Chapter 4

A generalization of Eisenstein-Schönemann Irreducibility Criterion

4.1 Motivation of the problem and statements of results

In 1846, Schönemann proved that if $f(x)$ is a monic polynomial with coefficients from the ring $\mathbb{Z}$ of integers which is irreducible modulo a prime number $p$ and if $g(x)$ belonging to $\mathbb{Z}[x]$ is a polynomial of the form $g(x) = f(x)^r + pM(x)$ where $M(x)$ belonging to $\mathbb{Z}[x]$ has degree less than that of $g(x)$ and is relatively prime to $f(x)$ modulo $p$, then $g(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers. This irreducibility criterion has been extended to polynomials with coefficients in discrete valued fields and arbitrary valued fields (see [Rib2], [Kh-Sa], [Bro2]). We state below the most general form of this criterion after introducing some notations.

As in Chapter 1, for an element $c$ in the valuation ring $R_v$ of $v$, $\bar{c}$ will denote its $v$-residue and for $f(x)$ belonging to $R_v[x]$, $\bar{f}(x)$ will have the same meaning as before. $v^x$ will denote the Gaussian extension of $v$ to $K(x)$ defined by (1.1). If $f(x)$ is a fixed monic polynomial with coefficients in an integral domain $R$, then each $F(x)$ belonging to $R[x]$ can be uniquely represented as a finite sum $F(x) = \sum_{i=0}^{n} F_i(x) f(x)^i$, where for any $i$, the polynomial $F_i(x) \in R[x]$ has degree less than that of $f(x)$. This
The following theorem is proved in [Bro2].

**Theorem 4.1.A.** Let $v$ be a Krull valuation of a field $K$ with value group $\Gamma$ and valuation ring $R_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m$ such that $f(x)$ is irreducible over the residue field of $v$. Assume that $g(x)$ belonging to $R_v[x]$ is a monic polynomial whose $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$ satisfies (i) $A_0(x) \neq 0$, $A_n(x) = 1$, (ii) $v^r(A_i(x)) \geq \frac{v^r(A_0(x))}{n-i} > 0$ for $0 \leq i \leq n-1$ and (iii) $v^r(A_0(x)) \notin d\Gamma$ for any number $d > 1$ dividing $n$. Then $g(x)$ is irreducible over $K$.

A polynomial $g(x)$ satisfying conditions (i), (ii), (iii) of the above theorem will be referred to as a Generalized Schönenmann polynomial with respect to $v$ and $f(x)$. Note that in case $v$ is a discrete valuation of $K$ with value group $\mathbb{Z}$, then condition (iii) of Theorem 4.1.A says that $v^r(A_0(x))$ and $n$ are coprime. Hence in this case, it is immediate from the above theorem that a polynomial $g(x)$ having $f(x)$-expansion $f(x)^n + \sum_{i=0}^{n-1} A_i(x)f(x)^i$ with $v^r(A_0(x)) = 1$, $v^r(A_i(x)) > 0$ for $0 \leq i \leq n-1$, is irreducible over $K$; such a polynomial is called a Schönenmann polynomial with respect to the discrete valuation $v$ and $f(x)$; in the particular case when $f(x) = x$, it will be referred to as an Eisenstein polynomial with respect to $v$.

In this chapter, we extend Theorem 4.1.A by proving the following result which yields generalizations of some known irreducibility criteria.

