$ to be diagonal for the sake of simplicity, $M \equiv \text{diag}(M_1, M_2, M_3)$, gives $\eta_{\alpha\beta} = \frac{1}{2} \sum_k M_{D\alpha k} M_k^{-2} M_{D\beta k}^*$, but it can be written explicitly for the degenerate case...
\[
(M_1 = M_2 = M_3 = M_R)
\]

\[
\eta = \frac{1}{M_R^2} \begin{pmatrix}
0.0904 & 0.3894 - 0.9476i & 8.8544 - 22.7730i \\
0.3894 + 0.9476i & 12.1314 & 289.22 + 0.00005i \\
8.8544 + 22.7730i & 289.22 - 0.00005i & 6950.43
\end{pmatrix}
\]

(5.12)

Figure 5.1: The contours of \(M_1\) in the plane of \(M_2\) and \(M_3\). The solid curves in the diagram represent \(M_3\) dependence of \(M_2\) for fixed values of \(M_1\) using eq. (5.13). The brightest top-right corner suggests that lightest \(M_1\) may exist for largest values of \(M_2\) and \(M_3\).

For the non-degenerate diagonal matrix \(M\), saturating the experimental bound for \(|\eta_{\tau\tau}| < 2.7 \times 10^{-3}\) [245, 246] gives

\[
\frac{1}{2} \left[ \frac{0.170293}{M_1^2} + \frac{23.8535}{M_2^2} + \frac{13876}{M_3^2} \right] = 2.7 \times 10^{-3},
\]

(5.13)

where the three numbers inside the square bracket are in GeV\(^2\). The correlation between \(M_2\) and \(M_3\) is shown in Fig. 5.1 where the allowed region in the brightest top right corner suggests the possibility of lightest \(M_1\) for large values of \(M_2\) and \(M_3\). It is clear from eq. (5.13) that \(M_i\) can not be arbitrary. Rather they are ordered with \(M_3 > M_2 > M_1\) and also they are bounded from below with \(M_1 > 5.6\) GeV, \(M_2 > 66.4\) GeV, \(M_3 > 1.6\) TeV. In the degenerate case \(M_1 = M_2 = M_3 = 1604.4\) GeV. If we assume equal contribution to non-unitarity from all three terms in the left hand side of eq. (5.13), we get \(M = \text{diag}(9.7, 115.1, 2776.6)\) GeV. Besides these constraints, we have used the primary criteria \(M_N > M >> M_D, \mu_S\) where \(M_N \leq O(v_R)\), the \(G_{2113}\) breaking scale in choosing the elements of \(M\).

The elements of \(\eta\) have been listed in the Tab. 5.1 for (a) degenerate \(M=\text{diag}(1604.4,
Table 5.2: Structure of $\mu_S$ from neutrino oscillation data for NH of light neutrino masses, $m_\nu = (0.00127, 0.008838, 0.04978)$ eV and different mass pattern of $M$: (a) $M = (1604.44, 1604.44, 1604.44)$ GeV, (b) $M = (100.0, 100.0, 2151.5)$ GeV, and (c) $M = (9.72, 115.12, 2776.57)$ GeV.

Table 5.3: Same as Tab. 5.2 but for IH of light neutrino masses $\tilde{m}_\nu = (0.04901, 0.04978, 0.00127)$ eV.
\( \mu_S = X^{-1} U_\nu \tilde{m}_\nu U_\nu^T (X^T)^{-1} \)

\[
\begin{pmatrix}
3.48 + 1.51i & -9.02 - 3.93i & 9.18 + 0.14i \\
. & 23.41 + 10.23i & -23.84 - 0.38i \\
. & . & 20.67 - 8.25i
\end{pmatrix}
\times 10^{-4} \text{ GeV}, \quad (5.14)
\]

where we have used normal hierarchy (NH) for light neutrino masses, \( \tilde{m}_\nu = (0.00127, 0.00885, 0.0495) \) eV in the non-degenerate case of \( M = \text{diag}(9.72, 115.12, 2776.6) \) GeV. For the sake of completeness, we have presented few solutions of \( \mu_S \) matrix for degenerate, partially-degenerate and non-degenerate values of \( M \) as shown in the Tab. 5.2 and Tab. 5.3 corresponding to NH and inverted hierarchy (IH) light neutrino masses, respectively. For the quasi-degenerate (QD) pattern of light neutrino masses the matrix \( \mu_S \) can be easily derived and all our analyses carried out in Sec. 5.2 to Sec. 5.4 can be repeated.

### 5.2 Amplitudes for 0ν2β decay and effective mass parameters

In this section we discuss analytically the contributions of various Feynman diagrams in \( W_{\nu}^- - W_{\nu}^- \) channel (with two LH currents), \( W_{\nu}^- - W_{\nu}^- \) channel (with two RH currents), and \( W_{\nu}^- - W_{\nu}^- \) channel (with one LH and one RH current) and estimate the corresponding amplitudes in the TeV scale asymmetric LR gauge theory with extended seesaw mechanism.

The charged current interaction Lagrangian for leptons in this model in the flavor basis is

\[
\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \sum_{\alpha = e, \mu, \tau} \left[ \bar{\ell}_{\alpha L} \gamma_\mu \nu_{\alpha L} W_{\mu L}^\mu + \bar{\ell}_{\alpha R} \gamma_\mu N_{C \alpha R} W_{\mu R}^\mu \right] + \text{h.c.} \quad (5.15)
\]

Following the masses and mixing for neutrinos in the extended seesaw scheme discussed in Sec. 5.1.2, LH and RH neutrino flavor states are expressed in terms of mass eigenstates \((\tilde{\nu}_i, \tilde{S}_i, \tilde{N}_{C Ri})\)

\[
\nu_{\alpha L} \sim \mathcal{V}_{\alpha i}^{\nu L} \tilde{\nu}_i + \mathcal{V}_{\alpha i}^{\nu S} \tilde{S}_i + \mathcal{V}_{\alpha i}^{\nu N} \tilde{N}_{C Ri}, \quad (5.16)
\]

\[
N_{C \alpha R} \sim \mathcal{V}_{\alpha i}^{N L} \tilde{\nu}_i + \mathcal{V}_{\alpha i}^{N S} \tilde{S}_i + \mathcal{V}_{\alpha i}^{N N} \tilde{N}_{C Ri}. \quad (5.17)
\]

In addition, there is a possibility where left-handed and right-handed gauge bosons mix with each other and, hence, the physical gauge bosons are linear combinations...
of $W_L$ and $W_R$ as

$$
\begin{align*}
W_1 &= \cos \zeta_{LR} W_L + \sin \zeta_{LR} W_R \\
W_2 &= -\sin \zeta_{LR} W_L + \cos \zeta_{LR} W_R
\end{align*}
$$

(5.18)

with

$$
|\tan 2\zeta_{LR}| \sim \frac{v_u v_d}{v_R^2} \sim \frac{v_d g_{2R}^2}{v_u g_{2L}^2} \left(\frac{M_{WL}^2}{M_{WR}^2}\right) \leq 10^{-4}.
$$

(5.19)

As it is evident from the charged-current interaction given in eq. (5.15) and taking left- and right-handed gauge boson mixings into account given in eq. (5.18), there can be several Feynman diagrams which contribute to neutrinoless double beta decay transition in the TeV scale left-right gauge theory. They can be broadly classified as due to $W_L^--W_L^-$ mediation purely due to two left-handed currents, $W_R^--W_R^-$ mediation purely due to two right-handed currents, and $W_L^--W_R^-$ mediations due to one left-handed current and one right-handed current which are denoted by LL, RR, and LR in the superscripts of the corresponding amplitudes. These diagrams are shown in Fig. 5.2 - Fig. 5.5.

![Feynman diagrams for neutrinoless double beta decay](image)

Figure 5.2: Feynman diagrams for neutrinoless double beta decay ($0\nu 2\beta$) contribution with virtual Majorana neutrinos $\hat{\nu}_i$, $\hat{S}_i$, and $\hat{N}_{Ri}^C$ along with the mediation of two $W_L$-bosons.

### 5.2.1 $W_L^--W_L^-$ mediation

The most popular standard contribution is due to $W_L^--W_L^-$ mediation by light neutrino exchanges. But one of our major contribution in this work is that even with $W_L^--W_L^-$ mediation, the sterile neutrino exchange allowed within the extended seesaw mechanism of the model can yield much more dominant contribution to $0\nu 2\beta$ decay rate than the standard one. With the exchange of left-handed light neutrinos ($\hat{\nu}_i$), sterile neutrinos ($\hat{S}_j$), and RH heavy Majorana neutrinos ($\hat{N}_{Rk}^C$), the diagrams
shown in Fig. 5.2.(a), Fig. 5.2.(b), and Fig. 5.2.(c) contribute

\[ A_{LL}^L \propto \frac{1}{M_{W_L}^4} \sum_{i=1,2,3} \left( \frac{\nu_{ei}}{\nu_{ei}} \right)^2 m_{\nu_i} P_L, \]  
(5.20)

\[ A_{LL}^S \propto \frac{1}{M_{W_L}^4} \sum_{j=1,2,3} \left( \nu_{ej}^S \right)^2 m_{S_j} P_L, \]  
(5.21)

\[ A_{LL}^N \propto \frac{1}{M_{W_L}^4} \sum_{k=1,2,3} \left( \nu_{ek}^N \right)^2 m_{N_k} P_L, \]  
(5.22)

where \(|p^2| \approx (190 \text{ MeV})^2\) represents neutrino virtuality momentum [248–253].

![Feynman diagram for neutrinoless double beta decay contribution by $W_L^{-1}W_L^{\perp}$ mediation and by the exchange of virtual sterile neutrinos (S). The Majorana mass insertion has been shown explicitly by a cross.](image)

Figure 5.3: Feynman diagram for neutrinoless double beta decay contribution by $W_L^{-1}W_L^{\perp}$ mediation and by the exchange of virtual sterile neutrinos (S). The Majorana mass insertion has been shown explicitly by a cross.

To understand the origin and the role of the relevant Majorana mass insertion terms as source of \(|\Delta L| = 2\) lepton number violation in the new contribution to $0\nu2\beta$ process, we briefly discuss the example of sterile fermion (S) exchange corresponding to Fig. 5.2.(b) and Fig. 5.3. At first we note that, in contrast to the inverse seesaw framework with pseudo-Dirac type RH neutrinos [45, 246] where the only source of \(|\Delta L| = 2\) lepton number violation is $\mu_S$, in the present case of extended seesaw the Majorana mass for $S$ gets an additional dominant contribution $MM_S^{-1}M^T$ as shown explicitly in eq. (5.4) and eq. (5.7). The expanded form of the Feynman diagram with both the mass insertion terms is shown in Fig. 5.3 which gives

\[ A_{LL}^S \propto \frac{1}{M_{W_L}^4} P_L \left[ \frac{\nu_{ei}^S}{p - \hat{m}_S} \frac{1}{p - \hat{m}_S} \nu_{ei}^S \right] P_L, \]  
(5.23)
where we have used $m_S = \mu_S - M M_N^{-1} M^T$. Within the model approximation and allowed values of parameters, $|m_S| \simeq |M M_N^{-1} M^T| \gg |p| \gg |\mu_S|$ resulting in

$$A_{LL}^{S} \propto \frac{1}{M_{WL}^4} \left[ \nu^S \left( \frac{\mu_S}{m_S^2} + \frac{1}{m_S} \right) \nu^S \right]_{ee}, \quad (5.24)$$

where the first term is negligible compared to the second term, and we get eq. (5.21).

