CHAPTER-I

INTRODUCTION

The question of multiplier goes back to late fifties of last century. Given a trigono-
metric series \( \sum_{n=-\infty}^{\infty} a_n e^{inx} \) of a class \( P \). The sequence \( (\lambda_n)_{n=-\infty}^{\infty} \) is said to be of class \( \mathcal{M}(P,Q) \) if \( \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx} \) is of class \( Q \). In particular if \( \sum_{n=-\infty}^{\infty} a_n e^{inx} \) is a Fourier series of a class of functions \( P \), the question is one of finding the class \( (\lambda_n)_{n=-\infty}^{\infty} \) of multipliers for which \( \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx} \) is a Fourier series of a class of functions \( Q \). From here onwards the class of multipliers from \( P \) to \( Q \) is denoted by \( \mathcal{M}(P,Q) \). It has been known that the necessary and sufficient condition for the sequence of multipliers \( (\lambda_n)_{n=-\infty}^{\infty} \) to belong to \( \mathcal{M}(C,C), \mathcal{M}(L_1,L_1), \mathcal{M}(L_{\infty},L_{\infty}) \) is that \( \sum_{n=-\infty}^{\infty} \lambda_n e^{inx} \) is in fact a Fourier-Stieltjes series i.e. there is a function of bounded variation \( F \) such that

\[
\lambda_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-int} dF(t).
\]

One can show that \( \mathcal{M}(L_2,L_2) \) is in fact \( l_\infty \). This means given a function \( f \in L_2 \), with Fourier series \( \sum_{n=-\infty}^{\infty} a_n e^{inx} \), a necessary and sufficient condition that the series \( \sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx} \) is also a Fourier series of an \( L_2 \) function is that \( (\lambda_n)_{n=-\infty}^{\infty} \in l_\infty \).

Recall that if a function \( f \in L_2 \) with Fourier series \( \sum_{n=-\infty}^{\infty} a_n e^{inx} \) then \( \sum_{n=-\infty}^{\infty} |a_n|^2 \) is convergent (Parseval identity) and conversely if \( \sum_{n=-\infty}^{\infty} |a_n|^2 \) is convergent then there is a function \( f \in L_2 \) such that \( a_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(t)e^{-int} dt \) (Riesz - Fischer Theorem).
So what it amounts to is that the sequence \((\lambda_n a_n)_{n=-\infty}^\infty \in l_2\) whenever the sequence \((a_n)_{n=-\infty}^\infty \in l_2\) if and only if the sequence \((\lambda_n)_{n=-\infty}^\infty \in l_\infty\). Now the question automatically goes to ask to which class the sequence \((\lambda_n a_n)_{n=-\infty}^\infty \in l_q\) given that \((a_n)_{n=-\infty}^\infty \in l_p\). This question has already been settled. In fact we know that

\[
\mathcal{M}(l_p(Z), l_q(Z)) \approx l_{\frac{p}{p-q}}(Z).
\]

We have generalized this to the cases \(\mathcal{M}(l_p(Z, X), l_q(Z, X))\). But the question of \(\mathcal{M}(L_p(T), L_q(T))\) is more involved. We have already discussed above the case when \(p = q = 2\), where the question is completely answered. The questions about \(\mathcal{M}(L_p(T), L_q(T))\) are far from being completely answered for other cases of \(p, q\). We state below a few of the known interesting results (cf. Larsen [21] p.146).

**THEOREM-A:** Let \(G\) be a locally compact Abelian group and suppose
\[
1 \leq p < \infty, 1 < q \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1.
\]
Then there exists an isometric linear isomorphism of \(\mathcal{M}(L_p(G), L_q(G))\) onto \(\mathcal{M}(L_{q'}(G), L_{p'}(G))\).

**THEOREM-B:** Let \(G\) be a locally compact Abelian group and suppose
\[
1 \leq p, q \leq \infty.
\]
If \(T \in \mathcal{M}(L_p(G), L_q(G))\) then there exists a unique quasimeasure \(\sigma \in Q(G)\) such that \(Tf = \sigma * f\) for each \(f \in C_c(G)\), where \(Q(G)\) is the class of quasimeasures on \(G\).

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THEOREM-C : Let $G$ be a locally compact Abelian group and suppose $1 \leq p_i, q_i \leq \infty, i = 1, 2$. If $T \in \mathcal{M}(L_{p_i}(G), L_{q_i}(G)), i = 1, 2,$ then $T$ defines a unique element of $\mathcal{M}(L_p(G), L_q(G))$ for each pair $(p, q)$ for which there exists an $\alpha, 0 < \alpha < 1$, such that $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{(1-\alpha)}{p_2}, \frac{1}{q} = \frac{\alpha}{q_1} + \frac{(1-\alpha)}{q_2}$.

THEOREM-D : Let $G$ be an infinite locally compact Abelian group and suppose $1 < p < q < \infty$. If $r > 1$ is such that $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}$ then $L_r(G) \subset \mathcal{M}(L_p(G), L_q(G))$ and the inclusion is proper.

Now this can be thought of as ” given a class of functions $F$ and $G$ on the compact abelian group $\mathbb{T}$, we are trying to find a class of functions $\varphi$ on its dual group $\mathbb{Z}$, such that, $\varphi \hat{f} \in \hat{G}$ for every $f \in F$, where $\hat{G} = \{ \hat{f} : \mathbb{Z} \to \mathbb{C}; f \in G \}$”.

The obvious extension of this idea is to find a generalization to more general groups $G$. For a locally compact group $G$ Larsen, Liu and Wang [20] defined a class of functions $f$ in $L_1(G)$ whose Fourier transforms belongs to $L_p(\hat{G})$. This is denoted by $A_p(G)$. Now it becomes interesting to study the question of multipliers from $A_p(G)$ to $A_q(G)$. Figa-Talamanca and Gaudry [10], Martin and Yap [23], Reiter [32], Gupta and Tewari [12] studied this question. Gupta, Madan and Tewari [11] studied the conjugation operator on $A_p(G)$. There can be many interesting generalization of this. For example let us consider the following:
Let $X$ be a normed linear space. We define the space $A_p(G, X)$ as a space of all $X$-valued functions on $G$ whose norms are integrable and Fourier transform belong to $L_p(\hat{G}, X)$ . We can now define $A_p(G, X)$ like wise . We then attempt to obtain a similar results about multipliers from $A_p(G, X)$ to $A_q(G, X)$. It is easily seen that the multipliers must be operator valued . One of the reasons for studying this is as follows :

As stated above it was a question of finding the class of sequences $(\lambda_n)_{n=-\infty}^{\infty}$ for which the weighted Fourier series $\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}$ belongs to a certain other class given that $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is a Fourier series of a function belonging to a certain class. Now if $(\lambda_n)_{n=-\infty}^{\infty}$ instead of being a sequence of complex numbers were a sequence of random variables the question now boils down to the question of convergence of the Random Fourier series $\sum_{n=-\infty}^{\infty} \lambda_n a_n e^{inx}$.

Such questions have been studied in detail by many authors begining with Rademacher [31] . The best generalization of this is provided by Billard [1] . We want to look at it instead in the following way .

Suppose the space $X$ is the space $L_2(\Omega, \mathcal{A}, \mathcal{P})$ of all square integrable random variables on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$ . For the space $l_p(\mathbb{Z}, X)$ we can look at the class of multipliers $\mathcal{M}(l_p(\mathbb{Z}, X), l_q(\mathbb{Z}, X))$ . The question of $\mathcal{M}(l_p(\mathbb{Z}, \mathbb{C}), l_q(\mathbb{Z}, \mathbb{C}))$ have already been studied.
In fact it can be shown that for \( 1 \leq p < 2 \), \( 2 < q < \infty \), \( \mathcal{M}(l_p(\mathbb{Z}, \mathbb{C}), l_q(\mathbb{Z}, \mathbb{C})) \) is isometrically isomorphic to the space \( l_{\frac{pq}{q-p}}(\mathbb{Z}, \mathbb{C}) \). Now when we consider \( l_p(\mathbb{Z}, X) \) where \( X \) is a Banach space we should look at multipliers which are operator valued functions on \( \mathbb{Z} \). That means \( \mathbf{a} : \mathbb{Z} \to \mathcal{L}(X) \) will be called a multiplier from the space \( l_p(\mathbb{Z}, X) \) to the space \( l_q(\mathbb{Z}, X) \) if \( \mathbf{a}f \in l_q(\mathbb{Z}, X) \) for \( f \in l_p(\mathbb{Z}, X) \).

We have in the Chapter II discussed these questions in generality, when \( X \) is a Hilbert space and generalised the results of Gupta and Tewari [12]. We consider the space of multipliers from \( \mathbf{A}_p(G, X) \) to \( \mathbf{A}_q(G, X) \) the \( X \)-valued integrable functions whose Fourier transforms are in \( L_p(\widehat{G}, X) \). In this Chapter we have shown that the space of multipliers from \( l_p(I, X) \) to \( l_q(I, X) \) is isometrically isomorphic to \( l_{\frac{pq}{q-p}}(I, \mathcal{L}(X)) \) where \( I \) is the index set and for \( 1 \leq q < p < 2 \).

In the sequel, suppose, we define a multiplier from \( l_p(I, \mathbb{C}) \) to \( l_q(I, X) \), where \( X \) is a Hilbert space. We are looking for the \( X \)-valued sequence \((f(j))_{j \in \mathbb{Z}}\) such that

\[
\mathbf{a}_j f(j) \in l_q(I, X), \text{ whenever } (\mathbf{a}_j)_{j \in I} \in l_p(I, \mathbb{C}),
\]

i.e., to find \( X \)-valued sequence \( f(i) \) such that

\[
\sum_{i \in I} \|\mathbf{a}_i f(i)\|^q < \infty
\]

whenever \( \sum_{i \in I} |\mathbf{a}_i|^p < \infty \). One gets that this happens if and only if

\[
\sum_{i \in I} \| f(i) \|_{\frac{pq}{q-p}} < \infty.
\]
Now suppose we take $X$ to be $L_2(\Omega, \mathcal{A}, \mathcal{P})$, where $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space. Then this amounts to saying that for a sequence of random variables $(X_n)_{n \in \mathbb{Z}}$ with finite second moment

$$(X_n a_n)_{n=-\infty}^\infty \in l_q(\mathbb{Z}, X) \text{ for } (a_n)_{n=-\infty}^\infty \in l_p(\mathbb{Z}, \mathbb{C})$$

if and only if

$$\sum_{n=-\infty}^\infty \|X_n\|^{\frac{pq}{p-q}} < \infty.$$ 

So under the condition $\sum_{n=-\infty}^\infty \|X_n\|^{\frac{pq}{p-q}} < \infty$, we have

$$\sum_{n=-\infty}^\infty \|X_n a_n e^{int}\|^q < \infty \text{ whenever } \sum_{n=-\infty}^\infty |a_n|^p < \infty.$$ 

So we now ask the question of convergence of the random trigonometric series

$$\sum_{n=-\infty}^\infty a_n X_n e^{int}.$$ 

This leads to the question of convergence of the trigonometric series

$$\sum_{n=-\infty}^\infty b_n e^{int} \text{ when } \sum_{n=-\infty}^\infty |b_n|^r < \infty ?$$ 

For $r = 2$ this is answered by the Riesz-Fischer theorem which asserts that if $\sum_{n=-\infty}^\infty |b_n|^2 < \infty$ then there is a function $g \in L_2$ such that

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt.$$
So $\sum_{n=-\infty}^{\infty} b_n e^{int}$ is in fact, the Fourier series of an $L_2$ function. For $r \neq 2$ we may invoke the Hausdorff-Young theorem which asserts that for $1 < r \leq 2$,

$$\sum_{n \in \mathbb{Z}} |b_n|^r < \infty,$$

implies that there is an $f \in L_r$, such that

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

and

$$\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^r dt \right)^{\frac{1}{r}} \leq c \left( \sum_{n=-\infty}^{\infty} |b_n|^r \right)^{\frac{1}{r}},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ and $c$ is independent of the sequence $(b_n)_{n \in \mathbb{Z}}$. In other words the trigonometric series $\sum_{n \in \mathbb{Z}} b_n e^{int}$ is a Fourier series of an $L_r$ function if $\sum_{n \in \mathbb{Z}} |b_n|^r < \infty$.

Coming back to the question of convergence of the Random Fourier series $\sum_{n=-\infty}^{\infty} a_n X_n e^{int}$, can we say if $\sum_{n=-\infty}^{\infty} a_n X_n e^{int}$ is a Fourier series of $X$-valued $L_q$ function? A look at Hilbert space valued functions would read as follows:

Let $X$ be a Hilbert space. Let

$$L^2_X(\mathbb{T}) = \{ f : \mathbb{T} \to X; \int_{-\pi}^{\pi} \| f(t) \|^2 dt < \infty \},$$

where $\mathbb{T}$ is the circle group. $L^2_X(\mathbb{T})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \langle f(t), g(t) \rangle dt.$$

For $f \in L^2_X(\mathbb{T})$ and let for $n \in \mathbb{Z}$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt = a_n$.  


One can show that

\[ \sum_{n \in \mathbb{Z}} \|a_n\|^2 = \int_{-\pi}^{\pi} \|f(t)\|^2 dt \quad (\text{Parseval's identity}). \]

Riesz-Fischer theorem on \( L^2_X(T) \) will read as follows:

If \((a_n)_{n \in \mathbb{Z}}\) is a sequence in \( X \) with \( \sum_{n \in \mathbb{Z}} \|a_n\|^2 < \infty \) then there is a function \( f \in L^2_X(T) \) such that

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt = a_n. \]

One can show this easily. Let

\[ s_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}. \]

Then

\[ \int \|s_n(t) - s_m(t)\|^2 dt = \int_{|k| \leq n} \|a_k e^{ikt}\|^2 dt = \sum_{|k| \leq n} \|a_k\|^2. \]

Because of \((a_n)_{n \in \mathbb{Z}} \in l_2\) we have \((s_n)_{n=1}^{\infty}\) is a Cauchy sequence in \( L^2_X(T) \) hence convergent. Let \( f = \lim_{n \to \infty} s_n \). It is easily seen that \( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt = a_n. \)

If we take \( X = L_2(\Omega, \mathcal{A}, \mathcal{P}) \), where \((\Omega, \mathcal{A}, \mathcal{P})\) is a probability space and \((X_n)_{n=-\infty}^{\infty}\) is a sequence of random variables in \( L_2(\Omega, \mathcal{A}, \mathcal{P}) \) with \( E|X_n|^2 = 1 \), then we would have that

\[ \sum_{n=-\infty}^{\infty} a_n X_n e^{int} \]
is a Fourier series of $X$-valued function if

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty.$$  

By Fubini’s theorem we have

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} E|s_n(t) - s_m(t)|^2 dt = \lim_{n \to \infty} E \int_{-\pi}^{\pi} |s_n(t) - s_m(t)|^2 dt = 0,$$

where $s_n(t) = \sum_{k=-n}^{n} a_k X_k e^{ikt}$.

Carleson showed that for $f \in L^2(\mathbb{T})$ and $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$, the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{int}$ converges almost everywhere. Now if $f \in L^2_X(\mathbb{T})$ can one claim that the Fourier series of $f$ converges almost everywhere? If $X$ is a separable Hilbert space the answer should be easily provable in affirmative. So now the question is if the same should be true for any Hilbert space? Can we apply the result to settle the question of convergence of $\sum_{n=-\infty}^{\infty} a_n X_n e^{int}$ where $(X_n)_{n=-\infty}^{\infty}$ is a sequence of random variables with $E|X_n|^2 = 1$?

Suppose $X$ is separable. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $X$. Let $\sum_{n \in \mathbb{Z}} a_n e^{int}$ be the Fourier series of an $X$-valued function $f$. Then each $a_n \in X$. So $a_n = \sum_{k=1}^{\infty} a_{nk} e_k$, $a_{nk} \in \mathbb{C}$. For each $k$, $\sum_{n \in \mathbb{Z}} a_{nk} e^{int}$ is now a Fourier series of a scalar valued function in $L^2_{\mathbb{C}}(\mathbb{T})$. So according to Carleson [3],

$$\sum_{n \in \mathbb{Z}} a_{nk} e^{int}, \quad a_n \in X$$
converges almost everywhere. Let

\[ E_k = \{ e^{it} : \sum_{n \in \mathbb{Z}} a_{nk} e^{int} \text{ does not converge} \} . \]

By Carleson, we have \( m(E_k) = 0 \), where \( m \) is the normalized Lebesgue measure on \( \mathbb{T} \). Let \( \sum_{n \in \mathbb{Z}} a_{nk} e^{int} \) converge to \( f_k(e^{it}) \) for \( e^{it} \notin E_k \).

Now each \( f_k \in L^2_{\mathbb{T}}(\mathbb{T}) \). So let us consider Hilbert space valued function

\[ f(e^{it}) = \sum_{k=1}^{\infty} f_k(e^{it}) e_k \]

defined on the complement of \( \bigcup_{k=1}^{\infty} E_k \).

