Chapter 4

Measuring of Time-Frequency Representation (TFR)- using the Kapur’s entropy
4.1 Introduction.

It has been well understood that a given signal can be represented in an infinite number of different ways. No doubt, different signal representations can be used for different applications. Despite the fact, the number of ways of describing a given signal is countless, the most popular, important and fundamental variables are: time and frequency. The time domain indicates how a signal’s amplitude changes over time and the frequency domain indicates how often these changes take place. The key for the description of a signal was to find a form which would unite the variables.

The most common representation of signals and waveforms is in the time domain. However, most signal analysis techniques work only in the frequency domain. The concept of the frequency domain representation of a signal is quite difficult to understand when one is first introduced to it. The frequency plane is orthogonal to the time plane, and intersects with it on a line which is the amplitude axis.

By Fourier analysis, we can decompose signal into individual frequency components and establish the relative intensity of each component. The energy spectrum unable to tell us when those frequencies occurred. For instance, in sunset, the composition of the light reaching us is very different than what it is during most of the day. If we analyze the light from sunrise to sunset by fourier analysis, the energy spectrum would not tell us significant difference of the spectral composition in the last five minutes. If the light occurred considerable changes faster than five minutes, then we may shorten the time slab accordingly. This is the basic idea of short time fourier transform, which is currently the standard method for the study of time-varying signals.

However, there exists natural and man made signals whose spectral content is changing so rapidly that finding an appropriate short time slab is a problem, since there may not be any time interval for which the signal is more or less stationary. Also, decreasing the time slab so that one may locate events in time reduces the frequency resolution. Hence, there is an inherent trade off between the time and frequency resolution.
The prime example of signals whose frequency content is changing rapidly is a human speech. Speech is one of the most complex non-stationary signals and a natural application for these time-frequency distribution.

In this chapter, we study the properties of the Kapur’s entropy, with emphasis on the mathematical foundations for quadratic TFRs. In particular, for the Wigner distribution we establish some results that there exist signals for which the measures are well defined. The generalized entropy of Kapur inspire new measures for estimating signal information and complexity in the time-frequency plane. When applied to a time-frequency representation (TFR) from Cohen's class or the affine class, the Kapur's entropy confirm closely to the concept of complexity. Baraniuk et al. [2001] used Renyi’s entropy for measuring TFRs and give some results. However, we are using Kapur’s entropy and found that it will give us much better results for measuring TFRs than Renyi’s entropy.

In section (4.2), we will study Time-Frequency representation (TFR). Formulation of quadratic forms of Time-Frequency representation (TFR) and Time-Frequency distribution (TFD) will discuss in section (4.3). In section (4.4), we will measure Time-Frequency Representation (TFR), using the Kapur’s entropy. We validate some results for measuring Time-Frequency Representation (TFR) in sections (4.4.1) and (4.4.2).

4.2 Time-Frequency Representation (TFR).

The concept of time and frequency representation of signal dates back to the first notation for music: a sequence of notes to be played in time $s(t)$, each note being associated with a fundamental frequency and its higher harmonics. From a mathematical viewpoint, we can associate the time function $s(t)$ to its fourier transform $s(f)$. However, $s(f)$ is independent of time and gives no indication as the note to be played at each particular time instant. It would therefore be useful to introduce a representation of signals simultaneously in time and frequency. The time-frequency distributions (TFDs) contain more information about non-stationary signals.
The fundamental problem of the time-frequency analysis is to discover a good mathematical
device, able to simultaneously represent a given signal \( s(t) \) in terms of its intensity in time and
frequency.

Despite their simple interpretation of pure frequencies, the fourier transform is not always the
best tool to analyze real life signals. Real life signals are usually finite, short duration and have
frequency contents changes over time. That is why, joint time frequency transforms were
developed for the purpose of characterizing the time-varying frequency content of a signal. Many
transforms were developed and used at different applications. The developed transforms are
divided into two classes:

(i) Linear Time Frequency Transforms.

(ii) Quadratic (Bilinear) Transforms.