**Theorem 4.1.1.** Let $v$ be a henselian Krull valuation of a field $K$ with value group $\Gamma$ and valuation ring $R_v$ having maximal ideal $\mathcal{M}_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m$ such that $f(x)$ is irreducible over $R_v/\mathcal{M}_v$ and $\phi(x)$ belonging to $R_v[x]$ be a monic polynomial having $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$ with $A_0(x) \neq 0$. Assume that there exists $s \leq n$ such that (i) $v^r(A_i(x)) = 0$, (ii) $v^r(A_0(x)) \geq v^r(A_i(x)) > 0$ for $0 \leq i \leq s-1$ and (iii) $v^r(A_0(x)) \notin d\Gamma$ for any number $d > 1$ dividing $s$. Then $\phi(x)$ has an irreducible factor $g(x)$ of degree $sm$ over $K$.
such that \( g(x) \) is a Generalized Schönenmann polynomial with respect to \( v \) and \( f(x) \); moreover the \( f(x) \)-expansion of \( g(x) = f(x)^s + B_{s-1}(x)f(x)^{s-1} + \cdots + B_0(x) \) satisfies 
\[ v^a(B_0(x)) = v^a(A_0(x)). \]

It is immediate from the above theorem that if \( f(x) \) is as in this theorem, then a monic polynomial \( \phi(x) \) belonging to \( R_v[x] \) with \( f(x) \)-expansion 
\[ \sum_{i=0}^{n} A_i(x)f(x)^i \]
satisfying conditions (ii) and (iii) of Theorem 4.1, but not satisfying (i), must be reducible over \( K \).

The following corollaries will be deduced from Theorem 4.1.1. Corollary 4.1.2 extends Schönenmann Irreducibility Criterion [Rib2, §3.1, Theorem D] and Corollary 4.1.3 extends Akira’s criterion (cf. [Aki], [Pa-St]).

**Corollary 4.1.2.** Let \( v \) be a discrete valuation of \( K \) with value group \( \mathbb{Z} \) and \( \pi \) be an element of \( K \) with \( v(\pi) = 1 \). Let \( f(x), m \) be as in Theorem 4.1.1. Let \( F(x) \) belonging to \( R_v[x] \) be a monic polynomial having \( f(x) \)-expansion 
\[ \sum_{i=0}^{n} A_i(x)f(x)^i. \]
Assume that there exists \( s \leq n \) such that \( \pi \) does not divide the content of \( A_s(x) \), \( \pi \) divides the content of each \( A_i(x), 0 \leq i \leq s - 1 \) and \( \pi^2 \) does not divide the content of \( A_0(x) \). Then \( F(x) \) has an irreducible factor of degree \( sm \) over the completion \((\hat{K}, \hat{v})\) of \((K, v)\) which is a Schönenmann polynomial with respect to \( \hat{v} \) and \( f(x) \).

**Corollary 4.1.3.** Let \((K, v), \pi \) be as above and \( F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) be a polynomial over \( R_v \) satisfying the following conditions for an index \( s \leq n - 1 \).

(i) \( \pi \mid a_i \) for \( 0 \leq i \leq s - 1 \), \( \pi^2 \not\mid a_0, \pi \not\mid a_s \).

(ii) The polynomial \( x^{n-s} + \tilde{a}_{n-1}x^{n-s-1} + \cdots + \tilde{a}_s \) is irreducible over the residue field of \( v \).

(iii) \( \tilde{d} \neq \tilde{a}_s \) for any divisor \( d \) of \( a_0 \) in \( R_v \).

Then \( F(x) \) is irreducible over \( K \).
Recall that the resultant $R(G, H)$ of two polynomials $G(x)$ and $H(x)$ with $G(x)$
monic, equals $\prod_{j=1}^{q} H(y_j)$ where $q$ is the degree of $G(x)$ and $y_1, y_2, \ldots, y_q$ are the roots
of $G(x)$.

As usual, for non-negative elements $\lambda, \mu$ in a totally ordered abelian group $\Gamma$, we
write $\lambda \gg 0$ if $\lambda > 0$ and $\lambda \gg \mu > 0$ if $\lambda > \mu$ and $\lambda - \mu$ does not belong to the
largest convex subgroup of $\Gamma$ not containing $\mu$.

The following already known result proved in [Ribl] which is equivalent to
Hensel's Lemma will be used in the sequel.