Since for each \( k \), \( m(E_k) = 0 \), we have

\[ m \left( \bigcup_{k=1}^{\infty} E_k \right) = 0 . \]

But

\[ f_k(e^{it}) e_k = \sum_{n=-\infty}^{\infty} a_{nk} e_k e^{int} . \]

So

\[ f(e^{it}) = \sum_{k=1}^{\infty} f_k(e^{it}) e_k = \sum_{n=-\infty}^{\infty} \left( \sum_{k=1}^{\infty} a_{nk} e_k \right) e^{int} \]
\[ = \sum_{n=-\infty}^{\infty} a_n e^{int} . \]

The convergence here is in fact convergence in the norm on \( X \). This would in the case of \( X = L^2(\Omega, \mathcal{A}, \mathbb{P}) \) mean that the Random Fourier series \( \sum_{n=-\infty}^{\infty} a_n X_n e^{int} \) convergence in the mean for almost all \( t \) provided \( L^2(\Omega, \mathcal{A}, \mathbb{P}) \) is separable. This complements the Khintchine-Kolmogorov convergence theorem (c.f. Chow and
This suggests that the question of Convergence of Random Fourier series is fraught with many difficulties. The result \( M(l_\|l\|_p(Z, \mathbb{C}), l_\|l\|_q(Z, X)) \) being isometrically isomorphic to \( l_\|l\|_{p\cdot q}(I, X) \) would become, for \( p = q = 2 \), \( M(l_2(Z, \mathbb{C}), l_2(Z, X)) \) is isometrically isomorphic to \( l_\|l\|_{\infty}(Z, X) \). This means that the Random Trigonometric series \( \sum_{n=-\infty}^{\infty} X_n a_n e^{int} \) is in fact the Fourier series of an \( X \)-valued function if and only if \( \sup_n E|X_n|^2 < \infty \). We recall the results of Random Fourier series were studied by Paley, Wiener and Zygmund [25] way back in 1933.

Indication of these were in the work of Rademacher [31] in 1922 and in the work of H. Steinhaus [36]. The general question of convergence of Random Trigonometric series \( \sum_{n=-\infty}^{\infty} X_n a_n e^{int} \) has only limited answers.

We in Chapter III have obtained results analogous to the results of A. Pelczyński and F. Sukochev [30] on the Toeplitz Schur multipliers from the class of upper triangular trace class matrices to the class of absolutely summable matrices. We have obtained a necessary and sufficient condition for the weighted Toeplitz Multiplier \( \tilde{T}_\lambda \) to induce a bounded linear operator from \( S_1 \) into \( M_1 \). We have also got a condition for a sequence to be in the Fefferman multiplier class and the corresponding Toeplitz multiplier mapping the upper triangular trace class matrices to the class of absolutely summable matrices.
We observe here that, the question of multipliers from the Hardy space $H_p$ to
the Hardy space $H_q$ is really interesting. So also the question of multipliers from
$H_p$ to the sequence space $l_q$. It is known that (cf., Duren [7]):

**THEOREM E:** For $0 < p < 1$, the sequence $(\lambda_n)_{n=0}^\infty$ is a multiplier of
$H_p$ into $l_\infty$ if and only if

$$
\lambda_n = O(n^{1 - \frac{1}{p}}).
$$

From this it follows that:

**COROLLARY-A:** If $\{d_n\}$ is a sequence of positive numbers such that
$a_n = O(d_n)$ for every function $\sum_{n=0}^\infty a_n z^n$ in $H_p$ for $p \leq 1$, then there is an $\epsilon > 0$
such that

$$
d_n n^{1 - \frac{1}{p}} \geq \epsilon, \ n = 1, 2, \ldots.
$$

For $0 < p < 1$ and $p < q < \infty$ we know the following:

**THEOREM - F:** $(\lambda_n)_{n=1}^\infty$ is a multiplier of $H_p$ into $l_q$ if and only if

$$
\sum_{n=1}^N \frac{2}{n} |\lambda_n|^q = O(N^q).
$$

This gives a sufficient condition for $\sum_{n=1}^\infty a_n z^n$ to belong to $H_p$ for $0 < p < 1$.

An important result in this direction being:
THEOREM-G : The sequence $\{\lambda_n\}$ is a multiplier of $H_1$ into $H_2$ if and only if
\[
\sum_{n=1}^{N} n^2 |\lambda_n|^2 = O(N^2) .
\]

This leads to discovery by Charles Fefferman of characterization of multiplier from $H_1$ to $l_1(\mathbb{Z}_+)$ . Natural question now would be one of getting similar results for vector valued Hardy spaces . Blasco and Pełczyński [2] have obtained very interesting generalization of these to nuclear operator valued $H_1$ space.

We recall that for $a \in \mathcal{L}(l_2)$, the class of bounded linear operators on the Hilbert space $l_2$, $a^*a$ is a positive operator. That is $(a^*a \xi, \xi) \geq 0$ for every $\xi$ in $l_2$. So we can define a square root of $a^*a$ (cf. Douglas [6]). We denote the square root of $a^*a$ by $\sqrt{a}$. $a \in \mathcal{L}(l_2)$ is called a nuclear operator if
\[
\sum_{i \in \mathbb{Z}_+} |a|(i, i) < \infty .
\]
Every $a \in \mathcal{L}(l_2)$ has a matrix representation $a(i, j)_{i, j \in \mathbb{Z}_+}$ with the condition that
\[
\sum_{j=1}^{\infty} |a(i, j)|^2 < \infty \text{ for every } i
\]
and
\[
\sup \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a(i, j)\xi_i|^2 < \infty \text{ for } \sum_{i=1}^{\infty} |\xi_i|^2 < \infty \right\} .
\]
For a bounded linear operator \( a \in \mathcal{L}(l_2) \) with \( |a| = \sqrt{a^*a} \), is a nuclear operator if the sum of the diagonal elements of the matrix representation of \( |a| \) converges.

We denote by \( S_1 \) the class of all nuclear operators on \( l_2 \). For \( a \in S_1 \), the norm is defined as

\[
\|a\|_{S_1} = \sum_{i \in \mathbb{Z}_+} |a|(i,i).
\]

Let

\[
L_1(\mathbb{T}, S_1) = \{ f : \mathbb{T} \to S_1; \int_0^{2\pi} \|f(e^{it})\|_{S_1} dt < \infty \}.
\]

It is clear that \( L_1(\mathbb{T}, S_1) \) is a Banach space. We observe that each analytic trigonometric polynomial

\[
\sum_{k=0}^n a_k e^{ikt}, n \in \mathbb{Z}_+, a_k \in S_1 \text{ is in } L_1(\mathbb{T}, S_1).
\]

Let

\[
\mathbb{P}_+(\mathbb{T}, S_1) = \left\{ \sum_{k=0}^n a_k e^{ikt} : n \in \mathbb{Z}_+, a_k \in S_1 \right\}.
\]

Closure of \( \mathbb{P}_+(\mathbb{T}, S_1) \) is \( L_1(\mathbb{T}, S_1) \) is denoted by \( H_1(\mathbb{T}, S_1) \).

Blasco and Pełczyński [2] have shown that:

\((\lambda_n)_{n \in \mathbb{Z}_+}\) is a multiplier of \( H_1(\mathbb{T}, S_1) \) into \( l_1(S_1) \) is that

\[
|\lambda_0|^2 + |\lambda_1|^2 + \sup_{r \geq 1} \sum_{q=1}^\infty \left( \sum_{j=rq+1}^{rq+1} |\lambda_j| \right)^2 < \infty.
\]

In this context it becomes convenient to introduce the notation \( \rho(\lambda) \) for the sequence \((\lambda_n)_0^\infty\) as

\[
\rho(\lambda) = \left( |\lambda_0|^2 + |\lambda_1|^2 + \sup_{r \geq 1} \sum_{q=1}^\infty \left( \sum_{j=rq+1}^{rq+1} |\lambda_j| \right)^2 \right)^{\frac{1}{2}}.
\]
The results of Blasco and Pelczyński now reads as \( \rho(\lambda) < \infty \) if and only if there is a constant \( c \) such that

\[
\sum |\lambda_j||a_j|_1 \leq cp(\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(e^{it})\|_1 dt
\]

for \( f \in H_1(\mathbb{T}, S_1) \) where

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})e^{-ikt} dt, \quad k \in \mathbb{Z}_+.\]

Pelczyński and Sukochev [30] have obtained another equivalent condition for \( \rho(\lambda) < \infty \) in terms of Toeplitz multipliers. They have obtained that \( \rho(\lambda) < \infty \) if and only if Toeplitz multipliers \( T_\lambda \) maps the upper triangular \( S_1 \) operators to \( \mathbb{M}_1 \) the class of infinite matrices \( a(i,j)_{i,j\in\mathbb{Z}_+} \) such that \( \sum_{i,j\in\mathbb{Z}_+} |a(i,j)| < \infty \).

We in the chapter III have generalized the results of Pelczyński and Sukochev[30] to weighted Toeplitz multipliers \( \tilde{T}_\lambda \). A Toeplitz matrix being an infinite matrix which are constant on diagonals. That is there is a sequence \( (\lambda_j)_{j\in\mathbb{Z}} \) such that the entries \( a(i,j)_{i,j\in\mathbb{Z}_+} \) of the matrix \( a \) are given by

\[
a(i,j) = \lambda_{i-j} \quad i,j \in \mathbb{Z}_+.\]

We have considered in Chapter III the matrix \( a \) with entries

\[
a(i,j) = \frac{\sqrt{ij}}{i+j} \lambda_{i-j} \quad i,j \in \mathbb{Z}_+.\]

This type of matrices occur for Toeplitz operators with harmonic symbols on the Bergmann space.
We now return to the question of random multipliers. We may consider the series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)X_n,$$  \hspace{1cm} (1.1)

where \((X_n)_{n=1}^{\infty}\) is a sequence of independent random variables. Paley and Wiener [26] investigated the series

$$\sum_{n=1}^{\infty} a_n [X_n \cos nx + Y_n \sin nx],$$  \hspace{1cm} (1.2)

where \((X_n)\) and \((Y_n)\) are independent random variables which are normally distributed. Both the series can be looked upon as special case of the series

$$\sum_{n=1}^{\infty} R_n \cos[nx + \Phi_n],$$  \hspace{1cm} (1.3)

where \((R_n)\) and \((\Phi_n)\) are both sequences of independent random variables.

(R_m and \(\Phi_n\) are not necessarily independent).

Kahane [17] has shown (cf. Kawata [18]):

**THEOREM-H:** Let \((R_n)_{n=1}^{\infty}\) be such that \(R_n \geq 0, n = 1, 2, \ldots, \) and \((\Phi_n)_{n=1}^{\infty}\), be sequences of independent random variables. If the series

$$\sum_{n=1}^{\infty} E(\min(1, R_n^2))$$

converges, then

$$\sum_{n=1}^{\infty} R_n \cos[nx + \Phi_n]$$

converges for almost every \(x\) almost surely and \(\sum_{n=1}^{\infty} R_n \cos[nx + \Phi_n]\) is a Fourier series.
A condition for absolute convergence of the series (1.1) is as follows:

If the random variables $R_n$ and $\Phi_n$ are independent with $R_n$ nonnegative then the series (1.3) converges absolutely for all $x$ if the series

$$\sum_{n=1}^{\infty} E(\min(1, R_n))$$

converges.

Salem and Zygmund [34] considered the series

$$\sum_{n=0}^{\infty} c_n r_n(t) \cos(nx + \alpha_n), \quad - - - - - - - - (1.4)$$

where $r_n$'s are Rademacher functions and $0 \leq \alpha_n \leq 2\pi$.

They showed that

$$\sum_{j=2}^{\infty} \left( \frac{\sum_{n=j}^{\infty} \epsilon_n^2}{j(\log j)^{1/2}} \right)^{1/2} < \infty$$

is a sufficient condition for the uniform convergence almost surely of (1.4).

One of the reason where Paley, Wiener and Zygmund [25] investigated the series (1.3) is perhaps related to effort by Wiener in 1923 to understand the Brownian motion for mathematical points of view. It was shown by him that the Brownian motion can be represented by the series

$$X_0t + \sum_{n=1}^{\infty} \frac{X_n \sin nt}{n} + \frac{1 - \cos 2\pi nt}{n} Y_n,$$

where $X_0, X_1, ..., X_n, ..., Y_1, Y_2, ..., Y_n, .......$ are independent identically distributed standard normal random variables. If we write $W(t)$ for the sum of the
series, then it can be shown that it is in fact a Brownian motion. Brownian motion can be shown to be nowhere differentiable almost surely. Yet we can define the integral
\[ \int_{a}^{b} f(t) \, dW(t) \]
in some sense for \( f \in L_2[a, b] \). It can even be shown that the differentiated series for Brownian motion can be used to define the integral \( \int_{a}^{b} f(t)dW(t) \). Now when instead of the Brownian motion \( W(t) \) could be taken some other stochastic processes \( X(t) \) and define
\[ \int_{a}^{b} f(t) \, dX(t) \]
in some sense. This question is related to the Random Fourier-Stieltjes series considered in chapter IV.

We in Chapter IV have studied some particular classes of Random Fourier series \( \sum_{n=-\infty}^{\infty} a_n X_n e^{int} \). Samal [35] studied the ”Integrated Random Fourier-Stieltjes series”
\[ \sum_{n=-\infty}^{\infty} \frac{A_n e^{int}}{n} , \]
where \( \sum_{n=-\infty}^{\infty}' \) indicates that the term \( n = 0 \) is omitted from the summation. Here
\[ A_n = \int_{0}^{1} e^{-2\pi int} dX(t) \]
where \( X(t) \) is a continuous stochastic process with independent increments.
Nayak, Pattanayak and Mishra [24] considered the Random Fourier series
\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi i nt}, \]
where
\[ A_n = \int_0^{2\pi} e^{-2\pi i nt} dX(t), \quad a_n = \int_0^1 f(t) e^{-2\pi i nt} dt. \]

We in this Chapter have shown that the Random Fourier- Stieltjes (RFS) series associated with a stochastic process whose increments belong to the domain of attraction of the stable law converges in mean to a stochastic integral. Also we have shown that both the series are Abel summable. Pattanayak and Sharma [28] studied the question of convergence in the context of processes belonging to the domain of attraction of the stable law. But they were able to define the stochastic integral only in the sense of convergence in probability. So also the question of convergence of corresponding random Fourier series was proved in the sense of convergence in probability. We in this chapter are successful in defining the the stochastic integral in the sense of convergence in the mean. So also the question of convergence of the corresponding random Fourier Stieltjes series in the mean.
The general question of multiplier arise in many different contexts. We describe them below:

Let $G$ be a locally compact abelian group and $L_1(G)$, the space of complex valued functions on $G$ absolutely integrable with respect to the Haar measure $\lambda$ on $G$. Let the norm on $L_1(G)$ be defined as

$$\|f\|_1 = \int_G |f(t)| \, d\lambda(t).$$

It is well known that this is a Banach space under the norm defined above and is also a Banach algebra with the convolution as multiplication defined by

$$f * g(t) = \int_G f(ts^{-1}) g(s) \, d\lambda(s),$$

for $f, g \in L_1(G)$. Let $M(G)$ denote the Banach space of bounded regular complex valued Borel measures on $G$ with the norm defined as

$$\|\mu\| = |\mu|(G),$$

for $\mu \in M(G)$, where $|\mu|$ is the total variation of the measure $\mu$. $M(G)$ also can be given the structure of a Banach algebra with multiplication defined by

$$\mu * \nu(E) = \int_G \mu(Es^{-1}) \, d\nu(s),$$
for $\mu, \nu \in M(G)$. Let $\hat{G}$ be the dual group of $G$.

For $f$ in $L_1(G)$ and $\mu$ in $M(G)$, the Fourier and Fourier-Stieltjes transform are defined as follows:

$$\hat{f}(\gamma) = \int_G (t^{-1}, \gamma) f(t) d\lambda(t), \gamma \in \hat{G}$$

and

$$\hat{\mu}(\gamma) = \int_G (t^{-1}, \gamma) d\mu(t), \gamma \in \hat{G},$$

where $(., .)$ stands for the usual pairing between $G$ and $\hat{G}$ (cf Rudian [33], Page 13). We know that these transforms are indeed homomorphisms, i.e., to say

$$(\hat{f} \ast \hat{g}) = \hat{f \ast g} \quad \text{and} \quad (\hat{\mu} \ast \hat{\nu}) = \hat{\mu \ast \nu}.$$

Let $(L_1(G))^\Lambda = \{\hat{f} : f \in L_1(G)\}$ and $(M(G))^\Lambda = \{\hat{\mu} : \mu \in M(G)\}$.

For $s \in G$, the translation operator $\tau_s$ is defined by

$$(\tau_s f)(t) = f(ts^{-1}).$$

It is easy to see that translation operator is an isometry on $L_1(G)$. Let $L_\infty(G)$ denote the Banach algebra of essentially bounded measurable functions on $G$ under pointwise multiplications with the norm

$$\|f\|_\infty = \text{ess sup} \{|f(s)| : s \in G\}.$$ 

The following theorem gives the motivation for study of multipliers (cf. Larsen [21]).
THEOREM-A : Let $G$ be a locally compact abelian group and suppose $T : L_1(G) \rightarrow L_1(G)$ is a continuous linear transformation. Then the following are equivalent:

(i) $T$ commutes with the translation operators, that is, $T \tau_s = \tau_s T$ for each $s \in G$.

(ii) $T(f * g) = Tf * g$ for each $f, g \in L_1(G)$.

(iii) There exists a unique function $\varphi$ defined on $\hat{G}$ such that $(Tf)^\Lambda = \varphi \hat{f}$ for each $f \in L_1(G)$.

(iv) There exists a unique measure $\mu \in M(G)$ such that $(Tf)^\Lambda = \hat{\mu} \hat{f}$ for each $f \in L_1(G)$.

(v) There exists a unique measure $\mu \in M(G)$ such that $Tf = f * \mu$ for each $f \in L_1(G)$.

A multiplier on $L_1(G)$ may be defined as a continuous linear operator on $L_1(G)$ satisfying one of the equivalent conditions defined in the theorem above. It is plain to see that the translation operator is a multiplier.