The first class belong to transforms as Short Time Fourier Transform (STFT), Continuous
Wavelet Transform (CWT) and Adaptive Time-Frequency Representation where as, the second
class belongs the Wigner-Ville Distribution (WVD) as discussed by Hlawatch [1997]. Therefore,
WVD introduces cross terms for multi component signals. These cross terms are removed by
using smoothing kemels. The choice of smoothing kernel gives rise to different TFRs.

Signals can be modeled as linear combination of polynomial phase signals appear in several
applications including active noise cancellation, communication using chirp modulation, radar
etc.

A time-frequency representation (TFR) is a view of a signal (taken to be a function of time)
represented over both time and frequency. Time-frequency analysis means analysis into the time-
frequency domain provided by a TFR. This is achieved by using a formulation often called
Time-Frequency Distribution (TFD). TFRs are often complex-valued fields over time and
frequency, where the modulus of the field represents "energy density" (the concentration of the
root mean square over time and frequency) or amplitude, and the argument of the field represents phase.

A signal, as a function of time, may be considered as a representation with perfect time resolution. TFRs provide a bridge between these two representations in that they provide some temporal information and some spectral information simultaneously. Thus, TFRs are useful for the representation and analysis of signals containing multiple time-varying frequencies.

4.3 Formulation of Quadratic forms of TFRs and TFDs.

One form of TFR (or TFD) can be formulated by the multiplicative comparison of a signal with itself, expanded in different directions about each point in time. Such representations and formulations are known as quadratic TFRs or TFDs (QTFRs or QTFDs) because the representation is quadratic in the signal. This formulation was first described by Eugene Wigner in 1932 in the context of quantum mechanics and, later, reformulated as a general TFR by Ville in 1948 to form what is now known as the Wigner-Ville distribution, as it was shown that Wigner's formula needed to use the analytic signals for practical analysis.

Although quadratic TFRs offer perfect temporal and spectral resolutions simultaneously, the quadratic nature of the transforms creates cross-terms.

The basic idea to devise a joint function of time and frequency, a distribution that will describe the energy density or intensity of a signal simultaneously in time and frequency. In ideal case, such a joint distribution would be used and manipulated in the same manner as any density function of more than one variable.

For instance, if we had a joint density for the height and weight of humans, we could obtain the distribution of height by integrating out weight. The method related to joint time-frequency distribution to a signal will be a powerful tool for the construction of signals with desirable properties. Undoubtedly, time-frequency analysis has unique features, such as the uncertainty principle, which add the richness and challenge to the field.
From Standard Fourier analysis, the instantaneous energy of the signal $s(t)$ or the intensity per unit time at time $t$ is $|s(t)|^2$

or $|s(t)|^2 \Delta t = \text{fractional energy in time interval } \Delta t \text{ at time } t$.

The intensity per unit frequency at $f$ is $|s(f)|^2$

or $|s(f)|^2 \Delta f = \text{fractional energy in frequency interval } \Delta f \text{ at frequency } f$.

where, $s(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s(t) e^{-if\tau} \, dt$  \hspace{1cm} (4.3.1)

The normalization of the signal is

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} |s(f)|^2 \, df$$ \hspace{1cm} (4.3.2)

$$= \int \int_{-\infty}^{\infty} C(t, f) \, dt \, df$$ \hspace{1cm} (4.3.3)

$$= \text{total energy of the signal.}$$

For convenience, we always take the total energy equal to unity.

Signals that cannot normalize may be taken as limiting cases of normalized ones or by using generalized functions. Our main goal is to devise a joint function of time and frequency which represents the energy or intensity per unit time per unit frequency.

For a joint distribution $C(t, f)$, we have

$C(t, f) = \text{intensity at time } t \text{ and frequency } f$.

or $C(t, f) \Delta t \Delta f = \text{fractional energy in time-frequency interval } \Delta t \Delta f \text{ at } t, f$.

The total energy $E$, expressed in terms of distribution is given by
\[ E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(t, f) \, dt \, df \quad (4.3.4) \]

and will be equal to the total energy of the signal if the marginals are satisfied.

where,

\[ \int_{-\infty}^{\infty} C(t, f) \, df = |s(t)|^2 \quad (4.3.5) \]

and

\[ \int_{-\infty}^{\infty} C(t, f) \, dt = |s(f)|^2 \quad (4.3.6) \]

Hence, \(|s(t)|^2\) and \(|s(f)|^2\) are the marginals of \(C(t, f)\).