On the other hand, in the case of pseudo-Dirac RH neutrinos corresponding to $M_N = 0$ in eq. (5.3), the only Majorana mass insertion term in Fig. 5.3 is through $m_S = \mu_S$ with $|\mu_S| \ll |p|$. Then eq. (5.24) gives

$$A_{LL}^{S} \propto \frac{1}{M_{WL}^4} \left( \nu^S \right)^2 \mu_S \frac{m_\nu}{p^2} \approx \frac{1}{M_{WL}^4} \frac{m_\nu}{p^2}$$

which is similar to the standard contribution. This latter situation is never encountered in the parameter space of the present models.

![Figure 5.4](image)

Figure 5.4: Same as Fig. 5.2 but with $W_R-W_R$ mediation.

### 5.2.2 $W_R^{-}-W_R^{-}$ mediation

This contribution arising purely out of right-handed weak currents can also occur by the exchanges of $\hat{\nu}_i$, $\hat{S}_i$, and $\hat{N}_{C}^R$ and the corresponding diagrams are shown in Fig. 5.4.(a), Fig. 5.4.(b), and Fig. 5.4.(c) leading to the amplitudes

$$A_{LL}^{RR} \propto \frac{1}{M_{WL}^4} \left( \nu_{ei} \right)^2 \frac{m_\nu_i}{p^2}, \quad (5.25)$$

$$A_{S}^{RR} \propto \frac{1}{M_{WL}^4} \left( \nu_{ej} \right)^2 \frac{m_{S_j}}{m_{S_j}}, \quad (5.26)$$

$$A_{N}^{RR} \propto \frac{1}{M_{WL}^4} \left( \nu_{ej} \right)^2 \frac{m_{N_k}}{m_{N_k}}. \quad (5.27)$$
5.2.3 $W_L^- - W_R^-$ mediation

According to our observation, although these contributions arising out of mixed effects by the exchanges of light LH and heavy RH neutrinos and also by the exchange of sterile neutrinos are not so dominant compared to those due to $W_L^- - W_L^-$ mediation with sterile neutrino exchanges, as discussed in Sec. 5.2.1, the amplitudes are stronger than the standard one. The two types of mixed helicity Feynman diagrams are (i) $\lambda$-mechanism: coming from one left-handed and one right-handed current ($W_L^- - W_R^-$ mediation) shown in Fig. 5.5.(a), (ii) $\eta$-mechanism: arising because of additional possibility of $W_L^- - W_R^-$ mixing even though two hadronic currents are left-handed, as shown in Fig. 5.5.(b), leading to a suppression factor $\tan \zeta_{LR}$. The corresponding Feynman amplitudes for these mixed helicity diagrams are given below

\[
A^L_{\lambda} \propto \frac{1}{M^2_{W_L} M^2_{W_R}} \left( U_{\nu_i} \right)_{ei} \left( \frac{M_D}{M_N} \right)_{ei} \frac{1}{|p|}, \quad (5.28)
\]

\[
A^L_{\eta} \propto \frac{\tan \zeta_{LR}}{M^2_{W_L}} \left( U_{\nu_i} \right)_{ei} \left( \frac{M_D}{M_N} \right)_{ei} \frac{1}{|p|}, \quad (5.29)
\]

5.2.4 Doubly Charged Higgs contribution

Although we have ignored contributions due to exchanges of LH (RH) doubly charged Higgs bosons $\Delta^- \Delta^-$ in this work, we present the corresponding amplitudes for the sake of completeness,

(i) \[ A^{LL}_{\Delta} \propto \frac{1}{M^2_{W_L}} \frac{1}{M^2_{\Delta_L}} f_{LV} \nu_L, \]

(ii) \[ A^{RR}_{\Delta} \propto \frac{1}{M^2_{W_R}} \frac{1}{M^2_{\Delta_R}} f_{RV} \nu_R. \]
As stated in Sec. 5.1, the masses of $\Delta L^{--}$ and $\Delta L^{++}$ are of the order of the large parity restoration scale which damps out the induced VEV $v_L$ and the corresponding amplitude. The amplitude due to $\Delta R^{--}$ exchange is damped out compared to the standard amplitude as it is $\propto \frac{1}{M_{WR}}$.

### 5.2.5 Nuclear matrix elements and normalized effective mass parameters

By now it is well known that different particle exchange contributions for $0\nu2\beta$ decay discussed above are also modified by the corresponding nuclear matrix elements which depend upon the chirality of the hadronic currents involved [254–261]. Including all relevant contributions except those due to doubly charged Higgs exchanges, and using eq. (5.20) - eq. (5.29), we express the inverse half-life in terms of effective mass parameters with proper normalization factors taking into account the nuclear matrix elements [254–261] leading to the half-life prediction

$$T_{1/2}^{-1} = G_{01}^0 \left\{ |M_{\nu\nu}^0|^2 |\eta_\nu|^2 + |M_N^0|^2 |\eta_{NP}|^2 + |M_W^0|^2 |\eta_{RW}|^2 \right\} + |M_L^0|^2 |\eta_L|^2 + |M_R^0|^2 |\eta_R|^2 \right\} + \text{interference terms.} \quad (5.30)$$

where the dimensionless particle physics parameters are

$$|\eta_\nu| = \left| \sum_i \frac{\nu_{\nu i} \nu_{\nu i}^*}{m_e} m_{\nu i} \right|$$

$$|\eta_{NP}| = m_p \left( \frac{M_{W_L}}{M_{BR}} \right)^4 \left| \frac{\nu_{\nu i} N}{m_{N_i}} \right|$$

$$|\eta_{RW}| = m_p \left| \frac{V_{ei} N}{m_{N_i}} + V_{ei} S \right|$$

$$|\eta_L| = \left( \frac{M_{W_L}}{M_{WR}} \right)^2 \left| U_{ei} \left( \frac{M_D}{M_N} \right)_{ei} \right|$$

$$|\eta_R| = \tan \zeta_{LR} \left| U_{ei} \left( \frac{M_D}{M_N} \right)_{ei} \right| \quad (5.31)$$

In eq. (5.31), $m_e (m_i) =$ mass of electron (light neutrino), and $m_p =$ proton mass. In eq. (5.30), $G_{01}^0$ is the the phase space factor and besides different particle parameters, it contains the nuclear matrix elements due different chiralities of the hadronic weak currents such as ($M_{\nu\nu}^0$) involving left-left chirality in the standard contribution, and due to heavy neutrino exchanges ($M_{\nu\nu}^0$) involving right-right chirality arising out of
heavy neutrino exchange, \( (\mathcal{M}^0_{\nu}) \) for the \( \lambda \)-diagram, and \( (\mathcal{M}^0_{\eta}) \) for the \( \eta \)-diagram. Explicit numerical values of these nuclear matrix elements discussed in ref. \([254-261]\) are given in Tab. 5.4.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>( G^0_{01} ) ( [10^{-14} \text{ yrs}^{-1}] ) ( \text{refs. [254-260]} )</th>
<th>( \mathcal{M}^0_{\nu} )</th>
<th>( \mathcal{M}^0_{N} )</th>
<th>( \mathcal{M}^0_{\lambda} )</th>
<th>( \mathcal{M}^0_{\eta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{76}\text{Ge})</td>
<td>0.686</td>
<td>2.58–6.64</td>
<td>233–412</td>
<td>1.75–3.76</td>
<td>235–637</td>
</tr>
<tr>
<td>(^{82}\text{Se})</td>
<td>2.95</td>
<td>2.42–5.92</td>
<td>226–408</td>
<td>2.54–3.69</td>
<td>209–234</td>
</tr>
<tr>
<td>(^{130}\text{Te})</td>
<td>4.13</td>
<td>2.43–5.04</td>
<td>234–384</td>
<td>2.85–3.67</td>
<td>414–540</td>
</tr>
<tr>
<td>(^{136}\text{Xe})</td>
<td>4.24</td>
<td>1.57–3.85</td>
<td>160–172</td>
<td>1.96–2.49</td>
<td>370–419</td>
</tr>
</tbody>
</table>

Table 5.4: Phase space factors and nuclear matrix elements with their allowed ranges as derived in refs. \([254-261]\).

In order to arrive at a common normalization factor for all types of contributions, at first we use the expression for inverse half-life for \( 0\nu \beta \) decay process due to only light active Majorana neutrinos, \( \left[ T^0_{1/2} \right]^{-1} = G^0_{01} \left| \mathcal{M}^0_{\nu} \right|^2 \left| \eta \right|^2 \). Using the numerical values given in Tab. 5.4, we rewrite the inverse half-life in terms of effective mass parameter

\[
\left[ T^0_{1/2} \right]^{-1} = G^0_{01} \left| \mathcal{M}^0_{\nu} \right|^2 \left| m^e_{\nu} \right|^2 = 1.57 \times 10^{-25} \text{ yrs}^{-1} \text{ eV}^{-2} \left| m^e_{\nu} \right|^2 = K_{0\nu} \left| m^e_{\nu} \right|^2
\]

where \( m^e_{\nu} = \sum_i \left( \nu^{e\nu}_{e_i} \right)^2 m_{\nu} \). Then the analytic expression for all relevant contributions to effective mass parameters taking into account the respective nuclear matrix elements turns out to be

\[
\left[ T^0_{1/2} \right]^{-1} = K_{0\nu} \left| m^e_{\nu} \right|^2 + \left| m^e_{N} \right|^2 + \left| m^e_{S} \right|^2 + \left| m^e_{\lambda} \right|^2 + \left| m^e_{\eta} \right|^2 \right) + \cdots (5.32)
\]

where the ellipses denote interference terms and all other sub-dominant contributions. In eq. (5.32), the new effective interference terms and all other sub-dominant contributions. In eq. (5.32), the new effective mass parameters are

\[
m^e_{N,R} = \sum_i \left( \frac{M_{W_L}}{M_{W_R}} \right)^4 \left( \nu^{N}_{e_i} \right)^2 \left| p \right|^2 \left| m_{N_i} \right|^2 \right) (5.33)
\]

\[
m^e_{S,L} = \sum_i \left( \nu^{S}_{e_i} \right)^2 \left| p \right|^2 \left| m_{S_i} \right|^2 \right) (5.34)
\]

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\[ m^{ee}_\lambda = 10^{-2} \left( \frac{M_{W_L}}{M_{W_R}} \right)^2 \left| U_{\nu l} \left( \frac{M_D}{M_N} \cdots \right)_{\nu l} \right| |p| \quad (5.35) \]

\[ m^{ee}_\eta = \tan \zeta_{LR} \left| U_{\nu l} \left( \frac{M_D}{M_N} \cdots \right)_{\nu l} \right| |p| \quad (5.36) \]

where \( |p|^2 = m_e m_p \mathcal{M}^0_N/\mathcal{M}^0_\nu \simeq (200 \text{ MeV})^2 \). It is to be noted that the suppression factor \( 10^{-2} \) arises in the \( \lambda \)-diagram as pointed out in refs. [254–261].