In the context of classical Fourier series, the question becomes one of finding functions $\varphi$ on $\mathbb{Z}$ for which $\varphi \hat{f} \in \hat{Q}$, for $f$ in $P$, where $P$ and $Q$ are classes of functions on the circle group $\mathbb{T}$. We recall that if $G$ is the circle group then its dual group is $\mathbb{Z}$ and the question above now becomes one of finding a sequence $(\lambda_n)_{n \in \mathbb{Z}}$ such that $(\lambda_n a_n)_{n \in \mathbb{Z}}$ are the Fourier coefficients of a function in $Q$ whenever $(a_n)_{n \in \mathbb{Z}}$ are the Fourier coefficients of a function in $P$. In general, by $\mathcal{M}(P, Q)$ we mean
the class of multipliers \((\lambda_n)_{n \in \mathbb{Z}}\) such that \((\lambda_n a_n)_{n \in \mathbb{Z}}\) are the Fourier coefficients of a function in \(Q\) whenever \((a_n)_{n \in \mathbb{Z}}\) are the Fourier coefficients of a function in \(P\). Hence forward the class of multipliers from \(P\) to \(Q\) is denoted by \(\mathcal{M}(P, Q)\).

It is well known (cf. Zygmund [37], p.176, vol.I) that \(\mathcal{M}(C, C) = \mathcal{M}(L_1, L_1) = \mathcal{M}(L_\infty, L_\infty) = (M(\mathbb{T}))^\Lambda\).

Similar questions for Hardy spaces are also interesting. The question there is one of finding a sequence \((\lambda_n)_{n \in \mathbb{Z}_+}\) such that \((\lambda_n a_n)_{n \in \mathbb{Z}_+} \in l_q\), whenever \(\sum a_n z^n\) is the power series expansion of a function in \(H_p\). We shall talk about it in detail in the next Chapter and present our generalization.

For the circle group \(\mathbb{T}\) we know the following necessary conditions about \(\mathcal{M}(L_p(\mathbb{T}), L_q(\mathbb{T}))\) (cf. Edward [9], vol-II, p.267):

**THEOREM-B:** Suppose \(\varphi \in \mathcal{M}(L_p(\mathbb{T}), L_q(\mathbb{T}))\). Then

(i) if \(1 \leq p \leq \infty, 1 < q \leq 2\)

\[
\sum_{n \in \mathbb{Z}} (1 + |n|)^{\frac{1}{p} - \epsilon} |\varphi(n)|^{q'} < \infty, \text{ for each } \epsilon > 0,
\]

where \(\frac{1}{p} + \frac{1}{\bar{p}} = 1, \quad \frac{1}{q} + \frac{1}{\bar{q}} = 1\);
(ii) if $2 \leq p < \infty, \quad 1 \leq q \leq \infty$, then

$$\sum_{n \in \mathbb{Z}} (1 + |n|) \frac{p}{\pi} |\varphi(n)|^p < \infty \quad for \ each \ \epsilon > 0.$$ 

For general locally compact abelian groups the results of this type are similarly partial ( cf. Larsen [21], Ch-5). The study of multipliers for functions with Fourier transforms in $L_p(\hat{G})$ is more interesting.

Let $G$ be a locally compact abelian group. For $1 \leq p < \infty$, $A_p(G)$ is defined as:

$$A_p(G) = \{ f \in L_1(G) : \hat{f} \in L_p(\hat{G}) \}.$$

$A_p(G)$ can be given the structure of a Banach algebra with the norm

$$\|f\|_p = \|f\|_1 + \|\hat{f}\|_p = \int_G |f(t)|d\lambda(t) + \left( \int_G |\hat{f}(\gamma)|^p d\eta(\gamma) \right)^{\frac{1}{p}}$$

for $f \in A_p(G)$.

Gupta and Tewari [12] have shown:

**THEOREM-C**: Let $G$ be an infinite compact abelian group, $1 \leq q < \infty, 2 < p < \infty$ and $p > q$. Then

(i) $M(A_p, A_p) \cap C_0(\hat{G}) \subsetneq M(A_q, A_q) \cap C_0(\hat{G}),$

(ii) $\bigcup_{p > q} M(A_p, A_p) \subsetneq M(A_q, A_q).$
THEOREM-D: Let $1 \leq q < p \leq 2$. Then
\[
\mathcal{M}(A_p, A_q) = l_{\frac{p}{p-q}}(\hat{G}).
\]

THEOREM-E: Let $1 \leq q \leq 2 < p$. Then
\[
l_{\frac{p}{p-q}}(\hat{G}) \subseteq \mathcal{M}(A_p, A_q) \subsetneq l_{\frac{2}{2-q}}(\hat{G}).
\]

THEOREM-F: Let $G$ be a compact abelian group and $p > 2$. Then
\[
\mathcal{M}(A_p, A_p) \subseteq \bigcap_{q<p} \mathcal{M}(A_q, A_q).
\]

THEOREM-G: Let $1 \leq r \leq 2 < p < \infty$ and $\varphi$ a non-negative even sequence on $\mathbb{Z}$ such that $\varphi(n+1) \leq \varphi(n), n > 0$. Then $\varphi \in \mathcal{M}(A_p(\mathbb{T}), A_r(\mathbb{T}))$ if and only if $\varphi \in l_{\frac{p}{r}}(\mathbb{Z})$.

As we would need a vector valued version of the Hausdorff-Young inequality for Hilbert space valued functions we state the Hausdorff-Young inequality to be used later.

THEOREM-H (Hausdorff-Young Inequality)

Let $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose $(a_n)_{n \in \mathbb{Z}}$ is a sequence in the Hilbert space $X$ and $\sum_{n \in \mathbb{Z}} \|a_n\|^p < \infty$. Then there exists $f \in L_{p'}(\mathbb{T}, X)$ such that
\[
a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} dt.
\]
and

\[ \|f\|_p \leq c \left( \sum \|a_n\|^p \right)^{\frac{1}{p}}, \]

where \( c \) is a constant independent of \( a_n \).

As it had been indicated that the multipliers from \( A_p(G) \) to \( A_q(G) \) have been extensively studied for non discrete, noncompact, locally compact, infinite compact totally disconnected group \( G \) by Gupta and Tewari \([12]\) for abelian groups. Multipliers from \( A_p(G) \) to \( A_q(G) \) for scalar valued functions have been studied by Gupta and Tewari \([12]\) for abelian groups.

We in this chapter would like to study multipliers from \( A_p(G, X) \) to \( A_q(G, X) \). For this the following definitions are in order. Through out the definitions below \( X \) stands for a normed linear space.

\section*{2.1 Definitions}

\textbf{Definition-2.1.1} : Let \( X \) be a normed linear space. \( L_p(G, X) \) is the space of all \( X \)-valued measurable functions defined on \( G \) whose \( X \)-norms are in \( L_p \) space.

In other words,

\[ L_p(G, X) = \{ f : G \rightarrow X : \int_G \|f(g)\|^p dg < \infty \text{ for } g \in G \}. \]
DEFINITION-2.1.2 : \( l_p(I, X) \) is the space of all \( X \)-valued sequences defined on \( I \) whose \( X \)-norms are in \( l_p \) space. In otherwords, 
\[
\{ f : I \rightarrow X : \sum_{i \in I} \| f(i) \|^p < \infty \}.
\]

DEFINITION-2.1.3 : Let \( G \) be a locally compact abelian group with its dual group \( \hat{G} \). Then \( A_p(G, X) \) is defined as:
\[
A_p(G, X) = \{ f \in L_1(G, X) : \hat{f} \in L_p(\hat{G}, X) \}
\]
with the norm \( \| f \|_{A_p} = \| f \|_1 + \| \hat{f} \|_p \).

DEFINITION-2.1.4 : For \( X \) and \( Y \) normed linear spaces, \( \mathcal{L}(X,Y) \) is the space of all bounded linear transformations from \( X \) to \( Y \). \( \mathcal{L}(X,X) \) is written as \( \mathcal{L}(X) \).

DEFINITION-2.1.5 : A bounded linear operator valued function \( \varphi \) defined on \( \hat{G} \) is said to be a multiplier from \( A_p(G, X) \) to \( A_q(G, X) \) if
\[
\varphi \hat{f} \in A_q(G, X)^A \quad \text{for all} \quad f \in A_p(G, X).
\]
That is \( \varphi : \hat{G} \rightarrow \mathcal{L}(X) \) such that \( \varphi \hat{f} \in A_q(G, X)^A \) for all \( f \in A_p(G, X) \).

DEFINITION-2.1.6 : \( C_0(\hat{G}, X) = \{ f : \hat{G} \rightarrow X \) , continuous , such that for every \( \epsilon > 0 \) we can find a compact subset \( K \subset \hat{G} \) such that \( |f(\gamma)| < \epsilon \) for \( \gamma \notin K \) \).
DEFINITION- 2.1.7 Suppose $G$ is a locally compact abelian group and $H$ is a closed subgroup of $G$. For a function $f$ continuous on $G$ with compact support, let

$$\prod_H(f)(x) = \int_H f(xy) \, dy,$$

where $\int dy$ is Haar integral on $H$. It is clear that (cf. Reiter [32], CH-3) $\prod_H(f)$ is constant on cosets of $H$ and $\prod_H(f)$ is a continuous function on $G/H$ with compact support.

So we can define $\prod_H : \mathcal{C}_c(G) \Rightarrow \mathcal{C}_c(G/H)$ by $\prod_H(f)(x) = \int_H f(xy) \, dy$.

Here $\mathcal{C}_c(G)$ means the class of continuous functions on $G$ with compact support. $\prod_H$ extends to $L_1(G)$ and it takes $L_1(G)$ to $L_1(G/H)$.

§ 2.2 MAIN RESULTS

THEOREM 2.2.1:

For a Banach space $X$ and a set $I$, let

$$l_p(I, X) = \left\{ f : I \to X : \sum_{i \in I} \|f(i)\|^p < \infty \right\}.$$ 

The space $\mathcal{M}(l_p(I, X), l_q(I, X))$ of multipliers consists of functions $\varphi : I \to \mathcal{L}(X)$ such that

$$\varphi f \in l_q(I, X) \text{ for } f \in l_p(I, X).$$

For $1 \leq q < p < 2$, it is true that

i) $\mathcal{M}(l_p(I, X), l_q(I, X)) = l_\infty(I, \mathcal{L}(X))$ if $p \leq q,$

ii) $\mathcal{M}(l_p(I, X), l_q(I, X)) = l_{p/q}(I, \mathcal{L}(X)).$
\textbf{PROOF:} (i) Observe that $l_p(I, X) \subset l_q(I, X)$ for $p \leq q$. It is easy to show that if $\varphi \in l_\infty(I, \mathcal{L}(X))$, $\varphi f \in l_p(I, X)$ for every $f \in l_p(I, X)$, a fortiori, $\varphi f \in l_q(I, X)$. Conversely one has to show that if $\varphi f \in l_q(I, X)$ for every $f \in l_p(I, X)$, then $\varphi \in l_\infty(I, \mathcal{L}(X))$.

Let $\mathfrak{F} = \{ F : F \subset I, F \text{ finite} \}$.

Define for $F \in \mathfrak{F}$,

$$\varphi_F(i) = \begin{cases} 
\varphi(i), & i \in F \\
0, & i \notin F 
\end{cases}$$

Let $T_F : l_p(I, X) \rightarrow l_q(I, X)$ be defined by

$$T_F(f)(i) = \varphi_F(i) f(i), i \in I.$$ 

So we have:

$$\|T_F(f)\|_q \leq \sup_{i \in F} \|\varphi(i)\| \left( \sum_{i \in F} \|f(i)\|^q \right)^{\frac{1}{q}}$$

$$\leq \sup_{i \in F} \|\varphi(i)\| \left( \sum_{i \in F} \|f(i)\|^p \right)^{\frac{1}{p}}$$

$$\leq \sup_{i \in F} \|\varphi(i)\| \|f\|_p.$$ 

So, $\|T_F\| \leq \sup\{ \|\varphi(i)\| : i \in F \}$.

One can show that $\|T_F\| = \sup\{ \|\varphi(i)\| : i \in F \}$.

Thus by Uniform Boundedness principle we would have $\sup_{i \in F} \|\varphi(i)\| < \infty$.

Hence $\varphi \in l_\infty(I, \mathcal{L}(X))$.

(ii) Given that $\varphi(i) \in \mathcal{L}(X), \forall i \in I$ and $\varphi f \in l_q(I, X)$, where $f \in l_q(I, X)$, i.e.,
\[
\sum ||\varphi(i)f(i)||^q < \infty , \text{ whenever } \sum_{i \in I} ||f(i)||^p < \infty .
\]

We have to prove that \( \sum ||\phi(i)||^{\frac{pq}{p-q}} < \infty \).

For a finite subset \( F \subset I \), let \( T_F : l_p(I, X) \to l_q(I, X) \) defined by

\[
T_F f(i) = \begin{cases} 
\varphi(i)f(i), & i \in F \\
0, & i \notin F 
\end{cases}
\]

\[
||T_F||^q = \sum_{i \in F} ||T_F f(i)||^q 
= \sum_{i \in F} ||\varphi(i)f(i)||^q \leq \sum_{i \in F} ||\varphi(i)||^q ||f(i)||^q 
\leq \left( \sum_{i \in F} (||f(i)||^q)^r \right)^{\frac{1}{r}} \left( \sum_{i \in F} (||\varphi(i)||^q)^{r'} \right)^{\frac{1}{r'}} ,
\]

by Hölder's Inequality, where \( 1 < r < \infty \) and \( \frac{1}{r} + \frac{1}{r'} = 1 \).

\[
\therefore ||T_F||^q \leq \left( \sum_{i \in F} ||f(i)||^p \right)^{\frac{q}{p}} \left( \sum_{i \in F} ||\varphi(i)||^{\frac{pq}{p-q}} \right)^{\frac{p-q}{p}} ,
\]

taking \( r = \frac{p}{q} \), we have \( r' = \frac{p}{p-q} \).

This easily gives

\[
||T_F|| \leq \left( \sum_{i \in F} ||\varphi(i)||^{\frac{pq}{p-q}} \right)^{\frac{p-q}{p}} .
\]

We shall prove the reverse inequality. To do that we proceed thus.

We recall that for \( T \in L(X) \),

\[
||T|| = \sup \{ ||Tx|| : ||x|| = 1 \} .
\]
That is $\|Tx\| \leq \|T\|$ for $\|x\| = 1$. But for $\epsilon > 0$, there is an $x \in X$, such that $\|x\| = 1$ and

$$\|Tx\| \geq \|T\|(1 - \epsilon).$$

So for $\epsilon > 0$, we can find $h(i) \in X$ such that

$$\|h(i)\| = 1 \quad \text{and} \quad \|\varphi(i)h(i)\| \geq \|\varphi(i)\| (1 - \epsilon'),$$

where $\epsilon' = 1 - (1 - \epsilon)^{\frac{1}{q}}$.

Choose

$$f(i) = \frac{h(i)\|\varphi(i)\|^{\frac{q}{p-q}}}{\left[\sum_{i \in F} \|\varphi(i)\|^{\frac{pq}{p-q}}\right]^{\frac{1}{p}}} \quad (2.2)$$

so that

$$\sum_{i \in F} \|f(i)\|^p = 1.$$

We have $(TFf)(i) = \varphi(i)f(i)$, $i \in F$.

From (2.2), we get

$$\varphi(i)f(i) = \frac{\varphi(i)h(i)\|\varphi(i)\|^{\frac{q}{p-q}}}{\left[\sum_{i \in F} \|\varphi(i)\|^{\frac{pq}{p-q}}\right]^{\frac{1}{p}}}, \quad i \in F.$$
\[ \Rightarrow \| T_F f \|^q = \sum_{i \in F} \| T_F f(i) \|^q \]
\[ \geq \left[ \sum_{i \in F} \| \varphi(i) \|^\frac{pq}{p-q} \right]^{\frac{p-q}{p}} (1 - \epsilon). \]

This gives
\[ \| T_F f \| \geq \left( \sum_{i \in F} \| \varphi(i) \|^\frac{pq}{p-q} \right)^{\frac{p-q}{pq}} (1 - \epsilon)^{\frac{1}{q}}. \]

Since \( \epsilon \) is arbitrary, we have
\[ \| T_F \| \geq \left( \sum_{i \in F} \| \varphi(i) \|^\frac{pq}{p-q} \right)^{\frac{p-q}{pq}}, \quad (2.3) \]
since \( \| f \| = 1 \). Combining (2.1) and (2.3), we get
\[ \| T_F \| = \left( \sum_{i \in F} \| \varphi(i) \|^\frac{pq}{p-q} \right)^{\frac{p-q}{pq}}. \]

We have for every \( f \)
\[ \| T_F f \|^q = \sum_{i \in F} \| \varphi(i) f(i) \|^q. \]
\[ \sup_{F \in \mathcal{F}} \| T_F f \|^q = \sum_{i \in I} \| \varphi(i) f(i) \|^q < \infty \quad \text{(by hypothesis).} \]

By the Uniform Boundedness Principle
\[ \sup_{F \in \mathcal{F}} \| T_F f \|^q < \infty \text{ for all } f \in l_q(I, X) \Rightarrow \sup_{F \in \mathcal{F}} \| T_F \|^q < \infty. \]

Since
\[ \| T_F \| = \left( \sum_{i \in I} \| \varphi(i) \|^\frac{pq}{p-q} \right)^{\frac{p-q}{pq}}, \]
\[ \sup_{F \in \mathcal{F}} \| T_F \|^q = \sup_{F \in \mathcal{F}} \left( \sum_{i \in F} \| \varphi(i) \|^\frac{pq}{p-q} \right)^{\frac{p-q}{pq}} < \infty. \]
This amounts to saying
\[
\left( \sum_{i \in I} \| \varphi(i) \|^\frac{p}{r-q} \right)^{\frac{r-q}{p}} < \infty.
\]
Hence \( \varphi \in l_{\frac{p}{r-q}}(I, X) \).

Converse follows by Hölder’s Inequality. Conversely suppose \( \varphi \in l_{\frac{p}{r-q}}(I, \mathcal{L}(X)) \).