However, it is possible for a distribution to give correct value for the total energy without satisfying the marginals.

Gabor [1946] and Ville in 1948, addressed the joint distribution function in its two original papers. Gabor developed a mathematical method closely connected to the coherent states in quantum mechanics and also introducing the concept of analytical signal. Ville derived a distribution that Wigner gave in 1932 to study quantum statistical mechanics.

### 4.4 Measuring of Time-Frequency Representation (TFR)- using the Kapur’s entropy.

The term *component* is encountered in the signal processing. In general, a component is a concentration of energy in some domain, but it is very difficult to translate this concept into a quantitative concept explained by Williams et al. ([1991], [1996]), Orr [1991], Michael [1994] and Cohen [1992]. In other words, the concept of a signal component may not be clearly defined and this term is particularly used in the analysis of time-frequency. Time-frequency representations (TFRs) generalize the concept of the time and frequency
domains to a function $C_s(t, f)$. Cohen [1992] give the concept that how the frequency of a signal $s$ changes over a time $t$.

Let us consider $|s(t)|^2$ and $|s(f)|^2$ be the unidimensional densities of signal energy in time and frequency. It is very difficult for TFRs to act as bidimensional densities of signal energy in time-frequency.

In particular, there exist some marginal properties of TFRs parallel to those of probability densities:

$$\int C_s(t, f) \, df = |s(t)|^2,$$(4.4.1)

$$\int C_s(t, f) \, dt = |s(f)|^2,$$(4.4.2)

$$\iint C_s(t, f) \, dt \, df = \int |s(t)|^2 \, dt = \|s\|_2^2.$$ (4.4.3)

Cohen [1989], the quadratic TFRs of the large and useful Cohen’s Class can be obtained by convolution as:

$$C_s(t, f) = \iint W_s(u, v) \Phi(t - u, f - v) \, du \, dv = (W_s * \Phi)(t, f)$$ (4.4.4)

where, $\Phi$ is a kernel function.

The Wigner distribution $W_s$ of the signal is as:

$$W_s(t, f) = \int s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi \tau f} \, d\tau$$ (4.4.5)

and the associated spectrum of the signal $s(t)$ is

$$W_s(t, f) = \int s\left(\frac{\theta}{2}\right) s^*\left(\frac{\theta}{2}\right) e^{-j2\pi \theta t} \, d\theta$$ (4.4.6)
Also, taking $\phi_w(\theta, \tau) = 1$

The classical Shannon Entropy for unit-energy signals:

$$H(C_s) = -\iiint C_s(t, f) \log C_s(t, f) \, dt \, df$$  \hspace{1cm} (4.4.7)

(4.4.7) is a measure for the complexity of a signal through TFRs.

Here, complexity measure (Shannon’s Entropy) is only applied to TFRs, which acts as a probability density function not on a signal or process. The peaky TFRs of signals consists of small numbers of elementary components would give small entropy values, where as, the diffuse TFRs of more complicated signals would give large entropy values.

However, the negative values taken by most of the TFRs (including all fixed-kernel Cohen’s class TFRs satisfying (4.4.1) and (4.4.2)) are unable to give the application of the Shannon entropy due to the logarithm in (4.4.7).

Baraniuk [2001] used the concept of Renyi’s entropy for unit-energy signal to sideline the issue of negativity as:

$$H_\alpha(C_s) = \frac{1}{1 - \alpha} \log \iiint C_s^\alpha(t, f) \, dt \, df$$  \hspace{1cm} (4.4.8)

In order to keep the same thing in mind, we are employing the generalized Kapur’s entropy of order $\alpha$ and type $\beta$ (Kapur [1967])) (again for unit-energy signals)

$$H_{\alpha, \beta}(C_s) = \frac{1}{\beta - \alpha} \log \left( \frac{\iiint C_s^\alpha(t, f) \, dt \, df}{\iiint C_s^\beta(t, f) \, dt \, df} \right), \alpha \neq \beta, \hspace{0.2cm} \alpha, \beta > Z^+$$  \hspace{1cm} (4.4.9)

In the limiting case, when $\alpha \to 1$ and $\beta = 1$, $H_{\alpha, \beta}(C_s)$ reduces to $H(C_s)$ and when $\beta = 1$, $H_{\alpha, \beta}(C_s)$ reduces to $H_\alpha(C_s)$.
The class of information measures is obtained by taking the mean value property of the Shannon entropy from an arithmetic to an exponential mean. In several empirical studies, Williams et al. [1991] found that the appearance of negative TFR values invalidate the Shannon approach, the 3rd-order Renyi’s entropy seemed to measure signal complexity.