### 5.3 Numerical estimation of effective mass parameters

Using analytic expression for relevant effective mass parameters given in eq. (5.30)-eq. (5.36) and our model parameters discussed in Sec. 5.1.1, we now estimate the relevant individual contributions numerically.

#### 5.3.1 Nearly standard contribution

In our model the new mixing matrix \( \mathcal{N}_{\nu l} \equiv \mathcal{V}^\nu_{\nu l} = (1 - \eta) U_{\nu l} \) contains additional non-unitarity effect due to non-vanishing \( \eta \) where

\[
\begin{align*}
\mathcal{N}_{e1} &= (1 - \eta_e) U_{11} - \eta_e U_{21} - \eta_e U_{31} \\
\mathcal{N}_{e2} &= (1 - \eta_e) U_{12} - \eta_e U_{22} - \eta_e U_{32} \\
\mathcal{N}_{e3} &= (1 - \eta_e) U_{13} - \eta_e U_{23} - \eta_e U_{33}
\end{align*}
\quad (5.37)
\]

We estimate numerical values of \( \mathcal{N}_{\nu l} \) using all allowed values of \( \eta \) discussed in Sec. 5.1 and also by using \( U_{\nu l} \equiv U_{\text{PMNS}} \). Then the effective mass parameter for the \( W_L-W_L \) mediation with light neutrino exchanges is found to be almost similar to the standard prediction

\[
|m^{ee}_\nu| \simeq \begin{cases} 
0.004 \text{eV} & \text{NH}, \\
0.048 \text{eV} & \text{IH}, \\
0.23 \text{eV} & \text{QD}.
\end{cases} \quad (5.38)
\]

This nearly standard contribution on effective mass parameter is presented by the dashed-green colored lines of Fig. 5.6 and Fig. 5.7 for NH neutrino masses, but it is presented by the dashed-pink colored lines of the same figures for IH neutrino masses. In our numerical estimations presented in Fig. 5.6 we have used \( M_D \) values.
Figure 5.6: Variation of effective mass parameters with lightest neutrino mass. The standard contributions are shown by dashed-green (pink) colored lines for NH (IH) case. The non-standard contribution with $W_L^-W_L^-$ mediation and sterile neutrino exchanges is shown by the upper blue solid line whereas the one with $W_L^-W_R^-$ mediation and sterile neutrino exchanges is shown by the lower black solid line.

including RG corrections as given in eq. (5.11) but with $M = (50, 200, 1712)$ GeV, $\hat{M}_N = (1250, 3000, 5000)$ GeV, and $\hat{m}_S = (2, 13, 532)$ GeV. Similarly, in Fig. 5.7 we have utilized $M_D$ values including RG corrections from eq. (5.11) but with $M = (100, 100, 2151.6)$ GeV, $\hat{M}_N = (5000, 5000, 5000)$ GeV, and $\hat{m}_S = (2, 2, 800)$ GeV.

5.3.2 Dominant non-standard contributions

Before estimating the non-standard effective mass parameters, we present the mixing matrices numerically. As discussed in eq. (5.10) of Sec. 5.1.2, the mixing matrices $X = M_D M^{-1}, Y = M M_N^{-1}, Z = M_D M_N^{-1}$, and $y = \mu S M^{-1}$ all contribute to non-standard predictions of $0\nu2\beta$ amplitude in the extended seesaw scheme.

Using eq. (5.10) and the diagonal structures of the RH Majorana neutrino mass matrix $M_N = \text{diag}(M_{N_1}, M_{N_2}, M_{N_3})$ as well as $N-S$ mixing matrix $M = \text{diag}(M_1, M_2, M_3)$, and the Dirac neutrino mass matrix $M_D$ with RG corrections given in eq. (5.11), we derive the relevant elements of the mixing matrices $\mathcal{N}, \nu^{\nu\tilde{N}}, \nu^{\nu\tilde{S}}, \nu^{\tilde{S}\tilde{S}}, \nu^{\tilde{S}\tilde{N}}, \nu^{N\tilde{S}}$, and $\nu^{N\tilde{N}}$ for which one example is

$\mathcal{N}_{ei} = \{0.8135, 0.5597, 0.1278\}$

$\nu^{\nu\tilde{S}}_{ei} = \{4.5398 \times 10^{-4}, 4.93 \times 10^{-4}, 2.148 \times 10^{-4}\}$,

$\nu^{\nu\tilde{N}}_{ei} = \{1.8 \times 10^{-5}, 3.3 \times 10^{-5}, 6.7 \times 10^{-5}\}$,
Figure 5.7: Variation of effective mass parameters with lightest neutrino mass. The standard contributions are shown by dashed-green (pink) colored lines for NH (IH) case. The non-standard contribution with $W_L^{-}-W_L^{-}$ mediation and sterile neutrino exchanges is shown by the upper blue solid line whereas the one with $W_L^{-}-W_R^{-}$ mediation and sterile neutrino exchanges is shown by the lower black solid line.

\begin{align*}
\nu_{ei}^{S0} &= \{3.6 \times 10^{-3}, 3.3 \times 10^{-3}, 6.0 \times 10^{-3}\}, \\
\nu_{ei}^{SS} &= \{0.999, 0.0002, 5.0 \times 10^{-6}\}, \\
\nu_{ei}^{SN} &= \{0.04, 0.0, 0.0\}, \\
\nu_{ei}^{N\bar{N}} &= \{1.0, 0.0, 0.0\}, \\
\nu_{ei}^{N\bar{N}} &= \{9.33 \times 10^{-10}, 2.97 \times 10^{-9}, 1.0 \times 10^{-8}\}, \\
\nu_{ei}^{N\bar{S}} &= \{0.04, 0.0, 0.0\}. \\
\end{align*}

(5.39)

For evaluating these mixing matrix elements we have taken the input values, $M$, $M_N$, and $\hat{m}_S$ presented under column C1 of Tab. 5.5. These lead to the numerical results for effective mass parameter contributing to $0\nu2\beta$ decay rate presented under column C1 of Tab. 5.6. Similarly when we use the $M$, $M_N$, and $\hat{m}_S$ values from column C2 of Tab. 5.5 we obtain effective mass parameters given in column C2 of
Table 5.6: Rough estimation of effective mass parameters with the allowed model parameters. The results are for the Dirac neutrino mass matrix including RG corrections. The input values of mass matrices allowed by the current data for different columns are presented in Tab. 5.5.

<table>
<thead>
<tr>
<th>Effective mass parameter</th>
<th>C1 (eV)</th>
<th>C2 (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{ee}$</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>$m_{N}^e$</td>
<td>0.0085</td>
<td>0.0085</td>
</tr>
<tr>
<td>$m_{S}^e$</td>
<td>20.75</td>
<td>188.48</td>
</tr>
<tr>
<td>$m_{\lambda,\eta}^e$</td>
<td>0.0093</td>
<td>0.0274</td>
</tr>
</tbody>
</table>

The most dominant and new contribution to the effective mass parameters is found to emerge from the amplitude $A_{LL}^S$ of eq. (5.20) due to $W_L^-W_L^-$ mediation and sterile neutrino exchanges. This has been shown in Fig. 5.8 for various combinations of sterile neutrino mass eigenvalues and for $M_D$ values including RG corrections given in eq. (5.11). In Fig. 5.8 our estimated values range from $0.2 - 1.0$ eV. Looking to the results given in Tab. 5.6 and Fig. 5.6, Fig. 5.7, and Fig. 5.8, it is clear that the actual enhanced rate of $0\nu\beta\beta$ decay in this model depends primarily upon the sterile neutrino mass eigenvalues $m_{S_1}$ and $m_{S_2}$. If the decay rate corresponds to $|m_{\text{eff}}| \approx 0.21 - 0.53$ eV as claimed by the Heidelberg-Moscow experiment using $^{76}\text{Ge}$ [229, 236-239], our new finding is that the light neutrino masses could be still of NH or IH pattern, instead of necessarily being of QD pattern, but with $m_{S_1} \sim 10$ GeV and $m_{S_2} \sim 30$ GeV. Of course the the Dirac neutrino mass matrix having its high scale quark-lepton symmetric origin also contributes to the magnification of the effective mass parameter. The next dominant contributions coming from the Feynman amplitude $A_{LR}^S$ of eq. (5.29) due to $W_L^-W_R^-$ mediation and sterile neutrino exchanges with $m_{\lambda,\eta}^{ee,LR} = 0.04$ eV ($0.01$ eV) have been shown in Fig. 5.6 (Fig. 5.7).

5.4 Estimations on lepton flavor violating decays and $J_{CP}$

Besides the neutrinoless double beta decay process, the sterile and heavy neutrinos in this model can predominantly mediate different lepton flavor violating decays, $\mu \rightarrow e + \gamma$, $\tau \rightarrow e + \gamma$, and $\tau \rightarrow \mu + \gamma$. Since $\ell_\alpha \rightarrow \ell_\beta + \gamma$ ($\alpha \neq \beta$) is lepton flavor changing process, it is strictly forbidden in the SM when $m_\nu = 0$ and lepton number is conserved. In our model the underlying lepton flavor violating interactions and
Figure 5.8: Predictions of non-standard contributions to effective mass parameter with $W_L^- - W_L^-$ mediation and sterile neutrino exchange for $M = (120, 250, 1664.9)$ GeV (top solid line), $M = (250, 250, 1663.3)$ GeV (middle solid line), and $M = (250, 400, 1626.1)$ GeV (bottom solid line) keeping $M_N = (5, 5, 10)$ TeV fixed and for $M_D$ as in eq. (5.11).

non-unitarity effects contribute to lepton flavor violating decays by the mediation of heavy RH Majorana and sterile Majorana fermions.