For \( f \in l_p(I, X) \) we have by Hölder’s Inequality,
\[
\sum_{i=1}^{\infty} \| \varphi(i)f(i) \|^q \leq \left( \sum_{i=1}^{\infty} \| \varphi(i) \|^q \| f(i) \|^q \right)^\frac{1}{2} \left( \sum_{i=1}^{\infty} \| f(i) \|^q \right)^\frac{1}{2},
\]
where \( r = \frac{p}{q} \) and \( \frac{1}{r} + \frac{1}{r'} = 1 \) \(\text{i.e.} \frac{1}{r} = 1 - \frac{q}{p} = \frac{p - q}{p} \)
so that \( r' = \frac{p}{pq} \). So we have
\[
\sum_{i \in I} \| \varphi(i)f(i) \|^q \leq \left( \sum_{i=1}^{\infty} \| \varphi(i) \|^\frac{pq}{p} \right)^\frac{r}{p} \left( \sum_{i=1}^{\infty} \| f(i) \|^p \right)^\frac{q}{p}.
\]
Hence
\[
\left( \sum_{i \in I} \| \varphi(i)f(i) \|^q \right)^\frac{1}{q} \leq \left( \sum_{i=1}^{\infty} \| \varphi(i) \|^\frac{pq}{p} \right)^\frac{r}{p} \left( \sum_{i=1}^{\infty} \| f(i) \|^p \right)^\frac{1}{p} = \| \varphi \| \| f \|.
\]
This gives \( \varphi f \in l_q(I, X) \). This completes the proof of this theorem.

To prove the next result we need the Hausdorff-Young theorem in the following form:

Let \( G \) be a compact abelian group and \( X \) a Hilbert space. Then for \( 1 \leq p \leq 2 \).

(i) \( f \in L_p(G, X) \Rightarrow \hat{f} \in L_{p'}(\hat{G}, X) \) and \( \| \hat{f} \|_{p'} \leq \| f \|_p \).
(ii) If \( g \in L_p(\hat{G}, X) \) there exists \( f \in L_{p'}(G, X) \) such that \( \hat{f} = g \) and
\[
\|\hat{f}\|_{p'} \leq \|g\|_p,
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \)

**THEOREM 2.2.2 :** For a compact abelian group \( G \), a Hilbert space \( X \) and \( 1 < p \leq 2 \), \( A_p(G, X) \) is isomorphic to \( l_p(\hat{G}, X) \).

**PROOF :** \( A_p(G, X) = \{ f \in L_1(G, X) : \hat{f} \in L_p(\hat{G}, X) = l_p(\hat{G}, X) \} \).

We claim that if \( f = (a_n)_{n=-\infty}^{\infty} \in l_p(\hat{G}, X) \), then there exists a \( g \in L_1(G, X) \) such that \( \hat{g} = f \).

By Hausdorff-Young inequality, for \( 1 \leq p \leq 2 \), if \( f \in L_p(\hat{G}, X) \) there exists a \( g \in L_{p'}(\hat{G}, X) \) such that \( \hat{g} = f \). Since \( L_{p'} \subset L_1 \), we can say that if \( f \in L_p(\hat{G}, X) = l_p(\hat{G}, X) \), then there exists a \( g \in L_{p'}(G, X) \) such that \( \hat{g} = f \).

\( \therefore A_p(G, X) \simeq l_p(\hat{G}, X) \).

This completes the proof.

**§ 2.3 RESULTS ON A_p(G, X)**

**PROPOSITION-2.3.1 :** Let \( G \) be a nondiscrete locally compact abelian group and let \( 1 < p < 2 \). Let \( X \) be a Banach space. Then \( A_p(G, X) \subseteq A_q(G, X) \), provided \( p < q < \infty \).
PROOF: \( A_p(G, X) = \{ f \in L_1(G, X) : \hat{f} \in L_p(\widehat{G}, X) \} \). We know that if \( f \in L_1(G, X) \), then \( \hat{f} \in C_0(\widehat{G}, X) \).

Since \( f \in A_p(G, X) \), \( \hat{f} \in C_0(\widehat{G}, X) \cap L_p(\widehat{G}, X) \).

But
\[
\hat{f} \in C_0(\widehat{G}, X) \cap L_p(\widehat{G}, X) \\
\Rightarrow \hat{f} \in C_0(\widehat{G}, X) \cap L_q(\widehat{G}, X), \text{ since } p \leq q.
\]

Hence \( A_p(G, X) \subset A_q(G, X) \) for \( p < q \).

We know (cf. Gupta and Tewari [12], 1973) that there is an \( f \in A_q(G) \) such that \( f \notin A_p(G) \). Thus for any \( x \in X - \{0\} \) it is not hard to show that \( xf \in A_q(G, X) \) but \( xf \notin A_p(G, X) \). This completes the proof.

We know that for \( G = T, \quad \widehat{G} = \mathbb{Z} \).

\[
A_p(T, X) = \left\{ f : T \to X ; \int_T \| f(e^{it}) \| dt < \infty ; \sum_{n \in \mathbb{Z}} \| \hat{f}(n) \|^p < \infty \right\}.
\]

\[
A_p(T, \mathbb{C}) = \left\{ f : T \to \mathbb{C} ; \int_T |f(e^{it})| dt < \infty , \hat{f} \in l_p(\mathbb{Z}) \right\}.
\]

\( A_p(T) \) consists of all functions \( f \in L_1(T) \) whose Fourier coefficients are in \( l_p \).

\[
A_2(T) = \{ f \in L_1(T) : \sum_{n=\infty}^\infty |\hat{f}(n)|^2 < \infty \}.
\]

In fact one can show that \( A_2(T) \) is \( L_2(T) \):

Since \( \sum_{n=-\infty}^\infty |\hat{f}(n)|^2 < \infty \) implies , by Riesz- Fischer theorem , there is a function \( g \in L^2(T) \) such that \( \hat{g}(n) = \hat{f}(n) \). Now by uniqueness of Fourier series \( f = g \)
a.e. So it is clear that \( A_2(\mathbb{T}) \subset L_2(\mathbb{T}) \).

Reverse inclusion is by Parseval’s identity.

Suppose \( f \in L_2[\mathbb{T}] \) then \( \hat{f} \in l^2(\mathbb{Z}) \) by Parseval identity. Since \( L_2(\mathbb{T}) \subset L_1(\mathbb{T}) \) this shows that \( f \in A_2(\mathbb{T}) \). So \( L_2(\mathbb{T}) \subset A_2(\mathbb{T}) \). Hence \( A_2(\mathbb{T}) = L_2(\mathbb{T}) \).

What about \( A_p(\mathbb{T}) \) for \( p \neq 2 \)?

We know from Hausdorff-Young theorem that for \( 1 < p < 2 \), \( \sum_{n=-\infty}^{\infty} |c_n|^p < \infty \) implies that there is a function \( f \in L_p'(\mathbb{T}) \) such that

\[
    c_n = \hat{f}(n)
\]

and

\[
    \left( \int_{\mathbb{T}} |f(e^{it})|^p' \, dt \right)^{\frac{1}{p'}} \leq \left( \sum_{n \in \mathbb{Z}} |c_n|^p \right)^{\frac{1}{p}},
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \). So for \( 1 < p \leq 2 \)

\[
    f \in A_p(\mathbb{T}) \Rightarrow \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p < \infty.
\]

Now by Hausdorff-Young we have a function \( g \in L_p'(\mathbb{T}) \) such that \( \hat{g}(n) = \hat{f}(n) \) which by uniqueness of Fourier Series shows that \( f = g \) a.e. This amounts to saying that \( f \in L_{p'}(\mathbb{T}) \), whenever \( f \in A_p(\mathbb{T}) \). But converse is not guaranteed for \( p < 2 \). Indeed we have continuous functions with Fourier series \( \sum_{n \in \mathbb{Z}} c_n e^{int} \) yet \( \sum_{n \in \mathbb{Z}} |c_n|^p = \infty \). So we get, in case of \( 1 < p < 2 \), \( A_p(\mathbb{T}) \subsetneq L_p'(\mathbb{T}) \). For \( p > 2 \) we know there are sequences \( (c_n)_{n=-\infty}^{\infty} \) with \( \sum_{n \in \mathbb{Z}} |c_n|^p < \infty \) yet \( \sum_{n \in \mathbb{Z}} c_n e^{inx} \) is not a Fourier series. So there is no direct link of \( A_p(\mathbb{T}) \) and \( L_r(\mathbb{T}) \) for any \( r \) and for
$p > 2$. However we can see that $A_p(T) \subset A_q(T)$ for $0 < p < q$. This is because

$$\sum_{n \in \mathbb{Z}} |c_n|^p < \infty \Rightarrow \sum_{n \in \mathbb{Z}} |c_n|^q < \infty, \text{ whenever } p < q.$$  

COROLLARY 2.3.1: Let $G = \mathbb{T}$, the circle group, and $1 \leq p < q < \infty$. Then $A_p(G, X) \subseteq A_q(G, X)$, where $X$ is a Hilbert space.

**PROOF:** $f \in A_p(T, X)$ means $\sum_{n \in \mathbb{Z}} \|\hat{f}(n)\|^p < \infty$. But

$$\sum_{n \in \mathbb{Z}} \|\hat{f}(n)\|^p < \infty \Rightarrow \sum_{n \in \mathbb{Z}} \|\hat{f}(n)\|^q < \infty \text{ for } 0 < p < q.$$  

This shows that

$$A_p(T, X) \subset A_q(T, X) \text{ for } p < q.$$  

But to show a strict inequality one follows the following method.

It has been shown by Gupta and Tewari [12] that there is a function $f \in A_p(T)$ which is not in $A_q(T)$. This means $f \in L_1(T)$ and $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^q < \infty$ but there can be no function $g \in A_p(T)$ with $\hat{g}(n) = \hat{f}(n)$, i.e. although $\sum_{n \in \mathbb{Z}} |f(n)|^q < \infty$,

$$\sum_{n \in \mathbb{Z}} |f(n)|^p = \infty.$$  

All that we need to show is to consider the vector valued function

$F : \mathbb{T} \rightarrow X$ defined by

$$F(e^{it}) = f(e^{it})x, \text{ where } x \text{ is a fixed nonzero vector in } X.$$  

$$\sum_{n \in \mathbb{Z}} \|\hat{F}(n)\|^q = \|x\|^q \sum_{n \in \mathbb{Z}} \|\hat{f}(n)\|^q < \infty$$

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But
\[ \sum_{n \in \mathbb{Z}} \| \hat{F}(n) \|^p = \| x \|^p \sum_{n \in \mathbb{Z}} \| \hat{f}(n) \|^p = \infty. \]
Hence \( A_p(T, X) \subsetneq A_q(T, X) \). This completes the proof.

**PROPOSITION 2.3.2 :** Let \( G = \mathbb{R} \), the real line and \( 1 \leq p < q < \infty \).
Then \( A_p(G, X) \subsetneq A_q(G, X) \), where \( X \) is a Banach space.

**PROOF :** It has already been shown by Gupta and Tewari [12] for scalar valued case. There is an \( f \in A_q(G, \mathbb{C}) \) such that \( f \notin A_p(G, \mathbb{C}) \). Let \( x \in X - \{0\} \) and
\[ F(g) = f(g)x \quad \forall \ g \in G. \]
We can show that \( F \in A_q(G, X) \). But \( F \) cannot be in \( A_p(G, X) \) for that would mean \( f \in A_p(G, \mathbb{C}) \). This completes the proof.

**PROPOSITION 2.3.3** Let \( G \) be an infinite compact totally disconnected abelian group and \( 1 \leq p < q < \infty \). Then \( A_p(G, X) \subsetneq A_q(G, X) \), where \( X \) is a Banach space.

**PROOF :** We know that for compact group \( G \) its dual group \( \hat{G} \) is discrete (cf. Rudin [32] page - 9). So
\[ A_p(G, X) = \{ f \in L_1(G); \hat{f} \in l_p(\hat{G}) \}. \]
For a discrete set \( I \), we have
\[ \sum_{i \in I} \| a_i \|^p < \infty \Rightarrow \sum_{i \in I} \| a_i \|^q < \infty, \text{ where } a_i \in X, \text{ for } i \in I. \]
To show that the inclusion is proper, we use the results of Gupta and Tewari [12].

According to this there is at least one \( f \in A_q(G) \) but \( f \notin A_p(G) \).

Let \( x \in X - \{0\} \). Now consider \( F \in A_q(G, X) \) defined by

\[
F(g) = f(g)x \quad \text{for} \quad g \in G.
\]

It is clear that \( F \in A_p(G, X) \). Let \( h \) be defined by

\[
h(g) = f(g)x.
\]

It is easily seen that \( h \in A_q(G) \) and \( h \notin A_p(G) \).

\[
\therefore F \notin A_p(G, X). \quad \text{This completes the proof.}
\]

**Proposition 2.3.4** Let \( G_1 \) and \( G_2 \) be locally compact abelian groups and \( G = G_1 \times G_2 \). Let \( 1 \leq p < q < \infty \). If \( A_p(G_i, X) \neq A_q(G_i, X) \) for some \( i \), then \( A_p(G_1 \times G_2, X) \neq A_q(G_1 \times G_2, X) \), where \( X \) is a Banach space.

**Proof:** We are looking for an \( h \in A_q(G_1 \times G_2, X) \) such that \( h \notin A_p(G_1 \times G_2, X) \).

Let \( f \in A_q(G, X) \) such that \( f \notin A_p(G_1, X) \). Let \( g \in A_q(G_2, X) \). Define

\[
h(x, y) = f(x)g(y).
\]

We claim that \( h \in A_q(G, X) \) but \( h \notin A_p(G, X) \), because

\[
\hat{h}(\xi, \eta) = \hat{f}(\xi)\hat{g}(\eta), \quad \text{where} \quad \hat{f} \in L_q(\widehat{G_1}, X), \hat{g} \in L_q(\widehat{G_2}, X)
\]

and

\[
\hat{h} \in L_p(\widehat{G}) \quad \text{if and only if} \quad \hat{f} \in L_p(\widehat{G_1}, X), \hat{g} \in L_p(\widehat{G_2}, X).
\]
This completes the proof.

It was shown by A.K. Gupta and Tewari [12] that:

If $G$ is a locally compact abelian group and $H$ is a compact subgroup of $G$, then

$$\prod_H A_p(G) = A_p(G/H).$$

Observe that the results of Gupta and Tewari [12] easily generalised to $A_p(G,X)$ using the same techniques, we state the following Proposition without proof.

**PROPOSITION 2.3.5** Let $G$ be a locally compact abelian group and let $H$ be a compact subgroup of $G$. Then $\prod_H A_p(G,X) = A_p(G/H,X)$ where $X$ is a Banach space.

**THEOREM 2.3.1** Let $G$ be an infinite compact abelian group and let $1 \leq p < q < \infty$. Then $A_p(G,X) \subseteq A_q(G,X)$.

**PROOF**: The techniques are essentially the same as those in A.K. Gupta and Tewari [12].

§ 2.4 MULTIPLIERS FROM $A_p(G,X)$ TO $A_q(G,X)$

**PROPOSITION 2.4.1** Let $1 \leq q \leq 2 < p$ and $X$ a Hilbert space. Then

$$l_{\frac{p}{p-q}}(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(A_p(G,X), A_q(G,X)) \subseteq l_{\frac{q}{2q-q}}(\hat{G}, \mathcal{L}(X)).$$
Moreover, if $r > \frac{pq}{p - q}$, then $l_r(\hat{G}, \mathcal{L}(X)) \nsubseteq \mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X))$ and if $r > p$, then

$$l_{\frac{pq}{p - q}}(\hat{G}, \mathcal{L}(X)) \nsubseteq \mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X))$$

**Proof:** For $\varphi \in l_{\frac{pq}{p - q}}(\hat{G}, \mathcal{L}(X))$ and $f \in \mathbf{A}_p(G, X)$, we get by Hölder’s inequality

$$\varphi \hat{f} \in l_q(\hat{G}, \mathcal{L}(X)) = \hat{\mathbf{A}}_q(G, X).$$

Thus $l_{\frac{pq}{p - q}}(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X))$.

Since $\mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X)) \subseteq \mathcal{M}(\mathbf{A}_2(G, X), \mathbf{A}_q(G, X)) = l_{\frac{2q}{2 - q}}(\hat{G}, \mathcal{L}(X))$, by Theorem 2.2.2, we get

$$l_{\frac{pq}{p - q}}(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X)) \subseteq l_{\frac{2q}{2 - q}}(\hat{G}, \mathcal{L}(X)).$$

But to show $\mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X)) = l_{\frac{2q}{2 - q}}(\hat{G}, \mathcal{L}(X))$, it is sufficient to show that for $r > \frac{pq}{p - q}$,

$$l_r(\hat{G}, \mathcal{L}(X)) \nsubseteq \mathcal{M}(\mathbf{A}_p(G, X), \mathbf{A}_q(G, X)).$$

Now $r > \frac{pq}{p - q} \Rightarrow p > \frac{rq}{r - q}$. Then by Theorem 2.3.1, it follows that there exists $f \in \mathbf{A}_p(G, X)$ such that $f \notin A_{\frac{rq}{r - q}}(G, X)$ i.e.,

$$\sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^{\frac{rq}{r - q}} = \infty.$$ 

By our Theorem 2.2.1,

$$\mathcal{M}(l_p(I, X), l_q(I, X)) \approx l_{\frac{pq}{p - q}}(I, \mathcal{L}(X)).$$
Hence $\mathcal{M}(\mathbb{A}_p(G, X), \mathbb{A}_q(G, X)) = l_{\frac{p}{p-q}}(\widehat{G}, \mathcal{L}(X))$.

Thus there exist a function $\psi$ in $l_r(\widehat{G}, \mathcal{L}(X))$, such that $\psi \hat{f} \notin l_q(\widehat{G}, \mathcal{L}(X))$.