Since Kapur’s entropy of order $\alpha$ and type $\beta$ is the generalization of Renyi’s entropy. In our study, we are using the generalized Kapur’s entropy of order-3 and type $\beta$ to measure signal complexity. Also, we will discuss in detail the properties and some applications of the Kapur time-frequency information measures (4.4.9), which emphasis on the quadratic TFRs.

After reviewing the measure, we will examine the existence and show that for odd $\alpha$ and even $\beta$ or vice-versa and for $\alpha \neq \beta$, there exist signals $s$ for which (4.4.9) is not defined due to

$$\left( \frac{\iint C_s^\alpha(t,f) \, dt \, df}{\iint C_s^\beta(t,f) \, dt \, df} \right) < 0.$$  

Also, for odd integers $\alpha, \beta \geq 1$, (4.4.9) is defined for the 1st order Hermite function.

While using Renyi’s entropy, Baraniuk [2001] fails for sufficiently large order $\alpha$, but while using kapur’s entropy for measuring TFRs, we get promising results will discuss in next sections. The 3rd-order Renyi’s entropy is defined for a broad class of distributions including those taking locally negative values.

The properties of the Kapur time-frequency information measures are as follows:

1. $H_{\alpha,\beta}(C_s)$ counts the number of components in a multi component signal.

2. For odd orders $\alpha, \beta > 1$, $H_{\alpha,\beta}(C_s)$ does not count the no. of components, due to asymptotically invariant to TFR cross- components.

3. The range of $H_{\alpha,\beta}(C_s)$ values is bounded from below. For Wigner distribution, a single Gaussian pulse attains the lower bound.
The values of $H_{\alpha,\beta}(C_s)$ are invariant to time and frequency shifts of the signal.

Since $H_{\alpha,\beta}(C_s)$ is the generalization of $H_{\alpha}(C_s)$. Thus, for more general invariances, $H_{\alpha,\beta}(C_s)$ takes not only the TFRs of the affine class but also the generalized representations of the unitarily equivalent Cohen's and affine classes.

The application of entropy measures to TFRs to measure the complexity and information content of non stationary signals in time-frequency plane is discuss here. Primarily, the TFR tools lie in Cohen’s class (Cohen [1992]), which is expressed in (4.4.4) as the convolution between the wigner distribution and a real valued kernel $\Phi \in L^1(\mathbb{R}^2)$. To avoid confusion, we restrict ourself only to the wigner distribution and all the TFRs obtained from (4.4.4).

Since we are interested only in odd powers of TFRs, the kernel $\Phi$ and its inverse fourier transform $\phi$ completely determine the properties of the corresponding TFR.

For instance, A fixed kernel TFR possesses the energy preservation property (4.4.3) provided $\phi(0,0) = 1$ and the marginal properties (4.4.1) and (4.4.2) provided $\phi(\theta,0) = \phi(0,\tau) = 1, \forall \theta, \tau$

where, $\phi$ is uncertainty function of the time-reversed window function and $\phi(\theta,\tau) = h_1(\theta) h_2(\tau)$ is the smoothed pseudo-wigner distribution.

The similarity between TFRs and bi-dimensional probability densities ceases at atleast two points:

[1] The TFR of a given signal is non unique, due to the choice of kernel function.

[2] Most Cohen’s class TFRs are non positive and thus cannot be interpreted strictly as densities of signal energy, because here we consider only quadratic TFRs. There must exist non quadratic classes of TFRs which satisfy (4.4.1), (4.4.2) and (4.4.3).
These locally negative values creates a confusion with the logarithm in the Shannon entropy (4.4.7), while the Kapur’s entropy (4.4.9) makes interesting and encouraging the applications of TFRs (William et al. (1991), [1996]), Baraniuk et al. ([1993], [1994]). It is indeed a point of discussion, whether these measures deal successfully the locally negative of Cohen’s class TFRs.