<table>
<thead>
<tr>
<th>$M$ (GeV)</th>
<th>$M_N$ (TeV)</th>
<th>Heavy Mass Eigen Values (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(9.7, 115.2, 2776.5)</td>
<td>(5, 5, 5)</td>
<td>(0.018, 2.65, 1238, 5000, 5002, 6238)</td>
</tr>
<tr>
<td>(100, 200, 1702.67)</td>
<td>(5, 5, 5)</td>
<td>(1.99, 2.00, 800.5, 5001, 5002, 5800)</td>
</tr>
<tr>
<td>(50, 200, 1711)</td>
<td>(1.5, 2, 5)</td>
<td>(1.67, 19.8, 532.2, 1501, 2019, 5532)</td>
</tr>
<tr>
<td>(1604.4, 1604.4, 1604.4)</td>
<td>(5, 5, 10)</td>
<td>(252.4, 461.5, 470.6, 5471.6, 5471.4, 10252.4)</td>
</tr>
</tbody>
</table>

Table 5.7: The Heavy mass eigenvalues for the matrices of $M$ and $M_N$ which have been used to evaluate branching ratios.

### 5.4.1 Branching ratio

Keeping in mind the charged-current interaction in the neutrino mass basis for extended seesaw scheme given in eq. (5.15) - eq. (5.17), the dominant contributions are mainly through the exchange of the sterile and heavy RH neutrinos. The decomposition of eq. (4.44) in to heavy and sterile parts gives [45, 165, 167, 170, 171, 206, 262]

$$
\text{Br} (\ell_\alpha \rightarrow \ell_\beta + \gamma) = \frac{g_w^3 s_w^2 m_{\ell_\alpha}^5}{256 \pi^2 M_W^4 \Gamma_\alpha} |G_{\alpha \beta}^N + G_{\alpha \beta}^S|^2,
$$

(5.40)
where  
\[ G^N_{\alpha\beta} = \sum_k \left( V^{\nu N}_{\alpha k} \right)^* \frac{m^2_{N_k}}{M^2_{W_L}}, \]

\[ G^S_{\alpha\beta} = \sum_j \left( V^{\nu S}_{\alpha j} \right)^* \frac{m^2_{S_j}}{M^2_{W_L}}, \]  

(5.41)

and \( I(x) \) has already been defined in eq. (4.44). It is clear from the above equation and within the model parameter range, \( M_N \gg M \gg M_D \), that the first term in eq. (5.40) is negligible while second term involving the heavy sterile neutrinos gives dominant contribution which is proportional to \( \sum_j \left( V^{\nu S}_{\alpha j} \right)^* \left( V^{\nu S}_{\beta j} \right) \simeq 2\eta_{\alpha\beta} \).

<table>
<thead>
<tr>
<th>( M ) (GeV)</th>
<th>( M_N ) (TeV)</th>
<th>( \text{Br}(\mu \rightarrow e \gamma) ) ( \times 10^{-16} )</th>
<th>( \text{Br}(\tau \rightarrow e \gamma) ) ( \times 10^{-14} )</th>
<th>( \text{Br}(\tau \rightarrow \mu \gamma) ) ( \times 10^{-12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 200, 1711.8)</td>
<td>(1.5, 2, 5)</td>
<td>3.05</td>
<td>3.11</td>
<td>4.36</td>
</tr>
<tr>
<td>(100, 200, 2151.57)</td>
<td>(5, 5, 5)</td>
<td>1.28</td>
<td>1.39</td>
<td>1.95</td>
</tr>
<tr>
<td>(100, 200, 1702.67)</td>
<td>(5, 5, 5)</td>
<td>2.85</td>
<td>3.1</td>
<td>4.3</td>
</tr>
<tr>
<td>(1604.4, 1604.4, 1604.4)</td>
<td>(5, 5, 10)</td>
<td>2.18</td>
<td>2.32</td>
<td>3.25</td>
</tr>
</tbody>
</table>

Table 5.8: The three branching ratios in extended inverse seesaw for different values of \( M \) and \( M_N \) while \( M_D \) is same as in eq. (5.11).

Using the numerically computed mixing matrix, and using allowed mass scales presented in Tab. 5.7, our model estimations on branching ratios are given in Tab. 5.8. Recent experimental data gives the best limit on these branching ratios for lepton flavor violating decays coming from the MEG collaboration [208–210, 263]. Out of these \( \text{Br}(\mu \rightarrow e + \gamma) \leq 1.2 \times 10^{-11} \) [208–210, 263], is almost three orders of magnitude stronger than the limit \( \text{Br}(\tau \rightarrow e + \gamma) \leq 3.3 \times 10^{-8} \) or \( \text{Br}(\tau \rightarrow \mu + \gamma) \leq 4.4 \times 10^{-8} \) at 90% C.L. However, projected reach of future sensitivities of ongoing searches are \( \text{Br}(\tau \rightarrow e + \gamma) \leq 10^{-9} \), \( \text{Br}(\tau \rightarrow \mu + \gamma) \leq 10^{-9} \), and \( \text{Br}(\mu \rightarrow e + \gamma) \leq 10^{-18} \) [208–210, 263] which might play crucial role in verifying or falsifying the discussed scenario.

### 5.4.2 CP-violation due to non-unitarity

There are attempts taken in long baseline experiments [199–202] with accelerator neutrinos \( \nu_\mu \) and anti-neutrinos \( \bar{\nu}_\mu \) to search for \( CP \)-violating effects in neutrino oscillations. In the usual notation, the standard contribution to these effects is determined by the re-phasing invariant \( J_{CP} \) associated with the Dirac phase \( \delta_{CP} \) and matrix elements of the PMNS matrix is given in eq. (4.42). In this extended seesaw mechanism, the leptonic \( CP \)-violation can be written as in eq. (4.41), where
\[ \Delta J_{ij}^{\alpha \beta} \] is expanded in eq. (4.43). The extra contribution arises because of the non-unitarity mixing matrix which depends on both \( M_D \) and \( M \). Thus the new contribution to \( CP \)-violation is larger for larger \( M_D \) which is generated with quark-lepton symmetry and for smaller \( M \) while safeguarding the constraint \( M_N \gg M > M_D, \mu_S \). It is noteworthy that in our model even if the leptonic Dirac phase \( \delta_{CP} \simeq 0, \pi, 2\pi \), and/or \( \sin \theta_{13} \rightarrow 0 \), there is substantial contribution to \( CP \)-violation which might arise out of the imaginary parts of the non-unitarity matrix elements \( \eta_{\alpha \beta} \).

![Diagram](image.png)

Figure 5.9: \( CP \)-violation for the full allowed range of leptonic Dirac phase \( \delta_{CP} \). The left-panel corresponds to degenerate values of \( M \) with \( M_1 = M_2 = M_3 \simeq 1604.442 \) GeV, and the right panel is due to non-degenerate \( M \) with \( M_1 = 9.7 \) GeV, \( M_2 = 115.2 \) GeV, and \( M_3 = 2776.5 \) GeV.

<table>
<thead>
<tr>
<th>M</th>
<th>( \Delta J_{12}^{\mu \mu} )</th>
<th>( \Delta J_{23}^{\mu \tau} )</th>
<th>( \Delta J_{24}^{\mu \tau} )</th>
<th>( \Delta J_{31}^{\mu \tau} )</th>
<th>( \Delta J_{12}^{\tau e} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>( -2.0 \times 10^{-6} )</td>
<td>( -2.3 \times 10^{-6} )</td>
<td>( -1.2 \times 10^{-4} )</td>
<td>( -1.2 \times 10^{-4} )</td>
<td>( -1.1 \times 10^{-4} )</td>
</tr>
<tr>
<td>(b)</td>
<td>( -2.7 \times 10^{-6} )</td>
<td>( -3.2 \times 10^{-6} )</td>
<td>( -1.2 \times 10^{-4} )</td>
<td>( -1.2 \times 10^{-4} )</td>
<td>( -1.1 \times 10^{-4} )</td>
</tr>
<tr>
<td>(c)</td>
<td>( -2.1 \times 10^{-5} )</td>
<td>( -2.4 \times 10^{-5} )</td>
<td>( 1.1 \times 10^{-7} )</td>
<td>( -1.8 \times 10^{-4} )</td>
<td>( -7.9 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

Table 5.9: The \( CP \)-violating effects for (a) degenerate masses \( M=(1604.4, 1604.4, 1604.4) \) GeV, (b) partially degenerate masses \( M=(100, 100, 2151.6) \) GeV and (c) non-degenerate masses \( M=(9.7, 115.2, 2776.5) \) GeV, while \( M_D \) is same as in eq. (5.11).

Our estimations using RGE corrected Dirac neutrino mass matrix and both degenerate and non-degenerate matrix \( M \) are shown in the left-panel and right-panel of Fig. 5.9. If the leptonic Dirac phase \( \delta_{CP} \neq 0, \pi, 2\pi \), significant \( CP \)-violation up to \( |\Delta J|_{\text{max}} \simeq 1.5 \times 10^{-4} \) is found to occur for degenerate \( M \), but when \( M \) is non-degenerate we obtain \( |\Delta J|_{\text{max}} \simeq (2-4) \times 10^{-4} \). Also even if \( \delta_{CP} \simeq 0, \pi, 2\pi \), non-vanishing \( CP \)-violation to the extent of \( |\Delta J| \simeq (1-2) \times 10^{-4} \) is noted to emerge for non-degenerate \( M \). These results may be compared with \( CP \)-violation in the quark
sector where \( J_{\text{CKM}} \simeq 3.05^{+0.19}_{-0.20} \times 10^{-5} [149,150] \) which is nearly one order lower than the leptonic case. The horizontal lines in Fig. 5.9 represent absence of non-unitarity effects on \( CP \)-violation. In Fig. 5.9 we have plotted the \( \delta_{CP} \) dependence of \( \Delta J^{ij}_{\alpha \beta} \) and found that estimation of \( \Delta J^{ij}_{\alpha \beta} \) in different channel would give the amount of non-degeneracy together with non-unitarity.

5.5 Implementation in \( SO(10) \)

Our main goal in this section is to examine whether the TeV scale LR gauge model that has been shown to give rise to dominant contribution to \( 0\nu2\beta \) decay and LFV in Sec. 5.1 - Sec. 5.4 can emerge from a non-SUSY \( SO(10) \) grand unified theory. Although the search for low mass \( W^\pm_R \) bosons in non-SUSY GUTs has been attempted initially without [90-92, 264, 265] precision CERN-LEP data on \( \alpha_S(M_Z) \) and \( \sin^2 \theta_W(M_Z) [149,150] \), there are more recent results on physically appealing intermediate scales [58,81,83,127,145]. But the analyses in non-SUSY cases where the \( B-L \) breaking scale synonymous to \( W_R \) gauge boson mass much lower than \( 10^{10} \) GeV are ruled out because of the associated large contributions to light neutrino masses via type-I seesaw mechanism. In view of the rich phenomenological consequences of the extended seesaw mechanism that evades the discordance between dominant \( 0\nu2\beta \) decay and small neutrino mass predictions as discussed in Sec. 5.1 - Sec. 5.4, we explore the possibility of such low scale LR gauge theory in the minimally extended \( SO(10) \) grand unification model.