This shows that

$$l_r(\widehat{G}, \mathcal{L}(X)) \not\subseteq \mathcal{M}(\mathbb{A}_p(G, X), \mathbb{A}_q(G, X)).$$

Now the remaining part of this proposition is to show that

$$l_{\frac{p}{p-q}}(\widehat{G}, \mathcal{L}(X)) \not\subseteq \mathcal{M}(\mathbb{A}_p(G, X), \mathbb{A}_q(G, X)),$$

if $r > p$. By theorem 2.3.1, there exists a function $f$ in $\mathbb{A}_r(G, X)$ such that

$$\sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^p = \infty.$$

By a similar argument as before that for some $\psi$ in $l_{\frac{p}{p-q}}(\widehat{G}, \mathcal{L}(X))$ such that $\psi \hat{f} \notin l_q(\widehat{G}, \mathcal{L}(X))$.

That is $\psi \notin \mathcal{M}(\mathbb{A}_r(G, X), \mathbb{A}_q(G, X))$. This completes the proof.

**Lemma 2.4.1** Let $2 < q \leq p < \infty$ and $1 \leq r < \infty$. Then

$$l_r(\widehat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbb{A}_p(G, X), \mathbb{A}_q(G, X))$$

if and only if

$$l_r(\widehat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbb{A}_p(\widehat{G}, X), \mathbb{A}_2(G, X)).$$
PROOF: Since $\mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_2(G,X)) \subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_q(G,X))$, it follows that

$$l_r(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_2(G,X))$$

$$\Rightarrow l_r(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_q(G,X)).$$

Now to prove that

$$l_r(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_q(G,X))$$

$$\Rightarrow l_r(\hat{G}, \mathcal{L}(X)) \subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_2(G,X)).$$

We shall prove this by taking contrapositive of the above statement.

Suppose $l_r(\hat{G}, \mathcal{L}(X)) \not\subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_2(G,X))$.

We shall show that

$$l_r(\hat{G}, \mathcal{L}(X)) \not\subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_q(G,X)).$$

If $l_r(\hat{G}, \mathcal{L}(X)) \not\subseteq \mathcal{M}(\mathbf{A}_p(G,X), \mathbf{A}_2(G,X))$, then there exists $\psi \in l_r(\hat{G}, \mathcal{L}(X))$ and $f \in \mathbf{A}_p(G,X)$ such that

$$\psi \hat{f} \not\in \hat{\mathbf{A}}_2(G,X) = l_2(\hat{G}, X).$$

It is really not hard to prove that

$$\mathcal{M}(\Omega, L_1(G,X)) \approx l_2(\hat{G}, X),$$

where $\Omega = \{\omega : \hat{G} \to \mathbb{T}\}$. 

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So there exists a function \( w : \hat{G} \to \mathbb{C} \) such that \( |\omega(\hat{G})| = 1 \) such that

\[
\omega(\hat{G}) \psi(\gamma) \hat{f}(\gamma) \notin (L_1(G, X))^A.
\]

Then the function \( \omega(\hat{G})\psi(\gamma) \in l_r(\hat{G}, \mathcal{L}(X)) \), but does not belong to \( \mathcal{M}(A_p(G, X), A_q(G, X)) \). This completes the proof.
CHAPTER-III

TOEPLITZ MULTIPLIERS

As described in the previous Chapter the question of multipliers between spaces of functions and spaces of sequences are interesting. We have indicated some of the results we were able to get for spaces of vector-valued functions and spaces of vector valued sequences. These generalize to similar question of spaces of infinite matrices. The general question about infinite matrices is too general to handle. What we describe are spaces of infinite matrices with some structures. To introduce this questions we need the following definitions.

§3.1 DEFINITIONS

DEFINITION- 3.1.1 :

\[ \mathcal{M} = \{ a = (a(i,j))(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : a(i,j) \in \mathbb{C} \}. \]

That is \( \mathcal{M} \) is the space of all infinite matrices with entries \( a(i,j) \in \mathbb{C} \) for \( (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \).

DEFINITION- 3.1.2 : For \( a \in \mathcal{M} \) by support of \( a \) denoted by \( \text{supp } a \), we mean

\[ \text{supp } a = \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : a(i,j) \neq 0 \}. \]
DEFINITION- 3.1.3 :

\[ M_0 = \{ a \in M : a(i,j) \neq 0 \text{ for only finitely many pairs } (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \}. \]

Again by the notation of support of \( a \), \( M_0 \) can be written as

\[ M_0 = \{ a \in M : \text{supp } a \text{ is finite} \}. \]

DEFINITION- 3.1.4 : Let \( M_p = \{ a \in M : \sum_{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+} |a(i,j)|^p < \infty \}. \)

We can now give \( M_p \) a Banach space structure with norm defined by

\[ \|a\|_{M_p} = \left( \sum_{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+} |a(i,j)|^p \right)^{\frac{1}{p}} \]

for \( 1 \leq p < \infty \). Again for \( 0 < p < 1 \), \( M_p \) is a topological vector space with the metric

\[ d(a,b) = \sum_{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+} |a(i,j) - b(i,j)|^p. \]

DEFINITION- 3.1.5 :

\[ M_\infty = \{ a \in M : \sup_{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+} |a(i,j)| < \infty \}. \]

Recall that we defined \( \mathcal{L}(X) \) to be the algebra of bounded linear transformation from a normed linear space \( X \) into itself. Now

\[ l_2(\mathbb{Z}_+) = \{ (\xi_n)_{n \in \mathbb{Z}_+} : \sum_{n \in \mathbb{Z}_+} |\xi_n|^2 < \infty \} \]

is a Hilbert space with an orthonormal basis \( (e_k)_{k \in \mathbb{Z}_+} \), where

\[ e_k(j) = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases} \]
Every operator $a \in \mathcal{L}(l_2)$ has a matrix representation with respect to the orthonormal basis:

$$a(i, j) = \langle ae_i, e_j \rangle.$$ 

An element $\xi$ of $l_2(\mathbb{Z}^+)$ is a sequence of the form

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) \text{ with } \sum_{i=1}^{\infty} |\xi_i|^2 < \infty.$$ 

If $a\xi = \eta$ then $\eta_i = \sum_{j=1}^{\infty} a(i, j)\xi_j$.

$$\|\eta\|^2 = \sum_{j=1}^{\infty} |\eta_i|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a(i, j)\xi_i|^2.$$ 

So we have

$$\|a\| = \sup \left\{ \left( \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} |a(i, j)\xi_i| \right)^2 \right)^{\frac{1}{2}} : \sum_{j=1}^{\infty} |\xi_i|^2 = 1 \right\}.$$ 

**DEFINITION- 3.1.6:** $a \in \mathcal{L}(l_2)$ is called a compact operator if it takes a bounded set into a compact set. We denoted the class of all compact operators by $\mathcal{K}(l_2)$.

We know for $a \in \mathcal{K}(l_2)$, $\lim_{i, j \to \infty} a(i, j) = 0$.

**DEFINITION- 3.1.7:** An operator $T \in \mathcal{L}(H)$ for a Hilbert space $H$ is called a positive operator if

$$(Tf, f) \geq 0 \text{ for every } f \in H.$$ 

We know that (cf. Douglas [6]) if $T$ is a positive operator then there is a positive operator $S$ such that $T = S^2$. We denote this $S$ by $T^{\frac{1}{2}}$ or $\sqrt{T}$. It is obvious
that for $a \in L(l_2)$, $a^*a$ is a positive operator, where $a^*$ is the adjoint of $a$. We denote by $\|a\| = (a^*a)^{\frac{1}{2}}$.

**DEFINITION- 3.1.8 :**

$$S_p = \{a \in L(l_2) : Tr(|a|^p)^{\frac{1}{p}} < \infty\}.$$ By $Tr(a)$, i.e., trace of the matrix $a$, we mean the sum $\sum_{i=1}^{\infty} a(i,i)$.

$S_2$ is now easily identified as

$$\{a \in M : \sum_{i,j \in \mathbb{Z}} |a(i,j)|^2 < \infty\}$$

of Hilbert Schmidt matrices.

$S_1$ is the class of nuclear operators. The classes $S_p$ are called the Schatten class operators.

**DEFINITION- 3.1.9 :** A matrix $a \in M$ is called an upper triangular matrix if $a(i,j) = 0$ if $i > j$. We denote the class of all upper triangular matrices by $UT$.

Similarly the class of lower triangular matrices, denoted by $LT$, consists of $a \in M$ with $a(i,j) = 0$ for $i < j$.

**DEFINITION- 3.1.10 :** $a \in M$ is called a Toeplitz matrix if

$$a(i,j) = \lambda_{i-j}, \text{ where } (\lambda_n)_{n=-\infty}^{\infty} \text{ is a sequence and } i, j \in \mathbb{Z}_+.$$
That is a Toeplitz matrix looks like

\[
a = \begin{bmatrix}
\lambda_0 & \lambda_{-1} & \lambda_{-2} & \lambda_{-3} & \lambda_{-4} & \cdots & \\
\lambda_1 & \lambda_0 & \lambda_{-1} & \lambda_{-2} & \lambda_{-3} & \cdots & \\
\lambda_2 & \lambda_1 & \lambda_0 & \lambda_{-1} & \lambda_{-2} & \cdots & \\
\lambda_3 & \lambda_2 & \lambda_1 & \lambda_0 & \lambda_{-1} & \cdots & \\
\lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 & \lambda_0 & \cdots & \\
& & & & & & \\
& & & & & & \\
\end{bmatrix}
\]

which is constant on all the diagonals parallel to the main diagonal.

We recall that these matrices appear in some situations as follows. \( H_2(\mathbb{T}) \) consists of all \( f \in L_2(\mathbb{T}) \) with no negative Fourier coefficients, i.e.

\[
H_2(\mathbb{T}) = \left\{ f \in L_2(\mathbb{T}) : \int_{-\pi}^{\pi} f(e^{it})e^{int}dt = 0 \text{ for } n > 0 \right\}.
\]

It is well known that \( H_2(\mathbb{T}) \) is a closed subspace of \( L_2(\mathbb{T}) \). Let

\[
P : L_2(\mathbb{T}) \to H_2(\mathbb{T})
\]

be the orthogonal projection from \( L_2(\mathbb{T}) \) to \( H_2(\mathbb{T}) \).

Let \( \varphi \in L_\infty(\mathbb{T}) \) with Fourier series \( \sum_{n=-\infty}^{\infty} a_n e^{int} \). For \( f \in H_2(\mathbb{T}) \) one sees that \( \varphi f \in L_2(\mathbb{T}) \). So \( P(\varphi f) \in H_2(\mathbb{T}) \). Thus we have for every \( f \in H_2(\mathbb{T}) \), an element \( P(\varphi f) \in H_2(\mathbb{T}) \).
We define the Toeplitz operator with symbol $\varphi$

$$T_\varphi : H_2(\mathbb{T}) \to H_2(\mathbb{T})$$

by

$$T_\varphi f = P(\varphi f).$$

It is not hard to see that $T_\varphi$ is a bounded linear operator on $H_2(\mathbb{T})$. We have the orthonormal bases $\{\chi_n\}_{n=-\infty}^{\infty}$ on $L^2(\mathbb{T})$ and $\{\chi_n\}_{n=0}^{\infty}$ on $H_2(\mathbb{T})$ where $\chi_n(e^{it}) = e^{int}$.

The matrix representation of $T_\varphi$ with respect to this orthonormal basis is easily seen to be

$$
\begin{bmatrix}
  a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots & \\
  a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & \\
  a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \\
  a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots & \\
  a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 
\end{bmatrix}
$$

which is a Toeplitz matrix.

We saw above that a Toeplitz matrix arises of a sequence $(\lambda_k)_{k\in\mathbb{Z}}$. For our purpose we shall denote the matrix by $T_\lambda$ (Though in the matrix $T_\varphi$ above, $\varphi$ is a function where as in the matrix $T_\lambda$, $\lambda$ is a sequence).
Pelczynski and Sukochev [30] have generalized this further to Schur-Toeplitz multipliers in the following way:

**DEFINITION - 3.1.11** The Schur multiplier induced by an $a \in \mathbb{M}$ is the operator $\mathcal{S}_a : \mathbb{M} \rightarrow \mathbb{M}$ defined by $\mathcal{S}_a(b) = a \circ b$, where $a, b \in \mathbb{M}$ and $a \circ b(i, j) = a(i, j)b(i, j)$ for $i, j \in \mathbb{Z}_+$. 

**DEFINITION - 3.1.12** A Toeplitz multiplier is a Schur multiplier induced by a Toeplitz matrix. The induced Toeplitz multiplier is usually denoted by $T_\lambda$, with entries $T_\lambda(i, j) = \lambda_{j-i}$.

**DEFINITION - 3.1.13** For a sequence $(\lambda_n)_{n=1}^{\infty}$ the matrix $\tilde{T}_\lambda$ be defined by

$$\tilde{T}_\lambda(i, j) = \frac{\sqrt{ij}}{i+j} \lambda_{j-i}.$$ 

is called a weighted Toeplitz matrix induced by $\lambda$. Recall that matrix representation of a Toeplitz operator with bounded harmonic symbol in Bergman space looks like this.

**DEFINITION - 3.1.14** Let $f$ be an analytic function in the open unit disc $\mathbb{D}$. Define

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

and

$$M_\infty(r, f) = \sup_{0 \leq \theta < 2\pi} \{|f(re^{i\theta})|\}.$$
We say that $f \in H_p(\mathbb{D})$ if $M_p(r, f), 0 < p \leq \infty$ remains bounded as $r \to 1^-$. Thus $H_p(\mathbb{D})$ is the space of all analytic functions $f$ defined in $\mathbb{D}$ such that $M_p(r, f), 0 < p \leq \infty$ remains bounded as $r \to 1^-$.

§3.2 OPERATORS AND MULTIPLIERS ON HARDY SPACE AND BERGMAAN SPACE

We defined earlier the Hardy Space $H_2$ as the class of analytic functions with unit disk $\mathbb{D} = \{ z : |z| < 1 \}$ with power expansion $\sum a_n z^n$ with $\sum |a_n|^2 < \infty$. It is possible to identify this class of analytic functions with the class $H_2(\mathbb{T})$ of functions in $L_2(\mathbb{T})$ with vanishing negative Fourier coefficient.

It is well known (cf. Duren [7], page 17) that for $f \in H_p(\mathbb{D}), p > 0$ the radial limit

$$\lim_{r \to 1^-} f(re^{i\theta})$$

exists almost everywhere and the limit function belongs to $L_p(\mathbb{T})$. For $p \geq 1$

$$\int_0^{2\pi} f(e^{it})e^{int}dt = 0 \text{ for } n > 0.$$

That is radial limit of $f \in H_p(\mathbb{D})$ for $p \geq 1$ is a function in $L_p(\mathbb{T})$ with no negative Fourier coefficients.

There is larger class of analytic functions in the unit disk $\mathbb{D}$ called the Bergmaan Space. The Bergmaan space $A^2(\mathbb{D})$ consists of class of all analytic function on $\mathbb{D}$.
which are square integrable. That is

$$A^2(\mathbb{D}) = \left\{ f : \mathbb{D} \rightarrow \mathbb{C}; \text{analytic, } \int \int_{\mathbb{D}} |f(z)|^2 dx \, dy < \infty \right\}.$$  

The space $A^2(\mathbb{D})$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int \int_{\mathbb{D}} f(z) \overline{g(z)} dx \, dy$$

for $f, g \in A^2(\mathbb{D})$ (The above integral is finite for $f, g \in A^2(\mathbb{D})$).

Let $e_n(z) = \frac{z^{n-1}}{\sqrt{n}}$ for $n = 1, 2, \ldots$ . $\{e_n\}_{n>0}$ is an orthonormal basis for $A^2(\mathbb{D})$. $A^2(\mathbb{D})$ is a closed subspace of $L_2(\mathbb{D})$ the space of all square integrable function on $\mathbb{D}$ . So one can define a projection $\mathbb{P} : L_2(\mathbb{D}) \rightarrow A_2(\mathbb{D})$ . Now if $\varphi \in L_\infty(\mathbb{D})$ then $\varphi f \in L^2(\mathbb{D})$ for $f \in A^2(\mathbb{D})$ . We define, as before,

$$T_\varphi : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D}) \text{ defined by}$$

$$T_\varphi f = \mathbb{P}(\varphi f) \text{ for } f \in A^2(\mathbb{D}).$$  

$T_\varphi$ now is a bounded linear operator on $A^2(\mathbb{D})$ . This too is called Toeplitz operator on the Bergmaan space $A^2(\mathbb{D})$ with symbol $\varphi$ . Particularly interesting Toeplitz operators on the Bergmaan spaces are those with harmonic symbol . We know if $\varphi$ is bounded harmonic function on $\mathbb{D}$ then it thus an expansion like

$$\varphi(z) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{i n \theta} , \text{ where } z = r e^{i \theta}.$$  

The matrix representation of $T_\varphi$ for this $\varphi$ is given by

$$T_\varphi(i, j) = a_{i-j} \frac{\sqrt{ij}}{i+j} \text{, for } i, j \in \mathbb{Z}_+.$$
This matrix looks like

\[
\begin{bmatrix}
\frac{1}{2}a_0 & \frac{\sqrt{2}}{3}a_{-1} & \frac{\sqrt{2}}{4}a_{-2} & \frac{2}{5}a_{-3} & \cdots & \cdots \\
\frac{\sqrt{2}}{3}a_1 & \frac{1}{2}a_0 & \frac{\sqrt{2}}{5}a_{-1} & \frac{2}{3}a_{-2} & \cdots & \cdots \\
\frac{\sqrt{3}}{4}a_2 & \frac{\sqrt{3}}{5}a_1 & \frac{1}{2}a_0 & \frac{2\sqrt{3}}{7}a_{-1} & \cdots & \cdots \\
\frac{\sqrt{2}}{3}a_3 & \frac{\sqrt{2}}{5}a_2 & \frac{2\sqrt{3}}{7}a_1 & \frac{1}{2}a_0 & \cdots & \cdots \\
& & & & \cdots & \cdots \\
& & & & \cdots & \cdots \\
& & & & \cdots & \cdots \\
& & & & \cdots & \cdots \\
\end{bmatrix}
\]

It is clear to see that this is our old Toeplitz matrix but for the weights. It can be shown that this is a bounded linear operator on the sequence space \( l^2(\mathbb{Z}^+) \) defined by \( \{ (\xi_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} \frac{|\xi_n|^2}{n+1} < \infty \} \).