For (4.4.9), we need \( C_s^\alpha (t, f), C_s^\beta (t, f) \) to be real for a signal \( s \) changes over a time \( t \) such that

\[
\left( \frac{\iint C_s^\alpha (t, f) \, dt \, df}{\iint C_s^\beta (t, f) \, dt \, df} \right) > 0
\]  

(4.4.10)

For non integer \( \alpha, \beta, C_s^\alpha (t, f), C_s^\beta (t, f) \) possess complex values and have limited utility. Also, for even integer \( \alpha, \beta, C_s^\alpha (t, f), C_s^\beta (t, f) \) always possess positive values and pose no such hazards. But, odd integer orders are not so strong in giving the positive value of \( C_s^\alpha (t, f), C_s^\beta (t, f) \).

For each odd integer \( \alpha, \beta \geq 3 \), there exist signals in \( L^2(\mathbb{R}) \) and TFRs such that (4.4.10) is not satisfied and by virtue of this (4.4.9) is not defined. Here, we develop some results in the form of theorems for the wigner distribution \( W_s (t, f) \), which invalidate the proofs of existence in the time-frequency plane based on kapur’s entropy.

For any odd integer \( \alpha, \beta \geq 3 \), a signal \( s \) for the wigner distribution \( W_s (t, f) \) such that

\[
\left( \frac{\iint W_s^\alpha (t, f) \, dt \, df}{\iint W_s^\beta (t, f) \, dt \, df} \right) > 0
\]  

(4.4.11)

For such signal \( s \), \( H_{\alpha, \beta}(W_s) \) is not defined.

The \( n^{th} \) order Hermite function is:
\[ h_n(t) = (-1)^n \ 2^{1/4} \ (n!)^{-1/2} \ (4\pi)^{-n/2} e^{\pi \frac{t^2}{2}} \left( \frac{d}{dt} \right)^n e^{-2\pi \ t^2} \]  \hspace{1cm} (4.4.12)  

The \( n^{th} \) order Hermite function has a wigner distribution that can be expressed in terms of a Laguerre polynomial as:

\[ W_n(t, f) = W_{h_n}(t, f) = 2(-1)^n \ e^{-2\pi \ t^2} L_n(4\pi \ r^2) \]  \hspace{1cm} (4.4.13)  

where, \( r^2 = t^2 + f^2 \) and \[ L_n(x) = \sum_{j=0}^{n} \frac{(-x)^j}{j!} \]  \hspace{1cm} (4.4.14)  
is the \( n^{th} \) order Laguerre polynomial.

It is well known that, the Hermite function has wigner distribution as:

\[ 1 \] When the order \( n \) is odd, Hermite function is strongly peaked at the origin, with a negative sign.

\[ 2 \] When origin has radius larger than \( \left( \frac{n + \frac{1}{2}}{\pi} \right)^{1/2} \), Hermite function has small but non-negligible values away from the origin but inside the circle.

\[ 3 \] Hermite function has negligibly small values outside the circle.

Therefore, the odd-order Hermite functions are giving negative values in (4.4.11)

Let us validate the results by the help of following theorems as:
**Theorem 4.4.1.**

For sufficiently large order $\alpha, \beta$ for Wigner distribution, (4.4.11) is satisfied for odd $\alpha$ and even $\beta$ and vice-versa and for $\alpha \neq \beta$ for smooth, rapidly decaying odd signal.

**Proof.**

Let $s$ be a smooth, rapidly decaying odd signal of unit energy, then $W_s(t,f)$ is smooth and rapidly decaying as $t^2 + f^2 \to \infty$ and

$$|W_s(t,f)| < 2 = -W_s(0,0), \quad (t,f) \neq (0,0)$$  \hspace{1cm} (4.4.15)

It can be done by Cauchy-Schwarz inequality.

As $\alpha, \beta \to \infty$, the asymptotic behavior of (4.4.11) is determine by the behavior of $W_s(t,f)$ at $(t,f) = (0,0)$.