5.5.1 Symmetry breaking chain

We consider the symmetry breaking chain discussed in ref. [58]. Although this model, as such, is ruled out because of the TeV scale canonical seesaw that operates to give large neutrino masses in contravention of the oscillation data, here we modify this model by including the additional doublets \( (\chi_L, \chi_R) \subset 16_H \) of \( SO(10) \) and extending the minimal fermion content in \( 16_F \) with the addition of one \( SO(10) \) singlet neutral fermion per generation in order to implement the extended seesaw mechanism

\[
SO(10) \xrightarrow{M_{\text{GUT}}} \ G_{2214D} \xrightarrow{M_P} \ G_{224} \xrightarrow{M_C} \ G_{2213} \\
\quad \xrightarrow{M^+_R} \ G_{2113} \xrightarrow{M_R^0} \ G_{213} \xrightarrow{M_Z} \ G_{13} \quad (5.42)
\]
Above the energy scale $M_P$, the $D$-parity is restored therefore above this energy $g_{2L} = g_{2R}$. It was found in refs. [56–58] that the $G_{224}$-singlets in $54_H$ and $210_H$ of $SO(10)$ are $D$-parity even and odd, respectively. Also it was noted that the neutral components of the $G_{224}$ multiplet $(1,1,15)$ contained in $210_H$ and $45_H$ of $SO(10)$ have $D$-parity even and odd, respectively. In the first step, VEV is assigned along the $\langle (1,1,1) \rangle \subset 54_H$ which has even $D$-Parity to guarantee the LR symmetric Pati-Salam group to survive while at the second step $D$-parity is broken by assigning $\langle (1,1,1) \rangle \subset 210_H$ to obtain asymmetric $G_{224}$ with $g_{2L} \neq g_{2R}$. The spontaneous breaking $G_{224} \to G_{2213}$ is achieved by the VEV $\langle (1,1,15)_H^0 \rangle \subset 210_H$. The symmetry breaking $G_{2213} \to G_{2113}$ is implemented by assigning $O(M_R^+)$ VEV to the neutral component of the sub-multiplet $\langle (1,3,15)_H^0 \rangle \subset 210_H$, and the breaking $U(1)_R \times U(1)_{B-L} \to U(1)_Y$ is achieved by $\langle \Delta_{R}^0 (3,1,10) \rangle \subset 126_H$ while the VEV $\langle \chi_R^0 (1,2,4) \rangle \subset 16_H$ provides the $N-S$ mixing. As usual, the breaking of SM to low energy symmetry $U(1)_Q \times SU(3)_C$ is carried out by the SM doublet contained in the bi-doublet $\Phi(2,2,1) \subset 10_H$.

### 5.5.2 Gauge coupling unification

While SM is the symmetry of fundamental interactions near the $M_Z$ scale, in the conventional approach to investigation of gauge coupling unification, usually the semi simple gauge symmetry to which the GUT gauge theory breaks is a product of three or more groups. As a result the symmetry below the GUT-breaking scale involves three or more gauge couplings. The renormalization group (RG) evolution of gauge couplings thus may creates a triangular region around the projected unification scale (Similar to SM) making the determination of the scale more or less uncertain. Even though the region of uncertainty is reduced in the presence of intermediate scales, it exists in principle. Only in the case when $G_I = SU(2)_L \times SU(2)_R \times SU(4)_C \times D$, the Pati-Salam symmetry with LR discrete symmetry [47] (≡ $D$-Parity) [56, 266], there are two gauge couplings $g_{2L} = g_{2R}$ and $g_{4C}$, and the meeting point of the two

<table>
<thead>
<tr>
<th>$M_R^0$ (TeV)</th>
<th>$M_R^+$ (TeV)</th>
<th>$M_C$ (TeV)</th>
<th>$M_P$ (GeV)</th>
<th>$M_G$ (GeV)</th>
<th>$\alpha_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>$10^3$</td>
<td>$10^{14.2}$</td>
<td>$10^{17.64}$</td>
<td>0.03884</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>$10^{3.3}$</td>
<td>$10^{14.42}$</td>
<td>$10^{17.61}$</td>
<td>0.03675</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>$10^3$</td>
<td>$10^{14.08}$</td>
<td>$10^{17.84}$</td>
<td>0.03915</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>100</td>
<td>$10^{13.72}$</td>
<td>$10^{17.67}$</td>
<td>0.0443</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>500</td>
<td>$10^{13.93}$</td>
<td>$10^{17.55}$</td>
<td>0.0406</td>
</tr>
</tbody>
</table>

Table 5.10: Predictions of allowed mass scales and the GUT couplings in the $SO(10)$ symmetry breaking chain with low-mass $W_R^\pm$, $Z'$ bosons.
RG-evolved coupling lines determines the unification point exactly. Several interesting consequences of this intermediate symmetry have been derived earlier including vanishing corrections to GUT-threshold effects on $\sin^2 \theta_W$ and the intermediate scale [82, 224–226]. We find this symmetry to be essentially required at the highest intermediate scale in the present model to guarantee several observable phenomena as $SO(10)$ model predictions while safeguarding precision unification. We have evaluated the one-loop and two-loop coefficients of $\beta$-functions of renormalization group equations for the gauge couplings [51, 107], as given in eq. (2.20). The one and two loop $\beta$-coefficients are given in Tab. B.2 of Appendix B.

The Higgs spectrum used in different ranges of mass scales under respective gauge symmetries ($G$) are

(i) $\mu = M_Z - M_0^R : G = SM = G_{213}, \quad \Phi(2, 1/2, 1)$;

(ii) $\mu = M_0^R - M^+_R : G = G_{213}, \quad \Phi(2, 1/2, 0, 1) \oplus \Phi(2, -1/2, 0, 1) \oplus \chi_R(1, 1/2, -1, 1) \oplus \Delta_R(1, 1, -2, 1)$;

(iii) $\mu = M^+_R - M_C : G = G_{2213}, \quad \Phi(2, 2, 0, 1) \oplus \Phi(2, 2, 0, 1) \oplus \chi_R(1, 2, -1, 1) \oplus \Delta_R(1, 3, -2, 1) \oplus \Sigma_R(1, 3, 0, 1)$;

(iv) $\mu = M_C - M_P : G = G_{224}, \quad \Phi(2, 2, 1) \oplus \Phi(2, 2, 1) \oplus \chi_R(1, 2, 4) \oplus \Delta_R(1, 3, 10) \oplus \Sigma_R(1, 3, 15)$;

(v) $\mu = M_P - M_U : G = G_{224D}, \quad \Phi(2, 2, 1) \oplus \Phi(2, 2, 1) \oplus \chi_L(2, 1, 4) \oplus \chi_R(1, 2, 4) \oplus \Delta_L(3, 1, 10) \oplus \Delta_R(1, 3, 10) \oplus \Sigma_L(3, 1, 15) \oplus \Sigma_R(1, 3, 15)$.

Recently bounds on the masses of the charged and neutral components of the second Higgs doublet in the left-right symmetric model has been estimated to be

84
While searching for possible mass scales we have used the second Higgs doublet $\Phi_2$ only for $\mu \geq 10$ TeV.

We have used extended survival hypothesis in implementing spontaneous symmetry breaking of $SO(10)$ and intermediate gauge symmetries leading to the SM gauge theory [268, 269]. In addition to $D$-Parity breaking models [56–58], the importance of the Higgs representation $210_h$ has been emphasized in the construction of a minimal SUSY $SO(10)$ GUT model [30–35]. But the present non-SUSY $SO(10)$ symmetry breaking chain shows a departure in that the $G_{224D}$ symmetry essentially required at the highest intermediate scale has unbroken $D$-Parity which is possible by breaking the GUT symmetry through the Higgs representation $54_H \subset SO(10)$ that acquires GUT-scale VEV in the direction of its $D$-parity even $G_{224D}$-singlet. The importance of this $G_{224D}$ symmetry in stabilizing the values of $M_P$ and $\sin^2 \theta_W(M_Z)$ against GUT-Planck scale threshold effects has been discussed in refs. [224–226] and Sec. 5.5.4 below.

Using precision CERN-LEP data [149, 150] $\alpha_s(M_Z) = 0.1184$, $\sin^2 \theta_W(M_Z) = 0.2311$ and $\alpha^{-1}(M_Z) = 127.9$, different allowed solutions presented in Tab. 5.10. One set of solutions corresponding to low mass $W_R^\pm$ and $Z'$ gauge bosons is

Figure 5.10: Two loop gauge coupling unification in the $SO(10)$ symmetry breaking chain described in the text. These results are also valid with $G_{224D}$ symmetry near GUT-Planck scale.
\[ M_R^0 = 3 - 5 \text{ TeV}, \ M_R^+ = 10 \text{ TeV}, \ M_C = 10^2 \text{TeV} - 10^3 \text{TeV}, \]
\[ M_p \simeq 10^{14.17} \text{GeV} \text{ and } M_{GUT} \simeq 10^{17.8} \text{GeV}. \] (5.44)

For these mass scales the emerging pattern of gauge coupling unification is shown in Fig. 5.10 with GUT fine structure constant \( \alpha_G = 0.0388 \).

### 5.5.3 Physical significance of mass scales

The presence of \( G_{224D} \) symmetry above the highest intermediate scale plays a crucial role in lowering down the values of \( M_R^+ \) while achieving high scale gauge coupling unification. With the gauge couplings allowed in the region \( \mu \simeq 3 \text{ TeV} - 10 \text{ TeV} \) in the grand unified scenario with \( g_{B-L} \simeq 0.725 \), \( g_{2R} \simeq 0.4 \), we have estimated the predicted \( W_R \) and \( Z' \) masses to be \( M_{W_R} \simeq 4 \text{ TeV} \), \( M_{Z'} \simeq (2.3 - 3.6) \text{ TeV} \) for the allowed mass scales \( M_R^0 \simeq (3 - 5) \text{ TeV} \), and \( M_R^+ \simeq 10 \text{ TeV} \) of Tab. 5.10.