We have already indicated in Chapter I about the question of multipliers from the Hardy Space \( H_p(\mathbb{D}) \) to \( l_q \). We mention the following seminal work done in this direction ( cf. Duren [ 7 ] ).

**THEOREM A :** For \( 0 < p \leq 1 \) the sequence \( (\lambda_n)_{n=0}^{\infty} \) is a multiplier of \( H_p \) into \( l_\infty \) if and only if

\[
\lambda_n = O(n^{1-\frac{1}{p}}).
\]

As it is seen above this is an extreme case, there are some intermediate results also.
THEOREM B : For $0 < p < 1$ and $p \leq q < \infty$, $(\lambda_n)_{n=1}^{\infty}$ is a multiplier of $H_p$ into $l_q$ if and only if
\[
\sum_{n=1}^{N} n^{\frac{q}{p}} |\lambda_n|^q = O(N^q).
\]
A much strong result in this direction is summarized as follows:

THEOREM C : $(\lambda_n)$ is a multiplier of $H_1$ into $H_2$ if and only if
\[
\sum_{n=1}^{N} n^2 |\lambda_n|^2 = O(N^2).
\]

In 1975 Charles Fefferman found the necessary and sufficient condition for the sequence $\lambda = (\lambda_s)_{s \in \mathbb{Z}_+}$ to be a multiplier from $H_1(\mathbb{T})$ to $l_1(\mathbb{Z}_+)$ is
\[
\rho(\lambda) = \left( |\lambda_0|^2 + |\lambda_1|^2 + \sup_{r \geq 1} \sum_{q=1}^{\infty} \left( \sum_{s=rq+1}^{rq+1} |\lambda_s| \right)^2 \right)^{\frac{1}{2}} < \infty.
\]
We say $\lambda \in FM$ if and only if $\rho(\lambda) < \infty$, where $FM$ stands for "Fefferman multiplier". Blasco and Pelczynski [2] generalized the result to nuclear operator valued $H_1$ functions as follows:

THEOREM D : For a complex sequence $(\lambda_s)_{s \in \mathbb{Z}_+}$, the following are equivalent:
(i) \( \rho(\lambda) < \infty \),

(ii) there is a constant \( k > 0 \) independent of \( \lambda \) such that

\[
\sum_{s \in \mathbb{Z}_+} |\lambda_s| \| \hat{f}(s) \|_1 \leq k \rho(\lambda)(2\pi)^{-1} \int_{-\pi}^{\pi} \| f(e^{it}) \|_1 dt
\]

for \( f \in H_1(\mathbb{T}; \mathcal{S}_1) \).

### §3.3 MAIN RESULTS

**THEOREM 3.3.1:** A weighted Toeplitz multiplier \( \tilde{T}_\lambda \) is a bounded linear operator from \( \mathcal{S}_1 \) into \( \mathcal{M}_1 \) if and only if \( \lambda = (\lambda_s)_{s \in \mathbb{Z}} \in l_1(\mathbb{Z}) \).

Moreover \( \| \tilde{T}_\lambda : \mathcal{S}_1 \rightarrow \mathcal{M}_1 \| \geq \frac{1}{16} \| \lambda \|_{l_1(\mathbb{Z})} \).

**PROOF:** If \( \lambda \in l_1(\mathbb{Z}) \) we show that \( \tilde{T}_\lambda : \mathcal{S}_1 \rightarrow \mathcal{M}_1 \) defined above is a bounded linear operator. It is plain to see that the task is to find out what it does to \( a \in \mathcal{S}_1 \) with \( \| a \|_1 = 1 \). To do this we follow what Pelczynski and Sukochev [29] do.

For \( x, y \in l_2 \), \( x = x \otimes y \in \mathcal{S}_1 \), \( x = (\xi_k)_{k=1}^\infty \), \( y = (\eta_k)_{k=1}^\infty \) we have

\[
\| \tilde{T}_\lambda(a) \|_{\mathcal{M}_1} = \sum_{i,j \in \mathbb{Z}_+} \sqrt{ij} |\lambda_{i-j}||\xi_i||\eta_j| 
\]

\[
= \sum_{s \geq 0} |\lambda_s| \sum_{k=0}^\infty \sqrt{(j+s+1)(j+1)} |\xi_k||\eta_j| 
\]

\[
+ \sum_{s < 0} |\lambda_s| \sum_{i=0}^\infty \sqrt{(i-s+1)(i+1)} |\xi_i||\eta_{-s}| 
\]

(Taking \( i-j = s \))

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Let \( I_1 = \sum_{s \geq 0} |\lambda_s| \sum_{j=0}^{\infty} \frac{\sqrt{(j + s + 1)(j + 1)}}{2j + s + 2} |\xi_{j+s}| |\eta_j| \),

\[ I_2 = \sum_{s < 0} |\lambda_s| \sum_{i=0}^{\infty} \frac{\sqrt{(i - s + 1)(i + 1)}}{2i - s + 2} |\xi_i||\eta_{i-s}|. \]

We now have \( \|\tilde{T}_\lambda(a)\|_{M_1} = I_1 + I_2 \).

Using the fact 

\[ \frac{1}{2} - \frac{\sqrt{ij}}{i + j} = \frac{(\sqrt{i} - \sqrt{j})^2}{2(i + j)} \geq 0 \]

for \( i, j > 0 \), we have 

\[ \frac{\sqrt{ij}}{i + j} \leq \frac{1}{2}. \]

Using Cauchy’s Schwarz inequality 

\[ \sum_{j=0}^{\infty} \frac{\sqrt{(j + s + 1)(j + 1)}}{2j + s + 2} |\xi_{j+s}| |\eta_j| \]

\[ \leq \frac{1}{2} \left( \sum_{j=0}^{\infty} |\xi_{j+s}|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} \left( \sum_{i=0}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}} \]

for \( s > 0 \) and taking \( i - j = s \).

\[ \sum_{i=0}^{\infty} \frac{\sqrt{(i - s + 1)(i + 1)}}{2i - s + 2} |\xi_i||\eta_{i-s}| \]

\[ \leq \frac{1}{2} \left( \sum_{i=0}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=0}^{\infty} |\eta_{i-s}|^2 \right)^{\frac{1}{2}} \]
for $s < 0$ and taking $i - j = s$ .

\[ I_1 \leq \frac{1}{2} \sum_{s \geq 0} |\lambda_s| \left( \sum_{i=0}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{j=0}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}}, \]

\[ I_2 \leq \frac{1}{2} \sum_{s < 0} |\lambda_s| \left( \sum_{i=0}^{\infty} |\xi_i|^2 \right)^{\frac{1}{2}} \left( \sum_{j=\infty}^{\infty} |\eta_j|^2 \right)^{\frac{1}{2}}. \]

Therefore,

\[ I_1 + I_2 \leq \frac{1}{2} \sum_{s \in \mathbb{Z}} |\lambda_s| \|x\|_2 \|y\|_2 \]

\[ = \frac{1}{2} \sum_{s \in \mathbb{Z}} |\lambda_s| = \frac{1}{2} \|\lambda\|_{l_1(\mathbb{Z})}. \]

Thus \(|\tilde{T}_\lambda : \mathcal{S}_1 \to \mathcal{M}_1| \leq \frac{1}{2} \|\lambda\|_{l_1(\mathbb{Z})}\) \quad (3.1). \]

This proves that if $\lambda \in l_1(\mathbb{Z})$, then $\tilde{T}_\lambda$ is a bounded linear operator from $\mathcal{S}_1$ into $\mathcal{M}_1$.

Conversely, suppose that

\[ a^{(n)} = x^{(n)} \otimes x^{(n)}, \quad \text{where} \quad x^{(n)} = \left( (n+1)^{-\frac{1}{2}}, ..., (n+1)^{-\frac{1}{2}}, 0, 0, ..., \right). \]

Then $\|a^{(n)}\|_1 = 1$.

Also $\|\tilde{T}_\lambda (a^{(n)})\|_{\mathcal{M}_1}$

\[ = \frac{1}{2} |\lambda_0| + |\lambda_1| \left( \frac{\sqrt{2}}{3} + \frac{\sqrt{6}}{5} + \frac{\sqrt{12}}{7} + .... \right) \frac{1}{n+1} \]

\[ + .... + |\lambda_{-1}| \left( \frac{\sqrt{2}}{3} + \frac{\sqrt{6}}{5} + \frac{\sqrt{12}}{7} + .... \right) \frac{1}{n+1} + .... \]

\[ \geq \frac{1}{16} \sum_{s=-n}^{n} |\lambda_s| \frac{n+1-|s|}{n+1}. \]

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Thus
\[ \| \tilde{T}_\lambda : S_1 \to M_1 \| \geq \lim_{n \to \infty} \| \tilde{T}_\lambda (a^{(n)}) \|_{M_1} \geq \frac{1}{16} \sum_{s \in \mathbb{Z}} |\lambda_s| \quad (3.2) . \]
This shows that if $\tilde{T}_\lambda$ is a bounded linear operator from $S_1$ into $M_1$, then $\lambda \in l_1(\mathbb{Z})$.
Thus from (3.1) and (3.2), we obtain
\[ \| \tilde{T}_\lambda : S_1 \to M_1 \| \geq \frac{1}{16} \| \lambda \|_{l_1(\mathbb{Z})} . \]
This completes the proof of the Theorem 3.3.1 .

**THEOREM 3.3.2** : A complex sequence $\lambda = (\lambda_s)_{s \in \mathbb{Z}^+}$ is in FM only if weighted Toeplitz multiplier $\tilde{T}_\lambda$ maps $UTS_1$ into $M_1$. Precisely, the following inequality holds :
\[ (\sqrt{3})^{-1} \| \tilde{T}_\lambda (a) \|_{M_1} \leq \| \tilde{T}_\lambda : UTS_1 \to M_1 \| \leq \frac{1}{2} k \rho (\lambda) \quad (3.3) \]
where $k > 0$ is a constant independent of $\lambda$.

**PROOF** : To prove this theorem, we need to apply Theorem-D. We know that matrices whose all but finitely many entries are zero are dense in $UTS_1$. So if we can show that $\tilde{T}_\lambda (UTS_1) \subset M_1$, then we can invoke the Banach-Steinhaus Theorem to conclude that the operator $\tilde{T}_\lambda : UTS_1 \to M_1$ defined by
\[ \tilde{T}_\lambda a(i, j) = \frac{\sqrt{(ij)}}{i + j} \lambda_{j-i} a(i, j) \]
is a bounded operator.

(The right hand inequality of 3.3)

To show this we proceed as Pelczynski and Sukochev [29] do. We define for \( a \in UTS_1 \) the \( s \)-diagonal matrix

\[
d_s(a)(i,j) = \begin{cases} 
  a(i,j) & \text{for } j - i = s \\
  0, & \text{otherwise}
\end{cases}
\]

It is not hard to show that \( d_s \in \mathcal{S}_1 \) and

\[
\|d_s(a)\|_1 = \sum_{j=s+1}^{\infty} |a(j - s, k)|.
\]

So it is possible to define the \( \mathcal{S}_1 \)-valued function for \( T \) as

\[
f(e^{it}) = \sum_{s=0}^{\infty} d_s(a)e^{ist}, \quad (-\pi \leq t \leq \pi).
\]

We know that

\[
\|f(e^{it})\|_1 = \|a\|_1.
\]

Hence

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(e^{it})\|_1 dt = \|a\|_1.
\]

So \( f \in H_1(T; S_1) \) with \( \hat{f}(s) = d_s(a) \).

Thus, by the Theorem - D,

\[
\sum_{s=0}^{\infty} \|d_s(a)\|_1 \|\lambda_s\| \leq \rho(\lambda)(2\pi)^{-1} \int_{-\pi}^{\pi} \|f(e^{it})\|_1 dt \quad (3.4)
\]
Now

\[ \|\tilde{T}_\lambda(a)\|_{M_1} = \sum_{j=1}^{\infty} \sum_{i=1}^{j} |a(i,j)| \frac{\sqrt{ij}}{i+j} |\lambda_{j-i}| \]

\[ = \sum_{s<0} \sum_{j=1}^{\infty} |a(j-s,j)||\lambda_s| \frac{\sqrt{(j-s)j}}{2j-s} \]

\[ + \sum_{i=1}^{\infty} \sum_{s>0} a(i,i+s)|\lambda_s| \frac{\sqrt{i(i+s)}}{2i+s} \]

\[ = \sum_{s<0} \sum_{j=1}^{\infty} |a(j-s,j)||\lambda_s| \frac{\sqrt{(j-s)j}}{2j-s}, \]

(since \(a(i,j)\) is an upper triangular matrix)

\[ \leq \frac{1}{2} \sum_s |\lambda_s||a(j-s,j)| \]

\[ = \frac{1}{2} \sum_{s<0} |\lambda_s|\|d_s(a)\|_1 \]

\[ = \frac{1}{2} \sum_{s<0} \|\hat{f}(s)\|_1|\lambda_s| \]

\[ \leq \frac{1}{2} k\rho(\lambda) \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(e^{it})\|_1 dt = \frac{1}{2} k\rho(\lambda)\|a\|_1. \]

Therefore,

\[ \|\tilde{T}_\lambda : UT S_1 \to M_1\| \leq k\rho(\lambda) \]

This completes the proof of right hand inequality of (3.3).

(\text{The left hand inequality of 3.3})

Suppose that a complex sequence \(\lambda = (\lambda_s)_{s \in \mathbb{Z}_+}\) is in FM. Then \(\rho(\lambda) < \infty\). So

\[ \|\tilde{T}_\lambda : UT S_1 \to M_1\| < \infty. \]

Without loss of generality we may assume that \(\lambda_s \geq 0\) for \(s \in \mathbb{Z}_+\). Fix a positive integer \(r\) and an eventually zero non-negative sequence
$(\alpha_q)_{q \in \mathbb{Z}^+}$ with $\sum_{q=0}^{\infty} \alpha_q^2 = 1$.

Define upper triangular matrices $a, a_1, a_2$ by $a = a_1 + a_2$, where

$$a_1 = x \otimes y, x = (1, 0, 0, ...), y = (\alpha_0, \alpha_1, 0, 0,...),$$

$$u = \left((r + 1)^{-\frac{1}{2}r - \frac{1}{4}}, (r + 1)^{-\frac{1}{2}r - \frac{1}{4}}, ..., 0, 0, ... \right)_{r \text{ times}}$$

$$v_k = \begin{cases} 
0, & 1 \leq i \leq r \\
\alpha_k r^{\frac{1}{4}}, & r + 1 \leq j \leq 2r, \\
\frac{\max(\alpha_q, \alpha_q+1)}{r^{\frac{1}{4}}}, & rq + 1 \leq j \leq r(q+1), q = 2, 3, ...
\end{cases}$$

Let $t = \alpha_0^2 + \alpha_1^2$. It is observed that

$$\|x\|_{l_2} = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{\frac{1}{2}} = 1$$

$$\|y\|_{l_2} = \left(\sum_{j=1}^{\infty} |y_j|^2\right)^{\frac{1}{2}} = \sqrt{\alpha_0^2 + \alpha_1^2} = \sqrt{t}$$

$$\therefore \|u\|_{l_2} = \left(\sum_{j=1}^{\infty} |u_k|^2\right)^{\frac{1}{2}} = \left(\frac{1}{\sqrt{r+1} r^{\frac{1}{4}}}, \frac{1}{\sqrt{r+1} r^{\frac{1}{4}}}, ..., \frac{1}{\sqrt{r+1} r^{\frac{1}{4}}}\right)$$

$$= \left(\frac{1}{(r+1)^{\frac{1}{2}}}, \frac{1}{(r+1)\sqrt{r}} + \frac{1}{(r+1)\sqrt{r}} + \ldots \ldots + \frac{1}{(r+1)\sqrt{r}}\right)^{\frac{1}{2}} (r \text{ times})$$

$$= \left(\frac{r^{\frac{1}{2}}}{(r+1)^{\frac{1}{2}}} \right)^{\frac{1}{2}} = \frac{r^{\frac{1}{4}}}{\sqrt{r+1}}.$$

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We have \( v = (v_k)_{k \in \mathbb{Z}_+} \).

\[
\therefore \|v\|_{l_2} = \left( \sum_{j=1}^{\infty} |v_j|^2 \right)^{\frac{1}{2}} = \left( \frac{\alpha_2^2}{\sqrt{r}} + \frac{\alpha_2^2}{\sqrt{r}} + \ldots + \frac{\alpha_2^2}{\sqrt{r}} + \sum_{q=1}^{\infty} \left( \max(\alpha_{q+1}, \alpha_{q+2}) \right) \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{r + 1}{\sqrt{r}} \alpha_2^2 + \frac{r + 1}{\sqrt{r}} \sum_{q=1}^{\infty} \max(\alpha_{q+1}, \alpha_{q+2}) \right)^{\frac{1}{2}}
\]

\[
= \sqrt{r + 1} \left( \alpha_2^2 + \sum_{q=1}^{\infty} (\alpha_{q+1}^2 + \alpha_{q+2}^2) \right)^{\frac{1}{2}}
\]

\[
= \sqrt{r + 1} \left[ 2 (\alpha_2^2 + \alpha_3^2 + \ldots + \alpha_n^2 + \ldots) \right]^{\frac{1}{2}}
\]

\[
= \sqrt{r + 1} \left( 2 \sum_{q=2}^{\infty} \alpha_q^2 \right)^{\frac{1}{2}} = \sqrt{r + 1} \cdot \sqrt{2} \left( \sum_{q=0}^{\infty} (\alpha_q^2) - (\alpha^2 - 0 + \alpha_0^2) \right)^{\frac{1}{2}}
\]

\[
= \sqrt{r + 1} \sqrt{2(1-t)}.
\]

So \( a_2 = u \otimes v \)

\[
\Rightarrow \|a_2\|_1 \leq \|u\|_1 \|v\|_1 = \sqrt{2(1-t)}
\]

\[
\Rightarrow \|a_2\|_1 \leq \sqrt{2(1-t)}.
\]

Hence,

\[
\|a\|_1 \leq \|a_1\|_1 + \|a_2\|_1
\]

\[
\leq \sqrt{t} + \sqrt{2(1-t)} \leq \sqrt{3},
\]

since for \( 0 \leq t \leq 1 \), \( \sqrt{t} + \sqrt{2(1-t)} \) attains its maximum at \( t = \frac{1}{3} \).
Therefore

\[ \sqrt{3} \| \tilde{T}_\lambda : UT S_1 \to M_1 \| \geq \| \tilde{T}_\lambda(a) \|_{M_1} \] (3.6)

This completes the proof of the Theorem 3.3.2.