Since

$$\frac{\partial W_s}{\partial t} = \frac{\partial W_s}{\partial f} = \frac{\partial^2 W_s}{\partial t \partial f} = \frac{\partial^2 W_s}{\partial f \partial t} = 0$$  \hspace{1cm} (4.4.16)

$$\frac{\partial^2 W_s}{\partial t^2} = 8\|s^*\|^2, \quad \frac{\partial^2 W_s}{\partial f^2} = 32\pi^2 \|s(t)\|^2$$  \hspace{1cm} (4.4.17)

As $(t,f) = (0,0)$, we have

$$\log\left(-\frac{1}{2} W_s(t,f)\right) = -2t^2 \|s^*\|^2 - 8\pi^2 f^2 \|s(t)\|^2 + O(t^2 + f^2)$$  \hspace{1cm} (4.4.18)

As $t^2 + f^2 \to 0$, we have
\[
\int W_s^\alpha (t, f) \, dt \, df = \frac{(-2)^\alpha \int \exp \left( \alpha \log \left( -\frac{1}{2} W_s(t, f) \right) \right) \, dt \, df}{(-2)^\beta \int \exp \left( \beta \log \left( -\frac{1}{2} W_s(t, f) \right) \right) \, dt \, df}
\]

\[
= \frac{(-2)^\alpha}{4 \alpha \| s \| \| s(t) \|}
\]

\[
= \frac{(-2)^\beta}{4 \beta \| s \| \| s(t) \|}
\]

\[
= (-2)^{\alpha - \beta} \frac{\beta}{\alpha}
\]

(4.4.19)

As \( \alpha, \beta \to \infty \), (4.4.11) is satisfied for large odd integer \( \alpha \) and even \( \beta \) and vice-versa and for \( \alpha \neq \beta \) for smooth, rapidly decaying odd signal \( s \).

**Theorem 4.4.2.**

For all positive \( \alpha, \beta \geq 1 \), (4.4.11) is satisfied for the 1st order Hermite function.

**Proof.**

Let \( s \) be the 1st order Hermite function with wigner distribution as:

\[
s(t) = h_1(t) = 2^{\frac{5}{4}} \pi^{\frac{1}{2}} t e^{-\pi t^2}, \quad t \in \mathbb{R}
\]

\[
W_s(t, f) = -2 e^{-2\pi(t^2 + f^2)} (1 - 4\pi(t^2 + f^2))
\]

Using polar coordinates, we have for \( \alpha, \beta \geq 1 \)
\[
\frac{\iint W_x^{\alpha} (t, f) \, dt \, df}{\iint W_x^{\beta} (t, f) \, dt \, df} = \frac{\frac{2^{\alpha-1}}{\alpha} (-1)^{\alpha} \int_0^{\infty} e^{-x} \left(1 - \frac{2x}{\alpha}\right)^{\alpha} \, dx}{\frac{2^{\beta-1}}{\beta} (-1)^{\beta} \int_0^{\infty} e^{-x} \left(1 - \frac{2x}{\beta}\right)^{\beta} \, dx}
\]  \hspace{1cm} (4.4.20)

For odd \( \alpha \geq 1 \), we have

\[
\int_0^{\infty} e^{-x} \left(1 - \frac{2x}{\alpha}\right)^{\alpha} \, dx = \int_0^{\alpha/2} e^{-x} \left(1 - \frac{2x}{\alpha}\right)^{\alpha} \, dx - \int_{\alpha/2}^{\infty} e^{-x} \left(\frac{2x}{\alpha} - 1\right)^{\alpha} \, dx
\]  \hspace{1cm} (4.4.21)

The first term on the right hand side of the (4.4.21), increases in \( \alpha \geq 1 \), where as second term decreases in \( \alpha \geq 1 \) and can be evaluated as:

\[
\int_{\alpha/2}^{\infty} e^{-x} \left(\frac{2x}{\alpha} - 1\right)^{\alpha} \, dx = \left(\frac{4}{e}\right)^{\alpha/2} \frac{\alpha!}{\alpha^\alpha}.
\]

Thus (4.4.21) increases for odd \( \alpha \geq 1 \).