These low mass \( W_R \) and \( Z' \) bosons have interesting RH current effects at low energies including \( K_L - K_S \) mass difference and dominant 0\( \nu \)\( \beta \) rates as discussed in Sec. 5.2 - Sec. 5.4. The predicted low mass \( W_R^\pm \) and \( Z' \) bosons are also expected to be testified at the LHC and future accelerators for which the current bounds are \( M_{W_R} \geq 2.5 \text{ TeV} \) \([270-275]\) and \( M_{Z'} \geq 1.162 \text{ TeV} \) \([276, 277]\). The predicted mass scale \( M_C \sim (10^5 - 10^6) \text{ GeV} \) leads to experimentally verifiable branching ratios for rare kaon decay with \( \text{Br}(K^0_L \rightarrow \mu \overline{\nu}) \simeq (10^{-9} - 10^{-11}) \) \([278]\) via leptoquark gauge boson mediation \([279-281]\). Because of the presence of \( G_{224} \) symmetry for \( \mu \geq M_C (10^5 - 10^6) \text{ GeV} \), all the components of di-quark Higgs scalars in \( \Delta_R (3, 1, 10) \) mediating \( n-\bar{n} \) and \( H-\bar{H} \) oscillations also acquire masses at that scale whereas the di-lepton Higgs scalar carrying \( B-L = -2 \) is at the \( \simeq 1 \text{ TeV} \) scale. This gives rise to observable \( n-\bar{n} \) oscillation with mixing time \( \tau_{n\bar{n}} \simeq (10^8 - 10^{11}) \text{ secs} \) \([282-284]\). However because of the large value of the GUT scale \( M_{GUT} \simeq 10^{18} \text{GeV} \), which is close to the Planck scale, the predicted proton life time for \( p \rightarrow e^+ \pi^0 \) is large, i.e. \( \tau_p \geq 10^{40} \text{ yrs} \) which is beyond the accessible range of ongoing search experiments that have set the lower limit \( (\tau_p)_{\text{expt.}} \geq 1.1 \times 10^{34} \text{ yrs.} \) \([152]\).

### 5.5.4 Importance of \( G_{224D} \) intermediate symmetry

Near Planck scale unification of this model exposes an interesting possibility that grand unification can be also achieved by the Pati-Salam symmetry \( G_{224D} \) even without the help of the GUT-gauge group \( SO(10) \) since, above this scale, gravity effects are expected to take over \([285]\).
The most interesting role of $G_{224D}$ gauge symmetry at the highest intermediate scale has been pointed out in ref. [224–226]. Normally super-heavy Higgs scalars contained in larger representations like $210_H$ and $126_H$ introduce uncertainties into GUT predictions of $\sin^2 \theta_W(M_Z)$ on which CERN-LEP data and others have precise experimental results. But the presence of $G_{224D}$ at the highest scale achieves the most desired objective that the GUT scale corrections to $\sin^2 \theta_W(M_Z)$ vanish due to such sources as super-heavy particles or higher dimensional operators signifying the effect of gravity.

5.5.5 Determination of Dirac neutrino mass matrix

It is well known that within Pati-Salam gauge symmetry $G_{224D}$, the presence of $SU(4)_C$ unifies quarks and leptons treating the latter as fourth color and this relates the up-quark mass matrix ($M^0_u$) to the Dirac neutrino mass matrix $M^0_D$ at the unification scale. Such relations are also valid in $SO(10)$ at the GUT scale since $G_{224D}$ is its maximal subgroup. Over the recent years it has been shown that in a large class of $SO(10)$ model the fermion mass fits at the GUT scale gives $M^0_D \sim O(M^0_u)$ [45, 72, 246, 286, 287]. Since the predictions of lepton number and LFVs carried out in this work are sensitive to the Dirac neutrino mass matrix, it is important to derive $M_D$ at the TeV scale given in eq. (5.11). This question has been answered in non-SUSY $SO(10)$ [246] and SUSY $SO(10)$ [45] while utilizing renormalization group running of fermion masses analogous to ref. [186] and using their low energy data but in the presence of intermediate symmetries $G_{2113}$, $G_{2213}$, and $G_{2213D}$. In this analysis we will also use additional RGEs for Yukawa coupling and fermion masses in the presence of $G_{224}$ and $G_{224D}$ symmetries operating between $M_C \approx 10^5 \text{ GeV}$ to $M_{GUT} \approx 10^{17.5} \text{ GeV}$ [288–290].

The determination of the Dirac neutrino mass matrix $M_D(M_{Re})$ at the TeV seesaw scale is done in three steps [246]: (A.) Extrapolation of masses to the GUT-scale using low-energy data on fermion masses and CKM mixings through corresponding RGEs in the bottom-up approach, (B.) Fitting the fermion masses at the GUT scale and determination of $M_D(M_{GUT})$, (C.) Determination of $M_D(M_{Re})$ by top-down approach.

5.5.5.1 Extrapolation of fermion masses to the GUT scale

At first RGEs for Yukawa coupling matrices and fermion mass matrices are set up from which RGEs for mass eigenvalues and CKM mixings are derived in the presence
of $G_{2113}$, $G_{2213}$, $G_{224}$, and $G_{224D}$ symmetries.

Denoting $\Phi_{1,2}$ as the corresponding bidoublets under $G_{2213}$ their VEVs are taken as

$$
\langle \Phi_1 \rangle = \begin{pmatrix} v_u \\ 0 \\ 0 \\ 0 \end{pmatrix},
\langle \Phi_2 \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_d \end{pmatrix}.
$$

For mass scales $\mu \ll M_G$, ignoring the contribution of the super-heavy bi-doublet in 126$_H$, the bi-doublet $\Phi_1 \subset 10_{H_1}$ is assumed to give dominant contribution to up quark and Dirac neutrino masses $M_u$ and $M_D$ whereas $\Phi_2 \subset 10_{H_2}$ is used to generate masses for down quarks and charged leptons, $M_d$ and $M_\ell$

$$
M_u = Y_u v_u, \quad M_D = Y_\nu v_u, \quad M_d = Y_d v_d,
M_\ell = Y_\nu v_u.
$$

At $\mu = M_Z$ we use the input values of running masses and quark mixings as in eq. (4.19) [186] with the CKM Dirac phase $\delta^q = 1.20 \pm 0.08$. This results in the CKM matrix at $\mu = M_Z$ as given in eq. (4.20). We use RGEs of the SM [186] to evolve all charged fermion masses and CKM mixings from $\mu = M_Z$ to $M^0_R \simeq 10$ TeV. With two Higgs doublets at $\mu > 10$ TeV consistent with the current experimental lower bound on the second Higgs doublet [267], we use the starting value of $\tan \beta = v_u/v_d = 10$ and evolve the masses up to $\mu = M_C$ using RGEs derived in the presence of non-SUSY $SO(10)$ and intermediate symmetries $G_{2113}$ and $G_{2213}$ [246] with two Higgs bi-doublets. For $\mu \geq M_C$, we use the fermion mass RGEs in the presence of $G_{224}$ and $G_{224D}$ [288–290] modified including the corresponding RGEs of $v_u$ and $v_d$. The fermion mass eigen values $m_i$ and the $V_{CKM}$ at the GUT scale turn out to be

At $\mu = M_{GUT}$ scale:

$$
m_e^0 = 0.2168 \text{ MeV}, m_\mu^0 = 38.846 \text{ MeV}, m_\tau^0 = 0.9620 \text{ GeV},
m_d^0 = 1.163 \text{ MeV}, m_s^0 = 23.352 \text{ MeV}, m_b^0 = 1.0256 \text{ GeV},
m_u^0 = 1.301 \text{ MeV}, m_c^0 = 0.1686 \text{ GeV}, m_t^0 = 51.504 \text{ GeV},
$$

(5.47)
where, in deriving eq. (5.47), we have used “run and diagonalize” procedure. Then using eq. (5.47) and eq. (5.48), the RG extrapolated value of the up-quark mass matrix at the GUT scale is determined

\[
V^0_{\text{CKM}} = \begin{pmatrix}
0.9764 & 0.216 & -0.0017 - 0.0036i \\
-0.2159 - 0.0001i & 0.9759 - 0.00002i & 0.0310 \\
0.0084 - 0.0035i & -0.0299 - 0.0008i & 0.9995 
\end{pmatrix}, \tag{5.48}
\]

5.5.5.2 Determination of \( M_D \) at GUT scale

In order to fit the fermion masses at the GUT scale, in addition to the two bi-doublets originating from two different Higgs representations \( 10_{H_1} \) and \( 10_{H_2} \), we utilize the super-heavy bi-doublet in \( \xi(2, 2, 15) \subset 126_{H} \). We will show that even if \( \xi \) has to be at the intermediate scale \((10^{13} - 10^{14}) \text{ GeV}\) to generate the desired value of induced VEV needed for quark-lepton mass splitting, the precision gauge coupling unification is unaffected. This fermion mass requires the predicted Majorana coupling \( f \) to be diagonal and the model predicts experimentally testable RH neutrino masses. In the presence of inverse seesaw formula taking into account the small masses and large mixings in the LH neutrino sector in the way of fitting the neutrino oscillation data, this diagonal structure of \( f \) causes no problem. However we show that when we treat the intermediate scale for sub-multiplet to be \( \xi'(2, 2, 15) \) replacing \( \xi(2, 2, 15) \) but originating from a second Higgs representation \( 126'_{H} \) which has coupling \( f' \) to the fermions and all other scalar components at the GUT scale, the coupling \( f \) and hence \( M_N \) can have a general texture, not necessarily diagonal, although fermion mass fit needs only \( f' \) to be diagonal.

The VEV of \( \xi(2, 2, 15) \) is well known for its role in to splitting the quark and lepton masses through the Yukawa interaction \( f_{16.16.126_H}^{16} \) [32]. It is sometimes apprehended, as happens in the presence of only one \( 10_{H} \), that this new contribution may also upset the near equality of \( M^0_u \simeq M^0_D \) at the GUT scale. But in the presence of the two different \( 10_{H_1} \) and \( 10_{H_2} \) producing the up and down type doublets, the effective theory from the \( \mu \geq 10 \text{ TeV} \) acts like a non-SUSY two-Higgs doublet model with available large value of tan\( \beta = v_u/v_d \) that causes the most desired splitting between the up and down quark mass matrices but ensures \( M^0_u \sim M^0_D \). After having achieved this splitting a smaller value of of the VEV \( v_\xi \) is needed to implement fitting

\[
M^0_u(M_{\text{GUT}}) = \begin{pmatrix}
0.0097 & 0.0379 - 0.0069i & 0.0635 - 0.167i \\
0.0379 + 0.0069i & 0.2482 & 2.117 + 0.0001i \\
0.0635 + 0.167i & 2.117 - 0.0001i & 51.38 
\end{pmatrix} \text{ GeV}. \tag{5.49}
\]
of charged fermion mass matrices without substantially upsetting the near equality of $M^0_u \simeq M^0_D$ at the GUT scale \textsuperscript{a}.