**Remark:** We believe that it should be possible to show from this inequality that

\[ \| \tilde{T}_\lambda(a) \|_{M_1, \rho(\lambda)} . \]

But we have not been able to do this.
CHAPTER-IV

RANDOM FOURIER SERIES

We have already indicated that the question of random multipliers from the class $L_p(\mathbb{T})$ to $L_q(\mathbb{T})$ can be challenging. Basically the question is to characterised the sequence of random variables $(X_n)_{n \in \mathbb{Z}}$ for which $\sum_{n \in \mathbb{Z}} a_n X_n e^{int}$ would the Fourier series of random functions where $\sum_{n=\infty}^{\infty} a_n e^{int}$ is the Fourier series of $L_p$ function. The question of convergence of random series

$$\sum_{n \in \mathbb{Z}} a_n X_n e^{int}$$

has been addressed since twenties of the last century. Rademacher [31] studied the question of convergence of

$$\sum_{n=-\infty}^{\infty} r_n(t) a_n e^{int\theta},$$

where $a_n$ are Fourier coefficients of an $L_2$ function and

$$r_n(t) = \text{sgn} \sin 2^n \pi t, \quad t \in [0, 1].$$

The functions $r_n$ are called Rademacher functions.

It is possible to regard the Rademacher function $r_n$ as identically distributed independent random variables with zero mean and variance 1. Here the sample space is taken as the unit interval $[0, 1]$ with Lebesgue measure as the probability measure.
Steinhaus [36] considered the series
\[ \sum_{n=-\infty}^{\infty} a_n e^{2\pi i w_n} e^{inx}, \]
where \((\omega_n)_{n \in \mathbb{N}}\) are called the Steinhaus functions. They are defined on the interval \([0, 1]\) as follows:

For \(\omega \in [0, 1]\), let the binary expansion of \(\omega\) be equal to
\[ \sum_{n=1}^{\infty} \beta_n 2^{-n} \text{ where } \beta_n = 0 \text{ or } 1. \]

It is easy to see that \(\sum_{n=1}^{\infty} \beta_n = \infty\) except over a set of \(\omega\) of Lebesgue measure zero.

If we write
\[ m(j, n) = 2^{j-1}(2n - 1) \text{ for } j, n \in \mathbb{N}, \]
we may define the map \(\omega_j : [0, 1] \to \mathbb{R}\) by
\[ \omega_j(\omega) = \sum_{n=1}^{\infty} \beta_{m(j, n)} 2^{-n} \text{ for } \omega \in [0, 1]. \]

If we take the sample space \(\Omega = [0, 1]\), the \(\sigma\)-algebra \(\mathcal{A}\) as the class of all Lebesgue measurable subsets of \(\Omega\) and \(\mathcal{P}\) as Lebesgue measure on \(\mathcal{A}\) then \((\Omega, \mathcal{A}, \mathcal{P})\) is now a probability space. Each of the Steinhaus function \(\omega_j, j \in \mathbb{N}\) now is a random variable on the probability space just defined. In fact \(\omega_1, \omega_2, \ldots, \omega_n\) can be shown to be independent identically distributed random variables. So the series
\[ \sum_{n=-\infty}^{\infty} a_n e^{2\pi i w_n} e^{inx} \]
is now a random trigonometric series. It is interesting to note that Rademacher series...
\[ \sum_{n=1}^{\infty} r_n(t) a_n e^{inx} \]

can be thought of as an amplitude modulated trigonometric series, where the modulating factor is random. The Steinhaus series can be deemed as a frequency modulated trigonometric series.

Ever since many authors have studied the question of convergence of

\[ \sum_{n \in \mathbb{Z}} a_n X_n e^{int}, \]

where \( X_n \) are independent identically distributed random variables with some distributions e.g. normal.

The Rademacher series is a particular case of random Fourier series of the form

\[ \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) X_n, \]

where \( (X_n)_{n=1}^{\infty} \) is a sequence of independent of random variables. Paley, Winer and Zygmund [25] studied the series

\[ \sum_{n=1}^{\infty} a_n [X_n \cos nx + Y_n \sin nx], \]

where both \( (X_n)_{n=1}^{\infty} \) and \( (Y_n)_{n=1}^{\infty} \) are sequences of independent random variables.

Both of these two can be combined into the series

\[ \sum_{n=1}^{\infty} R_n \cos[nx + \Phi_n], \]

where \( (R_n)_{n=1}^{\infty} \) and \( (\Phi_n)_{n=1}^{\infty} \) are both sequences of independent random variables \( (R_m \) and \( \Phi_n \) are not necessarily independent \). We observe that both Rademacher and Steinhaus series are particular cases of the above series.
Hunt [16] considered the series

\[ \sum_{n=-\infty}^{\infty} X_n c_n e^{int}, \]

where \((X_n)_{n=-\infty}^{\infty}\) are arbitrary random variables with zero mean and infinite variance.

Very few people have considered the question when \((X_n)_{n\in\mathbb{Z}}\) are not independent. Those who have studied these questions have indicated how this could be related to the stochastic integrals. We have advanced the work done by others in this direction. In section I of this Chapter we give a brief introduction to what has been done hither to. We also indicate what we have done in this Chapter. In section II we introduce essential definitions and notations for this Chapter. In the section III we state our main results. The first one being defining stochastic integral with respect to a stochastic process whose increments belongs to the domain of attraction of the stable law. Then we demonstrate how Random Fourier - Stieltjes (RFS) series associated with the process are related to the stochastic integral.

Samal [35] considered the question of the convergence of the "integrated" RFS series

\[ \sum_{n=-\infty}^{\infty} \frac{A_n}{n} e^{2\pi int}, \]

where \(\sum_{n=-\infty}^{\infty}'\) means the term for \(n = 0\) is ommited with

\[ A_n = \int_{0}^{1} e^{-2\pi int} dX(t), \]

where \(X(t)\) is a continuous stochastic process with independent increments.
Nayak, Pattanayak and Mishra [24] considered the RFS series
\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int} \]
where
\[ A_n = \int_{0}^{2\pi} e^{-2\pi int} dX(t), \quad a_n = \int_{0}^{1} f(t) e^{-2\pi int} dt \]
for \( f \in L_p[0,1], \ X(t) \) a stochastic process with independent increments which are symmetric stable random variables with index \( \alpha \) with \( 1 < \alpha \leq 2 \) and \( p \geq \alpha \).

In this work they showed that it is possible to define stochastic integral
\[ \int_{0}^{1} F(t) dX(t) \]
for \( F \in L_p[0,1], \ X \) a symmetric stable process of index \( \alpha \) with \( 1 < \alpha \leq 2 \) and \( p \geq \alpha \), in the sense of convergence in probability. They also showed that the weighted RFS series
\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int} \]
converges in probability to
\[ \int_{0}^{1} f(t-u) dX(u) \]
for \( f \in L_p[0,1] \) for \( p > 1 \). The proof depends, crucially, on the fact that for \( f \in L_p[0,1], \ p > 1 \),
\[ a_n = \int_{0}^{1} f(t) e^{-2\pi int} dt \quad \text{and} \quad s_n(t) = \sum_{k=-n}^{n} a_k e^{2\pi ikt} \]
\[ \lim_{n \to \infty} \int_{0}^{1} |s_n(t) - f(t)|^p dt = 0. \]
Pattanayak and Sahoo [29] , Dash and Pattanayak [5 ] were able to prove that these series in fact converges in the mean . Pattanayak and Sharma [ 28 ] studied the question of convergence in probability of a RFS series associated with the process belonging to the domain of attraction of the stable law . We in this note are able to define a stochastic integral with respect to process whose increments belong to the domain of attraction of the stable law in the sense of convergence in the mean . Also we are able to show the convergence of the RFS series and conjugate RFS series in the mean and show that they are also Abel summable .

§ 4.1 DEFINITIONS

DEFINITION 4.1.1 : A stochastic process is said to have independent increment if 
\[ X(t_2) - X(t_1) \] is independent of 
\[ X(t_3) - X(t_2) , \] for 
\[ t_1 < t_2 < t_3 . \]

DEFINITION 4.1.2 : A sequence of random variables \( (X_n)_{n \in \mathbb{N}} \) is said to converge in the sense of probability to the random variable \( X \) if for all \( \epsilon > 0 , \)
\[
\lim_{n \to \infty} P ( \{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \} ) = 0 .
\]
DEFINITIONS 4.1.3: A sequence of Random variables \((X_n)_{n \in \mathbb{N}}\) converges in the mean to a random variable \(X\) if
\[
\lim_{n \to \infty} E|X_n - X| = 0.
\]

DEFINITION 4.1.4: If \(\sum_{n=0}^{\infty} a_n x^n\) has radius of convergence 1 and if the sum function
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad -1 < x < 1
\]
is such that \(\lim_{x \to -1^-} f(x)\) exists then we say that \(\sum_{n=0}^{\infty} a_n\) is Abel summable.

DEFINITION 4.1.5: For \(f \in L_1[a,b]\), let
\[
f_\beta(x) = \frac{1}{\Gamma(\beta)} \int_{a}^{x} (x-t)^{\beta-1} f(t)dt ,
\]
for \(0 < \beta < 1\) and \(a \leq x \leq b\). \(f_\beta\) is called the fractional integral of order \(\beta\) of the function \(f\).

DEFINITION 4.1.6: A stochastic process \(X\) with independent increments is said to be in the domain of attraction of the stable law if its increments belong to the domain of attraction of the stable law i.e., the increment \(X(t_1) - X(t_2)\) has characteristic function \(\exp(-|t_1 - t_2||u|^\alpha h(u))\) where \(h\) is a slowly varying function.
DEFINITION 4.1.7: For a positive function $\varphi$ we define

$$L_{\varphi}[a,b] = \{ f : [a,b] \rightarrow \mathbb{C} : \text{measurable and} \int_a^b \varphi(|f(t)|)dt < \infty \}.$$ 

It is well known that for $f \in L_{\varphi}[a,b]$ it is possible to define (cf. Kwapién and Woyczyński [19]) the stochastic integral $\int_b^a f(t)dX(t)$ for $X$ stochastic process whose increment belong to domain of attraction of stable laws. However we use our own techniques to define the stochastic integral and use the results to study the question of convergence of the Random Fourier - Stieltjes Series.

4.2 § MAIN RESULTS

To prove our main results we need the following:

**LEMMA 4.2.1:** (c.f Chow, Teicher[4]) For a random variable $X$ with characteristic function $\psi$ the absolute moment of the random variable $X$ is given by

$$E|X| = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - Re\{\psi(t)\}}{t^2} \, dt.$$ 

**LEMMA 4.2.2:** (cf. Zygmund [37]) If $f \in L_p, 1 < p < \infty$, and has the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{int}$ then the conjugate series $(-i) \sum_{n=-\infty}^{\infty} (\text{sgn})a_n e^{int}$ is also a Fourier series of a function $\tilde{f}$ in $L_p$ and for $\tilde{s}_n(t) = (-i) \sum_{k=-n}^{\infty} (\text{sgn})a_k e^{ikt}$,

$$\lim_{n \to \infty} \int_0^{2\pi} |\tilde{s}_n(t) - \tilde{f}(t)|^p \, dt = 0.$$
THEOREM-4.2.1: Suppose $X(t)$ is a stochastic process with independent increments belonging to the domain of attraction of the stable law. Let the characteristic function $\varphi$ of the increment $X(t_2) - X(t_1)$ be given by

$$\varphi(u) = \exp(-|t_2 - t_1||u|^\alpha h(u)),$$

where $h$ is a slowly varying function with the property

$$\lim_{t \to \infty} \frac{h(\gamma t)}{h(t)} = 1, \quad \lim_{t \to 0} \frac{h(\gamma t)}{h(t)} = 1$$

for every $\gamma > 0$, $2 \geq \alpha > 1$. For $f \in L_{\varphi}[a,b] \cap L_p[a,b]$ for $2 \geq \alpha > p > 1$ it is possible to define uniquely the stochastic integral

$$\int_a^b f(t) dX(t)$$

in the sense of convergence in the mean and has characteristic function

$$\exp\left(-|u|^\alpha h(u) \int_a^b |f(t)|^\alpha h(f(t)) dt\right).$$

PROOF: We recall (cf. Chow and Teicher [4], p. 289) that if $X_1$ and $X_2$ are independent Random variables with characteristic functions $\varphi_1(t)$ and $\varphi_2(t)$ respectively then the characteristic function of $c_1X_1 + c_2X_2$ is $\varphi_1(c_1t) \varphi_2(c_2t)$. 

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Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a step function with $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$ and

$$f(t) = \begin{cases} 
  c_1 & \text{for } a \leq t < t_1 \\
  c_2 & \text{for } t_1 \leq t < t_2 \\
  \vdots & \text{\ldots} \\
  c_{n-1} & \text{for } t_{n-2} \leq t < t_{n-1} \\
  c_n & \text{for } t_{n-1} \leq t \leq b.
\end{cases}$$

We define, for $X(t)$ defined as above,

$$\int_a^b f(t) dX(t) = \sum_{i=1}^n c_i (X(t_i) - X(t_{i-1})).$$

The characteristic function of $\int_a^b f(t) dX(t)$ now would be

$$\prod_{i=1}^n \exp (-|c_i|^{\alpha} h(c_i)|t_i - t_{i-1}| |u|^{\alpha} h(u))$$

$$= \exp \left( -\left( \sum_{i=1}^n |c_i|^{\alpha} h(c_i)|t_i - t_{i-1}| \right) |u|^{\alpha} h(u) \right)$$

$$= \exp \left( -|u|^{\alpha} h(u) \int_a^b |f(t)^{\alpha} h(f(t)) dt \right).$$

Let us define the stochastic integral of a step function as above. If $f_1$ and $f_2$ are two step functions there linear combination $\lambda_1 f_1 + \lambda_2 f_2$ too is a step function. It is plain to see that

$$\int_a^b (\lambda_1 f_1(t) + \lambda_2 f_2(t)) dX(t) = c_1 \int_a^b f_1(t) dX(t) + c_2 \int_a^b f_2(t) dX(t).$$
Now suppose $f \in L_\phi[a,b] \cap L_p[a,b]$ , $1 < p < \alpha$, then we can get a sequence of step functions $(f_n)_{n=1}^\infty$ such that

$$\lim_{n \to \infty} \int_a^b |f_n(t) - f(t)|^p dt = 0,$$

for $0 < \delta < \alpha - 1$. Now let

$$Y_n = \int_a^b f_n(t) dX(t)$$

as defined above. Characteristic function of $Y_m - Y_n$ is given by

$$\exp \left( -|u|^\alpha h(u) \int_a^b |f_m(t) - f_n(t)|^\alpha h(f_m(t) - f_n(t)) dt \right).$$

Therefore the expectation $E|Y_m - Y_n|$ of $Y_n - Y_m$ is given by Lemma A .

$$E|Y_m - Y_n| = \frac{2}{\pi} \int_{-\infty}^\infty \frac{1 - \exp \left( -|u|^\alpha h(u) \int_a^b |f_m(u) - f_n(u)|^\alpha h(f_m(u) - f_n(u)) dt \right)}{u^2} du$$

$$= \frac{2}{\pi} \int_{|u| \leq R} \frac{2}{\pi} \int_{-\infty}^\infty \frac{1 - \exp \left( -|u|^\alpha h(u) \int_a^b |f_m(u) - f_n(u)|^\alpha h(f_m(u) - f_n(u)) dt \right)}{u^2} du$$

$$+ \frac{2}{\pi} \int_{|u| > R} \frac{2}{\pi} \int_{-\infty}^\infty \frac{1 - \exp \left( -|u|^\alpha h(u) \int_a^b |f_m(u) - f_n(u)|^\alpha h(f_m(u) - f_n(u)) dt \right)}{u^2} du$$

$$\leq \frac{4}{\pi} \left( \int_{|u| < R} \frac{|u|^\alpha h(u)|}{u^2} \int_a^b |f_m(u) - f_n(u)|^\alpha h(f_m(u) - f_n(u)) dt \right) du$$

$$+ \frac{2}{\pi} \int_{|u| \geq R} \frac{1}{u^2} du$$

For $\epsilon > 0$ choose $R$ such that $\frac{4}{\pi R} < \frac{\epsilon}{2}$. Since $\lim_{t \to 0} \frac{h(\gamma t)}{h(t)} = 0$ we can, for every $\delta > 0$, get a $K_\delta$ such that

$$|h(t)| \leq K_\delta t^{-\delta},$$

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where $K_\delta$ is independent of $t$. So

$$|u^\alpha h(t)| \leq K_\delta |u|^{\alpha-\delta}.$$  

Let $0 < \delta < p - \alpha$ and $0 < \delta < \alpha - 1$. So we have

$$\frac{4}{\pi} \int_{|u| < R} \frac{|u|^\alpha h(u)}{u^2} du = \frac{8}{\pi} K_\delta \int_0^R u^{\alpha-\delta-2} du = \frac{8K_\delta}{\pi} R^{\alpha-\delta-1}.$$

$$\int_a^b |f_n(t) - f_m(t)|^\alpha h(f_n(t) - f_m(t)) dt \leq K_\delta \int_a^b |f_m(t) - f_n(t)|^{\alpha-\delta} dt .$$

Now since $f_m$ is Cauchy in $L_p$ we can choose $n_0$ such that for $m,n > n_0$

$$\frac{8K_\delta^2}{\pi} \int_a^b |f_m(t) - f_n(t)|^{\alpha-\delta} dt < \frac{\epsilon}{2} .$$

So we have for $m,n > n_0$

$$E|Y_m - Y_n| < \epsilon .$$

This shows that $(Y_n)_{n=1}^\infty$ is Cauchy, hence converges in the mean. We may call the random variable to which $(Y_n)_{n=1}^\infty$ converges in the mean as $\int_a^b f(t)dX(t)$.