For odd \( \beta \geq 1 \), we have

\[
\int_0^{\infty} e^{-x} \left(1 - \frac{2x}{\beta}\right)^{\beta} \, dx = \int_0^{\beta/2} e^{-x} \left(1 - \frac{2x}{\beta}\right)^{\beta} \, dx - \int_{\beta/2}^{\infty} e^{-x} \left(\frac{2x}{\beta} - 1\right)^{\beta} \, dx
\]  \hspace{1cm} (4.4.22)

The first term on the right hand side of the (4.4.22), increases in \( \beta \geq 1 \), where as second term decreases in \( \beta \geq 1 \) and can be evaluated as:

\[
\int_{\beta/2}^{\infty} e^{-x} \left(\frac{2x}{\beta} - 1\right)^{\beta} \, dx = \left(\frac{4}{e}\right)^{\beta/2} \frac{\beta!}{\beta^\beta}
\]

Thus, (4.4.22) increases for odd \( \beta \geq 1 \).
Since,

For $\alpha = \beta = 1$:
\[
\int_0^\infty e^{-x} (1 - 2x) \, dx = -1
\]

For $\alpha = \beta = 3$:
\[
\int_0^\infty e^{-x} \left(1 - \frac{2x}{3}\right)^3 \, dx = -\frac{1}{9}
\]
and

For $\alpha = \beta = 5$:
\[
\int_0^\infty e^{-x} \left(1 - \frac{2x}{5}\right)^5 \, dx = \frac{127}{625}
\]

We can see that (4.4.21) and (4.4.22), is positive for all odd integers $\alpha, \beta \geq 5$.

Rewrite (4.4.20), we get

\[
\frac{\iint W_s^{\alpha}(t, f) \, dt \, df}{\iint W_s^{\beta}(t, f) \, dt \, df} = \frac{2^{\alpha-\beta}}{\alpha} \cdot \frac{\beta}{\beta}
\]

Thus, for all positive $\alpha, \beta \geq 1$, (4.4.11) is satisfied for the 1st order Hermite function.

Also, in order to balance the fact, we will assume that all the signals under consideration are such that the Kapur’s entropy is well-defined.

The extension of the $H_{\alpha,\beta}^K(P)$ to continuous valued bivariate densities $P(x, y)$ is:

\[
H_{\alpha,\beta}^K(P) = \frac{1}{\beta - \alpha} \left[ \log_2 \left( \frac{\iint P^{\alpha}(x, y) \, dx \, dy}{\iint P(x, y) \, dx \, dy} \right) - \log_2 \left( \iint P^{\beta}(x, y) \, dx \, dy \right) \right]
\]

Kapur’s entropy is the class of Renyi’s entropy, it involves only the relaxation of mean value property from an arithmetic to an exponential mean.

We assume that (4.4.10) is well defined for all signals under consideration. The pre-normalization of the signal energy before raising the TFR to the power $\alpha$ and $\beta$ are:
\[ H_{\alpha,\beta}(C_s) = \]
\[
\frac{1}{\beta - \alpha} \left[ \log_2 \left( \int \int C_s(t,f) \right)^{\alpha} \, dt \, df \right] - \frac{1}{\beta - \alpha} \left[ \log_2 \left( \int \int (C_s(t,f))^{\beta} \, dt \, df \right) \right]
\]  
(4.4.25)

The measures \( H_{\alpha,\beta}^K(C_s) \) and \( H_{\alpha,\beta}(C_s) \) are related to each other as:

\[ H_{\alpha,\beta}^K(C_s) = H_{\alpha}(C_s) - H_{\beta}(C_s) - \log_2 \| s \|_2^2 \]  
(4.4.26)

Since information measure should be invariant to the energy of the signal being analyzed.

Therefore, \( H_{\alpha,\beta}^K(C_s) \) varies with the signal energy

**Conclusion.**

In this chapter, we studied a new class of signal analysis tool - the Kapur’s entropy. The Kapur’s entropy measure shows great promising results for estimating the complexity of signals via the time frequency plane as mentioned in sections (4.4.1) and (4.4.2). Also, the pre-normalized version of Kapur’s entropy, which is equivalent to normalizing the signal energy before raising the TFR varies with the signal energy.