The formulas for mass matrices at the GUT scale are \cite{45, 246}

\begin{align*}
M_u &= G_u + F, \quad M_d = G_d + F, \\
M_e &= G_d - 3F, \quad M_D = G_u - 3F. \tag{5.50}
\end{align*}

where $G_k = Y_k \langle 10^h_H \rangle$, $k = u, d$ and $F = f v_\xi$ leading to

\begin{equation}
 f = \frac{(M_d - M_e)}{4v_\xi}. \tag{5.51}
\end{equation}

Using a charged-lepton diagonal mass basis and eq. (5.47) and eq. (5.50) we have

\begin{align*}
M_e(M_{GUT}) &= \text{diag}(0.000216, 0.0388, 0.9620) \text{ GeV}, \\
G_{d,ij} &= 3F_{ij}, \quad (i \neq j). \tag{5.52}
\end{align*}

(i) Diagonal structure of RH neutrino mass matrix:

In refs. \cite{45, 246} dealing with TeV scale pseudo-Dirac RH neutrinos, a diagonal structure of $F$ was assumed with the help of higher dimensional non-renormalizable operators in order to fit the charged fermion masses and mixings at the GUT scale. In the present model renormalizable interaction of $126_H$ is available the diagonal structure of $F$ is a result of utilization of diagonal basis of down quarks as well.

This diagonal structure of $f$ would have caused serious problem in fitting the neutrino oscillation data if we had a dominant type-II seesaw formula \cite{65, 66}, but it causes no problem in our present model where type-II seesaw contribution to light neutrino mass matrix is severely damped out compared to inverse seesaw contribution which fits the neutrino oscillation data. Further, the resulting diagonal structure of RH neutrino mass matrix that emerges in this model has been widely used in SUSY and non-SUSY $SO(10)$ by a large number of authors, and this model creates no anomaly as there are no experimental data or constraints which are violated by this diagonal structure.

The quark mixings reflected through the CKM mixing matrix $V_{CKM} = U_L^T D_L = U_L$ has been parametrized at $\mu = M_Z$ in the down-quark diagonal basis and this

\textsuperscript{a}It is to be noted that the validity of our estimations of $0\nu2\beta$ decay and non-unitarity and lepton flavor violating effects do not require exact equality of $M_u$ and $M_D$ and a relation between them within less than an order of magnitude deviation would suffice to make dominant contributions at the TeV scale. But the present models, either with $G_{224D}$ or $SO(10)$ symmetry at the high GUT scale, give the high scale prediction $M^0_u \sim M^0_D$ up to a good approximation.
mixing matrix has been extrapolated to the GUT scale resulting in $V_{CKM}^0 \equiv U_L^0$ in eq. (5.48) provided $D_L^0 = I$ which can hold even at the GUT scale if we use down quark diagonal basis. In that case $M_d(M_{GUT}) = M_d^0 = \text{diag}(m_d^0, m_s^0, m_b^0)$ which is completely determined by the respective mass eigenvalues determined by the bottom-up approach. Then the second mass relation of eq. (5.50) gives,

$$G_{d,ij} = -F_{ij}, \ (i \neq j).$$

(5.53)

Now eq. (5.52) and eq. (5.53) are satisfied only if $F_{ij} = 0, \ (i \neq j)$ i.e, if $F$ is diagonal. This is also reflected directly through the eq. (5.51). In other words the diagonality of $F$ used in earlier applications of inverse seesaw mechanism in $SO(10)$ [45,246] is a consequence of utilization of down quark and charged lepton diagonal bases and vice-versa, although through non-renormalizable Yukawa interaction. In the present model it shows that even by restricting $F$ to its diagonal structure which eliminates at least six additional parameters which would have otherwise existed via its non-diagonal elements, the model successfully fits all the charged fermion masses and mixings including the Dirac phase of the CKM matrix at the GUT scale. Besides, as shown below, the model predicts the RH neutrino masses accessible to high energy accelerators including LHC. We have relations between the diagonal elements which, in turn, determine the diagonal matrices $F$ and $G_d$ completely.

$$G_{d,ii} + F_{ii} = m_i^0, \ (i = d, s, b),$$
$$G_{d, jj} - 3F_{jj} = m_j^0, \ (j = e, \mu, \tau).$$

(5.54)

$$F = \text{diag} \frac{1}{4} (m_d^0 - m_e^0, m_s^0 - m_\mu^0, m_b^0 - m_\tau^0),$$
$$= \text{diag}(2.365 \times 10^{-4}, -0.0038, +0.015) \text{ GeV},$$
$$G_d = \text{diag} \frac{1}{4} (3m_d^0 + m_e^0, 3m_s^0 + m_\mu^0, 3m_b^0 + m_\tau^0),$$
$$= \text{diag}(9.2645 \times 10^{-4}, 0.027224, 1.00975) \text{ GeV},$$

(5.55)

where we have used the RG extrapolated values of eq. (5.47). It is clear from the value of the mass matrix $F$ in eq. (5.55) that we need as small a VEV as $v_\xi \sim 10 \text{ MeV}$ to carry out the fermion mass fits at the GUT scale. In the Sec. 5.5.5.4 below we show how the $SO(10)$ structure and the Higgs representations given for the symmetry breaking chain of eq. (5.42) clearly predicts a VEV $v_\xi \sim (10 - 100) \text{ MeV}$ consistent with precision gauge coupling unification and the fermion mass values discussed in
this subsection.

The model ansatz for CKM mixings at the GUT scale matches successfully with those given by $V_{CKM}^0$ of eq. (5.48) and, similarly, the model predictions for up quark masses can match with those given in eq. (5.47) provided we can identify $M_u$ of eq. (5.50) with $M_0^u$ of eq. (5.49). This is done by fixing $G_u = M_0^u - F$ leading to

$$G_u(M_{GUT}) = \begin{pmatrix}
0.00950 & 0.0379 - 0.00693i & 0.0635 - 0.1671i \\
0.0379 + 0.00693i & 0.2637 & 2.117 + 0.000116i \\
0.0635 + 0.1672i & 2.117 - 0.000116i & 51.4436
\end{pmatrix} \text{ GeV. (5.56)}$$

Now using eq. (5.55) and eq. (5.56) in eq. (5.50) gives the Dirac neutrino mass matrix $M_D$ at the GUT scale

$$M_D^0(M_{GUT}) = \begin{pmatrix}
0.00876 & 0.0380 - 0.00693i & 0.0635 - 0.1672i \\
0.0380 + 0.00693i & 0.3102 & 2.118 + 0.000116i \\
0.0635 + 0.1672i & 2.118 - 0.000116i & 51.6344
\end{pmatrix} \text{ GeV. (5.57)}$$

The relation $F = f v_\xi = \text{diag}(f_1, f_2, f_3)v_\xi$ in eq. (5.53) with $v_\xi = 10$ MeV gives $(f_1, f_2, f_3) = (0.0236, -0.38, 1.5)$ b. Then the allowed solution to RGEs for gauge coupling unification with $M_{R}^0 = v_R = 5$ TeV determines the RH neutrino masses.

$$M_{N_1} = 115 \text{ GeV, } M_{N_2} = 1.785 \text{ TeV, } M_{N_3} = 7.5 \text{ TeV. (5.58)}$$

Here we note that $M_N$ in general is not a diagonal matrix. But, complete RGE analysis gives that the off-diagonal elements of $M_N$ are very small compared to the diagonal elements. Therefore, for simplicity we will use the eigenvalues of $M_N$ given in eq. (5.58) with the right phase (i.e. $M_{N_2} = -1.785 \text{ TeV}$) instead of complete $M_N$ matrix, for simplicity and intuitive predictions.

While the first RH neutrino is lighter than the current experimental limit on $Z'$ boson mass, the second one is in-between the $Z'$ and $W_R$ boson mass limits, but the heaviest one is larger than the $W_R$ mass limit. These are expected to provide interesting collider signatures at LHC and future accelerators. This hierarchy of the RH neutrino masses has been found to be consistent with lepton-number and LFVs discussed in Sec. 5.1, Sec. 5.3, and Sec. 5.4.

We estimate effective mass parameters for $0\nu 2\beta$ decay using this predicted di-
Figure 5.11: Estimations of effective mass parameter for $0\nu2\beta$ decay in the $W^-_L$-$W^-_L$ channel with sterile neutrino exchanges shown by top, middle, and bottom horizontal lines. The RH neutrino masses and the Dirac neutrino masses are derived from fermion mass fits and the sterile neutrino masses have been obtained through $M$ values consistent with non-unitarity constraints as described in the text.

agonal structure of $M_N$ and three sets of constrained $N$-$S$ mixing matrix $M_i = (40, 150, 1810)$ GeV, $M_i = (40, 200, 1720)$ GeV, and $M_i = (40, 300, 1660)$ GeV corresponding to the three sets of sterile neutrino mass eigenvalues $\hat{m}_S = (12.4, 12.5, 416)$ GeV, $\hat{m}_S = (12.5, 22.1, 377)$ GeV, and $\hat{m}_S = (12.4, 49, 350)$ GeV, respectively. The estimated values of the effective mass parameters in the $W^-_L$-$W^-_L$ channel due to sterile neutrino exchanges have been shown in Fig. 5.11 where the top, middle and the bottom horizontal lines represent $m^\text{ee,L}_S = 2.1$ eV, 1.3 eV, and 1.0 eV corresponding to the first, second and the third set, respectively. Thus the new values are found to be much more dominant compared to the standard predictions in this channel. Clearly the Heidelberg-Moscow results can be easily accommodated even for normally hierarchical or inverted hierarchical light neutrino masses.

(ii) General form of RH neutrino mass matrix:

Although we have shown the emergence of diagonal structure of $M_N$ from the successful fermion mass fits at the GUT scale, it is worthwhile to explore as to how this approach may also allow a general structure for the Yukawa coupling $f$ of $126_H$ and hence the RH neutrino mass matrix while giving a successful fit to charged fermion masses at the GUT scale. It is clear from the above discussions that this is not possible via renormalizable interaction if the model has only a single $126_H$. We introduce a second $126'_H$ with its coupling $f'$ and all its scalar sub-multiplets at the GUT-Plank scale except for the component $\xi'(2, 2, 15)$ which is tuned to have its mass at the in-
termediate scale $M_{\xi} \sim 10^{13}$ GeV-$10^{14}$ GeV. Also, as before, the VEV of the neutral component of $\Delta_R(1, 3, 10) \subset 126_H$ is used to contribute to the spontaneous breaking of $G_{2113} \rightarrow \text{SM}$, but the component $\xi(2, 2, 15)$ assumes its natural GUT scale mass without the necessity of being at the intermediate scale. All our results go through by redefining $F = f'v_{\xi'}$ and $v_{\xi'} = (10 - 100)$ MeV is realized in the same way as discussed below in Sec. 5.5.5.4. In this case the diagonal structure of $f'$ gives the same successful fit to charged fermion masses and mixings at the GUT scale without affecting the allowed general structure of $f$ and $M_N$. Unlike the case (i) with single $126_H$ discussed above, as $f_1$ is not constrained to be small, observable $n-\bar{n}$ oscillation is possible in this case for all di-quark Higgs scalar masses $M_\Delta \sim M_C \sim 10^5 - 10^6$ GeV already permitted by RGE solutions to precision gauge coupling unification.