But we are yet to show that this is independent of the choice of the sequence $(f_n)_{n=1}^\infty$. To do that let $g_n$ be another sequence $(g_n)_{n=1}^\infty$ such that

$$\lim_{n \to \infty} \int_a^b |g_n(t) - f(t)|^p dt = 0 .$$

So we have

$$\lim_{n \to \infty} \int_a^b |f_m(t) - g_m(t)|^p dt = 0 .$$
Let \( Z_n = \int_a^b g_n(t) dX(t) \). Now

\[
E|Z_n - Y_n| = \frac{2}{\pi} \int_{-\infty}^{\infty} 1 - \exp \left( -|u|^\alpha h(u) \int_a^b |g_m(t) - f_m(t)|^\alpha h(g_n(t) - f_n(t)) dt \right) du .
\]

We can now show as we have done before that the right hand side converges to zero as \( n \to \infty \). Thus \( \int_a^b f(t) dX(t) \) is uniquely defined. This completes the proof of this theorem.

**THEOREM-4.2.2:** Let \( \psi(t) = |t|^\alpha h(t) \), where \( h \) is a slowly varying function satisfying

\[
\frac{h(\gamma t)}{h(t)} \to 1 \quad \text{as} \quad t \to \infty \quad \text{also} \quad \frac{h(\gamma t)}{h(t)} \to 1 \quad \text{as} \quad t \to 0, \quad \text{for} \ \gamma > 0.
\]

Let \( X \) be a stochastic process with independent increments such that the increment \( (X(t_1) - X(t_2)) \) has characteristic function \( \exp(-|t_1 - t_2| \psi(t)) \).

For \( f \in L^\psi[0, 2\pi] \), let

\[
a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) dt \quad \text{and} \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} dX(t),
\]

then the random Fourier-Stieltjes series

\[
\sum_{n=-\infty}^{\infty} a_n A_n e^{int}
\]

converges in the mean to the stochastic integral

\[
\frac{1}{2\pi} \int_0^{2\pi} f(t - u) dX(u).
\]
**PROOF:** Let $S_n$ be the $n$th partial sum of the series

$$
\sum_{n=-\infty}^{\infty} a_n A_n e^{int}
$$

that is

$$S_n = \sum_{k=-n}^{n} a_k A_k e^{int} \quad \text{and} \quad s_n = \sum_{k=-n}^{n} a_k e^{ikt}.$$

We have

$$S_n(t) - S_m(t) = \int_0^{2\pi} (s_n(t-u) - s_m(t-u)) dX(u).$$

By the Lemma -A we have

$$E|S_n(t) - S_m(t)|$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - \exp\left(-\int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du\right)}{\xi^2} d\xi$$

$$= \frac{2}{\pi} \int_{|\xi| \leq 1} \frac{1 - \exp\left(-\int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du\right)}{\xi^2} d\xi$$

$$+ \frac{2}{\pi} \int_{|\xi| > 1} \frac{1 - \exp\left(-\int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du\right)}{\xi^2} d\xi.$$

Writing

$$I_1 = \frac{2}{\pi} \int_{|\xi| \leq 1} \frac{1 - \exp\left(-\int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du\right)}{\xi^2} d\xi$$

$$I_2 = \frac{2}{\pi} \int_{|\xi| > 1} \frac{1 - \exp\left(-\int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du\right)}{\xi^2} d\xi,$$
we have
\[ E|S_n(t-u) - S_m(t-u)| = I_1 + I_2. \]

It is easy to see that
\[
|I_1| \leq \frac{4}{\pi} \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \psi\left(\xi (s_n(t-u) - s_m(t-u))\right) \frac{du}{\xi^2} \, d\xi
\]
\[
= \frac{4}{\pi} \int_0^1 \xi^{\alpha-2} \frac{1}{2\pi} \int_0^1 |s_n(t-u) - s_n(t-u)|^{\alpha} h(\xi (s_n(t-u) - s_m(t-u))) \, du \, d\xi
\]

It is easy to see that for the function $h$ with property as stated above for every $\delta > 0$, $h|u|^\delta \leq 1$ for sufficiently small $u$.

We may take for every $\delta > 0$, we can get a constant $K_\delta$ such that
\[ h(u)|u|^{\delta} \leq K_\delta \text{ for } 0 \leq u \leq 1. \]

So
\[ \psi(\xi (s_n(t-u) - s_m(t-u))) \leq K_\delta \xi^{\alpha-\delta} |s_n(t-u) - s_m(t-u)|^{\alpha-\delta}. \]

Now choose $0 < \delta < 2 - \alpha$.

This gives
\[ |I_1| \leq \frac{4}{\pi} \frac{1}{\alpha - \delta - 1} \int_0^{2\pi} |s_n(t-u) - s_m(t-u)|^{\alpha-\delta} \, du. \]

It is now seen that
\[ \lim_{m,n \to \infty} |I_1| = 0. \]

As for $I_2$ we observe that for $|\xi| > 1$
\[ \frac{1 - \exp\left(-\int_0^{2\pi} \varphi(\xi (s_n(t-u) - s_m(t-u))) \, du\right)}{\xi^2} \leq \frac{1}{\xi^2} \]

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and
\[\int_{|\xi|>1} \frac{1}{\xi^2} \, d\xi < \infty.\]

So by Lebesgue dominated convergence theorem we have
\[\lim_{m,n \to \infty} |I_2| = 0\]

Hence \(\lim_{m,n \to \infty} E|S_n(t) - S_m(t)| = 0\), i.e., \(S_n(t)\) is Cauchy, hence converges in the mean. It is easy to show that \(S_n(t)\) converges to the stochastic integral
\[\int_0^{2\pi} f(t-u) \, dX(u).\]

This completes the proof of Theorem 4.2.2.

**THEOREM-4.2.3**: If \(f \in L_{\psi}[0, 2\pi]\) with
\[a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) \, dt \quad \text{and} \quad A_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} dX(t),\]
then the conjugate RFS series
\[\sum_{n=-\infty}^{\infty} \tilde{a}_n A_n e^{int} \, dt\]
where \(\tilde{a}_n = -i (sgn n) a_n\) converges in the mean to the stochastic integral
\[\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(t-u) \, dX(u).\]
PROOF: Let \( \tilde{S}_n \) be the nth partial sum of the conjugate RFS series

\[
\sum_{n=-\infty}^{\infty} a_n A_n e^{int}.
\]

That is

\[
\tilde{S}_n = \sum_{k=-n}^{n} \tilde{a}_k A_k e^{ikt} \quad \text{and} \quad \tilde{s}_n = \sum_{k=-n}^{n} \tilde{a}_k e^{ikt}.
\]

We see that

\[
E|\tilde{S}_n(t) - \tilde{S}_m(t)|
\]

\[
= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - \exp \left( - \int_0^{2\pi} \varphi (\xi(\tilde{s}_n(t-u) - \tilde{s}_m(t-u)))du \right)}{\xi^2} d\xi
\]

\[
= \frac{2}{\pi} \int_{|\xi| \leq 1} \frac{1 - \exp \left( - \int_0^{2\pi} \varphi (\xi(\tilde{s}_n(t-u) - \tilde{s}_m(t-u)))du \right)}{\xi^2} d\xi
\]

\[
+ \frac{2}{\pi} \int_{|\xi| > 1} \frac{1 - \exp \left( - \int_0^{2\pi} \varphi (\xi(\tilde{s}_n(t-u) - \tilde{s}_m(t-u)))du \right)}{\xi^2} d\xi.
\]

Using the techniques used in the Theorem 4.2.2 and by Lemma - B it is easy to see that

\[
\lim_{m,n \to \infty} E|\tilde{S}_n(t) - \tilde{S}_m(t)| = 0.
\]

Hence \( \tilde{S}_n(t) \) converges in mean. It is not hard to show that this, in fact, converges to the Stochastic integral

\[
\int_0^{2\pi} \tilde{f}(t-u) dX(u).
\]

This completes the proof of the Theorem 4.2.3.
THEOREM 4.2.4: Let $X(t)$ be a stochastic process whose increment belong
domain of attraction of stable laws. If

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) dX(t) \quad \text{and} \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) dt,$$

then the series

$$\sum_{n=\infty} a_n A_n e^{int}$$

is Abel summable to

$$\frac{1}{2\pi} \int_0^{2\pi} f(t-u) dX(u)$$
in the sense of mean.

PROOF: We know that for $0 \leq r \leq 1$ the series

$$\sum_{n=\infty} a_n r^{|n|} e^{int}$$

converges uniformly to a function $f \in L_p, p \geq \alpha$. Let us write

$$f_r(t) = \sum_{n=\infty} a_n r^{|n|} e^{int}, \quad 0 \leq r < 1.$$

Since $f \in L_p, f_r \in L_p, p \geq \alpha$. By Theorem 4.2.2 the series

$$\sum_{n=\infty} a_n A_n r^{|n|} e^{int}$$

converges to the stochastic integral

$$\frac{1}{2\pi} \int_0^{2\pi} f_r(t-u) dX(u)$$
in the sense of mean.
Now \( E \left| \int_0^{2\pi} f_r(t-u) dX(u) - \int_0^{2\pi} f(t-u) dX(u) \right| \)
\[ = E \left| \int_0^{2\pi} (f_r(t-u) - f(t-u)) dX(u) \right| \]
\[ = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - \exp\left( - \int_0^{2\pi} \psi(\xi(s_n(t-u) - s_m(t-u))) du \right)}{\xi^2} d\xi. \]

Because
\[ \lim_{r \to 1} \int_0^{2\pi} |f_r(t-u) - f(t-u)|^p du = 0 \]
for every \( p > \alpha \) we get
\[ \lim_{r \to 1} E \left| \int_0^{2\pi} (f_r(t-u) - f(t-u)) dX(u) \right| = 0 \]
by the same arguments used in Theorem 4.2 . Hence the RFS series
\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{int} \]
is Abel summable to
\[ \frac{1}{2\pi} \int_0^{2\pi} f(t-u) dX(u) \]
in the sense of mean. It is plain that this also works for the conjugate RFS series. This completes the proof of Theorem 4.2.3.

In the beginning of this Chapter we have described work of Samal [35], which investigated the "formaly integrated" random fourier Stieltjes series
\[ \sum_{n=-\infty}^{\infty} \frac{A_n}{n} e^{2\pi int}, \quad \text{where} \quad A_n = \int_0^1 e^{-2\pi int} dX(t) \]
led to the study of Random Fourier - Stieltjes series

\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{2\pi int} \]

where \((a_n)_{n \in \mathbb{Z}}\) are Fourier coefficients (cf. Nayak, Pattanayak and Mishra [24]).

It was shown that the RFS series (4.1) really converged in probability to a stochastic integral

\[ \frac{1}{2\pi} \int_{0}^{2\pi} f(t - u) dX(u) , \]

Natural curiosity would be to investigate the series

\[ \sum_{n=-\infty}^{\infty} \frac{a_n A_n}{(im)^\beta} e^{int} , \text{ where } 0 < \beta < 1 . \]

As we have indicated the random Fourier series

\[ \sum_{n=-\infty}^{\infty} a_n A_n e^{int} \]

"integrated " term by term would be

\[ \sum_{n=-\infty}^{\infty} \frac{A_n a_n}{n} e^{int} \quad \text{if } a_0 = 0 . \]

It can be shown that if

\[ a_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) e^{-int} dt , \quad a_0 = 0 \text{ and } \]
\[ A_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-int} dX(t) , \]

then

\[ \sum_{n=-\infty}^{\infty} \frac{A_n a_n}{n} e^{int} \text{ converges to } \frac{1}{2\pi} \int_{0}^{2\pi} F(t - u) dX(u) , \]
where \( F(t) = \int_0^t f(u)du \).

We should now ask if
\[
\sum_{n=-\infty}^{\infty} \frac{a_n A_n e^{int}}{(in)^\alpha}
\]
has a similar representation? We address it thus.

Suppose that for given \( f \in L^1[a,b] \) we define
\[
\int_a^x f(t)dt = f_1(x)
\]
and \( f_\alpha(x) \) is the integral of \( f_{\alpha-1} \) then the integral \( f_\alpha \) for values of \( \alpha = 2, 3, \ldots \) are defined by induction.

One sees that one can show for \( \alpha \) integer
\[
f_\alpha(x) = \frac{1}{(\alpha - 1)!} \int_a^x (x-t)^{\alpha-1} f(t)dt.
\]

We now generalize this for \( \alpha \) a proper fraction. In the definition
\[
f_\alpha(x) = \frac{1}{(\Gamma(\alpha))} \int_a^x (x-t)^{\alpha-1} f(t)dt.
\]

For \( 0 < \alpha < 1 \) it is true that \( f_\alpha \) is defined as it is convolution of two functions in \( L^1[a,b] \) namely \( g_\alpha \) defined by \( g_\alpha(t) = (x-t)^{\alpha-1} \) and \( f \). We can apply this idea as Zygmund [37], does to Fourier series. But we need some modification to apply this. Consider the series
\[
\sum_{n=-\infty}^{\infty} \frac{e^{int}}{(in)^\beta}.
\]

For \( 0 < \beta < 1 \) is in fact a Fourier series of a function \( \Psi_\alpha \in L^1 \).
It can be shown that if
\[ \sum_{n=-\infty}^{\infty} c_n e^{int} \]
is the Fourier series of \( f \in L_1 \) with \( \int_{-\pi}^{\pi} f(t)dt = 0 \) then the series
\[ \sum_{n=-\infty}^{\infty} c'_n (\frac{\alpha}{\pi} n)^{\alpha} e^{int} \]
is the Fourier series of the convolution
\[ \int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt. \]

But this is not quite the same as \( f_\alpha \) defined above. It can be shown that ( cf. Zygmund [37], Vol.II, p.135 )
\[ \Gamma(\beta) \Psi_\beta(x) = 2\pi \lim_{n \to \infty} (x^{\beta-1} + (x + 2\pi)^{\beta-1} + \cdots + (x + 2\pi n)^{\beta-1}) \]
and we may write for \( f \) periodic with \( \int_0^{2\pi} f(t)dt = 0 \).
\[ \frac{1}{2\pi} \int_0^{2\pi} f(x-t)\Psi_\beta(t)dt = \lim_{n \to \infty} \frac{1}{\Gamma(\beta)} \int_0^{2\pi} f(x-t) \left( \sum_{k=0}^{n} (t + 2k\pi)^{\beta-1} \right) dt \]

That is the same as saying
\[ \frac{1}{2\pi} \int_0^{2\pi} f_\beta(x-t)\Psi_\beta(t)dt = f_\beta. \]

It is known that the fractional integral \( f_\beta \) belongs to \( L_p \) for all \( p \geq 1 \) if \( f \) belongs to \( L_p[0,2\pi] \) for some \( p \geq 1 \) and \( \frac{1}{p} < \beta < 1 + \frac{1}{p} \) ( cf. Zygmund [37], vol-II, p.138 ). This fact is used to prove the following theorem.
THEOREM 4.2.5: Suppose $X(t)$ be a stochastic process with independent increments where increments belong to the domain of attraction of the stable law.

Suppose $f \in L_\psi \cap L_p$ for $1 < p < \alpha$ as in Theorem 4.2.1, and

$$\int_0^{2\pi} f(t) dt = 0, \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} dt,$$

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} dX(t).$$

The series

$$\sum_{n=-\infty}^{\infty} \frac{A_n a_n e^{int}}{(in)^\beta}$$

converges in the mean to

$$\frac{1}{2\pi} \int_0^{2\pi} f_\beta(x - t) dX(t).$$

PROOF: Let $f_\beta$ be the fractional integral of $f$ as defined above. Now $f_\beta \in \lambda_\beta - \frac{1}{p}$. This makes $f_\beta$ a continuous function hence an element of $L_p$ too. So then by Theorem 4.2.1,

$$\int_0^{2\pi} f_\beta(t) dX(t)$$

exists in the sense of convergence in the mean. We have stated above that $f_\beta$ has the Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{a_n e^{int}}{(in)^\beta}.$$
converges in the mean to

\[ \frac{1}{2\pi} \int_0^{2\pi} f_\beta(t-u) dX(u). \]

This completes the proof of the Theorem 4.2.5.
Bibliography


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PUBLICATIONS

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