So far we have discussed emergence of dominant $0\nu2\beta$ decay rates subject to non-unitarity constraints with either a purely diagonal or nearly diagonal $M_N$ matrix with small mixing. To test whether such results exist for a general structure, we consider a mass matrix,

$$M_N = \begin{pmatrix} 1853.6 + 320.5i & -2165.2 - 47.98i & 2064.69 + 364.44i \\ -2165.24 - 47.98i & 2818.92 - 210.57i & -2030.45 + 245.8i \\ 2064.69 + 364.436i & -2030.45 + 245.82i & 4610.57 - 2.68i \end{pmatrix} \text{GeV} \quad (5.59)$$

which has the eigenvalues $M_{N_i} = (115, 1750, 7500)$ GeV with the same mixings as the LH neutrinos. Using eq. (5.59), the non-unitarity constrained $N-S$ mixing matrix $M = \text{diag}(40, 150, 1810)$ GeV, and the derived value of the Dirac neutrino mass matrix from eq. (5.61) leads to the sterile neutrino mass eigenvalues $m_{S_i} \sim (0.77, 51, 878)$ GeV and the resulting effective mass parameters in the notations of Sec. 5.3- Sec. 5.5 are found to be

$$m^{eeL}_S = 6.3 \text{ eV}, \quad m^{eL}_N = 0.02 \text{ eV}, \quad m^{eLR}_S = 0.08 \text{ eV}. \quad (5.60)$$

and all other contributions are relatively insignificant. Thus, we see that the dominant contribution in the $W_L-W_L$ channel due to sterile neutrino exchanges dominates over all other contributions. The difference of $O(10^2)$ in the leading and next to leading contributions to effective mass resemble the results of diagonal structure of $M_N$, if $m_{S_i}$ are of same order, as presented in Sec. 5.3, Fig. 5.6 - Fig. 5.8 and Tab. 5.5 - Tab. 5.6. A general $M_N$, such as in eq. (5.59), is not restricted by GUT-scale fermion mass fits. Henceforth, we will use the diagonal form of $M_N$, eq. (5.58), as in Fig. 5.11.
\subsection*{5.5.5.3 Determination of $M_D(M_R^0)$ by top-down approach}

We use the RGEs in the top-down approach \cite{186,246,288-290} for $M_D$ in the presence of $G_{224D}$, $G_{224}$, $G_{2213}$, and $G_{2113}$ to evolve $M_D(M_{GUT})$ to $M_D(M_R^0)$ through $M_D(M_{M_P})$, $M_D(M_{M_C})$ and $M_D(M_{M_R^+})$ and obtain the ansatz given in eq. (5.11) as

\[
M_D = \begin{pmatrix}
0.02274 & 0.09891 - 0.01603i & 0.1462 - 0.3859i \\
0.09891 + 0.01603i & 0.6319 & 4.884 + 0.0003034i \\
0.1462 + 0.3859i & 4.884 - 0.0003034i & 117.8 \\
\end{pmatrix} \text{GeV. \ (5.61)}
\]

As can be noted from the determination of running mass eigenvalues at the high GUT scale of the model shown in eq. (5.47), $b$-$\tau$ unification is almost perfect, although $m_\mu^0 \simeq 2 m_s^0$. In view of the fact that $G_{224}$ symmetry with unbroken $SU(4)_C$ gauge symmetry is present in this model right from $M_C \simeq 10^6$ GeV up to the high GUT scale $M_{GUT} \sim 10^{17.5}$ GeV, the dominance of quark lepton symmetry has manifested in the fermion mass relations like $m_\tau^0 \simeq m_\mu^0 \simeq 1.06$ and $M_u^0 \simeq M_B^0$ at the GUT scale while making the $SU(4)_C$-breaking effects sub-dominant. The bi-doublet $\xi(2,2,15) \subset 126_H$ has been found to make a small contribution resulting in the mass matrix $F$ in eq. (5.55) which plays an important role in our present model. The impressive manner in which the underlying quark-lepton symmetry manifests in exhibiting $M_u(M_{GUT}) \simeq M_D(M_{GUT})$ can be noted from the explicit forms of the two mass matrices derived at the GUT scale and shown in eq. (5.49) and eq. (5.57).

Thus, the present non-SUSY $SO(10)$ model, having predicted $M_D$ value given in eq. (5.11), all our discussions using TeV scale inverse see-saw mechanism including neutrinoless double beta decay, non-unitarity effects leading to LFVs, and new $CP$-violating effects discussed in Sec. 5.1 - Sec. 5.4, where this mass matrix has been used, are also applicable in this GUT model.

\subsection*{5.5.5.4 Determination of induced vacuum expectation value of $\xi(2,2,15)$}

Now we show how a small induced VEV $v_\xi \sim 10$ MeV of the sub-multiplet $\xi(2,2,15) \subset 126_H$, which has been found to be necessary for fitting the charged fermion masses at the GUT scale, originates from the the present $SO(10)$ model. The Higgs representations needed for the symmetry breaking chain permits the following term in

\footnote{While running mass eigenvalues are extrapolated up to the non-SUSY $SO(10)$ unification scale in the presence of $G_{2113}$ and $G_{2213}$ intermediate scales \cite{246}, it has been noted that at the GUT scale $m_\mu^0/m_s^0 \simeq 1.3$, $m_\mu^0/m_s^0 \simeq 2.5$, and $m_d^0/m_e^0 \simeq 4$. Compared to refs. \cite{45,246} where a non-renormalizable $dim.6$ operator has also been used for fermion mass fits at the GUT scale, all the interactions used in this work are renormalizable.}
the Higgs potential

$$\lambda_\xi M' 210_H 126^+_H 10_H \supset \lambda_\xi M' (2, 2, 15)_{126} (1, 1, 15)_{210} (2, 2, 1)_{10}$$

(5.62)

where $M'$ is a mass parameter appropriate for trilinear scalar coupling which is naturally of the order of the GUT scale $\sim O(10^{18})$ GeV. For allowed solutions of the mass scales in our model, we have found $\langle (1, 1, 15) \rangle = M_C \simeq 10^6$ GeV, a criteria necessary for observable $n-\bar{n}$ oscillation and rare kaon decay. The induced $v_\xi$ then turns out to be

$$v_\xi = \lambda_\xi M' M_C v_{ew}/M_\xi^2$$

(5.63)

Using $M_C \simeq 10^6$ GeV which is required as model predictions for observable $n-\bar{n}$ oscillation and rare kaon decay, and $v_{ew} \sim 100$ GeV, we find that for $\lambda_\xi = 0.1 - 1.0$, the eq. (5.63) gives the induced VEV $v_\xi \simeq (10 - 100)$ MeV provided $M_\xi \sim 10^{13}$ GeV-$10^{14}$ GeV. When $\xi(2, 2, 15)$ is made lighter than the GUT scale having such an
Table 5.11: Allowed solutions in the $SO(10)$ symmetry breaking chain shown in eq. (5.42) but with the scalar component $\xi(2,2,15) \subset 126_H$ lighter than the GUT scale and consistent with the determination of the induced VEV $v_\xi \sim (10 - 100)$ MeV needed to fit charged fermion masses at the GUT scale. For all solutions we have fixed $M_R^0 = 5$ TeV, $M_R^+ = 10$ TeV, and $M_C = 10^6$ GeV.

intermediate mass, $M_\xi = 10^{13.4}$ GeV, the precision gauge coupling is found to occur as shown in Fig. 5.12 but now with nearly two times larger GUT scale and larger GUT fine-structure constant than the minimal case. Our numerical solutions are shown in Tab. 5.11 where the Parity violating scale is close to the minimal case. It is interesting to note that the precision unification with $\xi(2,2,15) \subset 126_H$ at the intermediate scale is possible without upsetting low mass $W_R, Z', M_C$ and other mass scale predictions of the model. The fermion mass evolutions and the emerging value of $M_D$ remain close to the value derived in Sec. 5.5.5. The unification pattern and model predictions including GUT-scale fermion mass fit are essentially unchanged when the second $126'_H$ is introduced with its Yukawa coupling $f'$ and the component $\xi'(2,2,15) \subset 126'_H$ at the intermediate scale replacing $\xi(2,2,15) \subset 126_H$ and the latter is assigned its natural GUT scale mass. In this case the mass scales of the model give $v_{\xi'} = (10 - 100)$ MeV. As the additional threshold contributions to $\sin^2 \theta_W$ and $M_P$ due to the super-heavy components of second $126'_H$ vanish [279–281], the only change that can occur is the GUT-scale threshold effects on $M_{GUT}$. However as the unification scale is close to the Plank scale with large proton lifetime prediction, this will not have any additional observable effects.

Thus, we have shown that the small induced VEV $v_\xi$ or $v_{\xi'}$ needed for GUT scale fit to the charged fermion masses and prediction of $M_D$ which is crucial for low-energy estimation of $0\nu2\beta$ decay rate can be easily derived from the present $SO(10)$ structure. It is possible to have a diagonal structure or a general structure for the RH neutrino mass matrix $M_N$ for which dominant contributions to $0\nu2\beta$ decay, experimentally accessible lepton flavor violating decays, and non-unitarity and $CP$-violating effects have been discussed in Sec. 5.3 and Sec. 5.4.
5.5.6 Suppressed induced contribution to $\nu$-$S$ mixing

In our model the $\nu$-$S$ mixing term has been chosen to be vanishingly small in eq. (5.3). However, because of the presence of non-minimal Higgs fields including LH and RH doublets carrying $B-L = -1$, triplets carrying $B-L = -2$, two bi-doublets each with $B-L = 0$, and Parity odd singlet, it is necessary to evaluate if such a term can arise through the induced VEV of $\langle \chi_L \rangle$. We find that without taking recourse to any severe fine tuning of parameters, minimization of the scalar potential gives

$$\langle \chi_L \rangle \simeq K \frac{\langle \chi_R \rangle v}{M_P}, \tag{5.64}$$

where the ratio of parameters $K = \mathcal{O}(0.1-.01)$ and $M_P \simeq 10^{14}$ GeV. When eq. (5.64) is used in the corresponding correction to the light neutrino mass predictions [291, 292], $m_\nu \simeq M_D \frac{\langle \chi_L \rangle}{\langle \chi_R \rangle}$, this gives $m_{\nu_{33}} << 0.001$ eV and negligible contributions to all the three light neutrino masses. With fine-tuning of parameters this contribution can be reduced further. Thus the predictions of the model carried out using eq. (5.3) are found to hold up to a very good approximation as the small induced contribution $\langle \chi_L \rangle$ does not affect the results substantially. Fine tuning of model parameters would result in further reduction of this contribution.