Rychlik's Lemma. Let $\nu$ be a henselian Krull valuation of a field $K$. Let $F(x), G(x), H(x)$
belonging to $R_{\nu}[x]$ be non-zero polynomials such that

(i) $\deg G(x) > 0$, $\deg F(x) = \deg G(x) + \deg H(x)$, $G(x)$ is monic, $\nu^0(F(x)) = 0$
and $F(x)$ and $H(x)$ have the same leading coefficient.
(ii) $\nu(R(G, H)) = \rho < \infty$.
(iii) $\nu^\rho(F(x) - G(x)H(x)) \gg 2\rho$.

Then there exist polynomials $g(x)$ and $h(x)$ belonging to $R_{\nu}[x]$ such that

a) $\nu^\rho(G(x) - g(x)) > \rho$, $\nu^\rho(H(x) - h(x)) > \rho$.
b) $\deg g(x) = \deg G(x)$, $\deg h(x) = \deg H(x)$, $g(x)$ is monic.
c) $F(x) = g(x)h(x)$.

In what follows, $\nu$ is a henselian Krull valuation of a field $K$ with value group $\Gamma$, valuation ring $R_{\nu}$ and $\tilde{\nu}$ is the unique prolongation of $\nu$ to the algebraic closure $\tilde{K}$
of $K$ with value group $\tilde{\Gamma}$. For an element $\xi$ in the valuation ring of $\tilde{\nu}$, $\tilde{\xi}$ will denote
its $\tilde{\nu}$-residue. By the degree of an element $\alpha \in \tilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ which will be denoted by $\deg \alpha$. A pair $(\alpha, \delta)$ belonging to $\tilde{K} \times \tilde{\Gamma}$
is said to be a minimal pair (more precisely $(K, \nu)$-minimal pair) if whenever $\beta$ belongs to $\tilde{K}$ with $\deg \beta < \deg \alpha$, then $\tilde{\nu}(\alpha - \beta) < \delta$. For example, if $f(x)$ is a monic
polynomial with coefficients in $R_{\nu}$ such that $\tilde{f}(x)$ is irreducible over the residue field
of $v$ and $\alpha$ is a root of $f(x)$, then $(\alpha, \delta)$ is a $(K, v)$-minimal pair for each positive $\delta$ in $\tilde{\Gamma}$, because whenever $\beta$ belonging to $\tilde{K}$ has degree less than $m = \deg f(x)$, then $\tilde{v}(\alpha - \beta) \leq 0$, for otherwise $\tilde{\alpha} = \tilde{\beta}$, which in view of the fundamental inequality [En-Pr, Theorem 3.3.4] would lead to $[K(\beta) : K] \geq [\tilde{K}(\tilde{\beta}) : \tilde{K}] = m$.

Let $(K, v)$, $(\tilde{K}, \tilde{v})$ be as above and $(\alpha, \delta)$ belonging to $\tilde{K} \times \tilde{\Gamma}$ be a $(K, v)$-minimal pair. The valuation $\tilde{w}_{\alpha, \delta}$ of $\tilde{K}(x)$ defined on $\tilde{K}[x]$ by

$$\tilde{w}_{\alpha, \delta}(\sum c_i(x - \alpha)^i) = \min_i \{\tilde{v}(c_i) + i\delta}\}, \ c_i \in \tilde{K}$$

will be referred to as the valuation with respect to the minimal pair $(\alpha, \delta)$. The valuation of $K(x)$ obtained by restricting $\tilde{w}_{\alpha, \delta}$ will be denoted by $w_{\alpha, \delta}$.

The description of $w_{\alpha, \delta}$ is given by the already known theorem stated below (cf. [A-P-Z, Theorem 2.1], [Khal, Theorem 1.4]).

**Theorem 4.2.A.** Let $(\alpha, \delta)$ be a $(K, v)$-minimal pair. If $f(x)$ is the minimal polynomial of $\alpha$ over $K$, then for any polynomial $F(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum_i A_i(x)f(x)^i$, we have

$$w_{\alpha, \delta}(F(x)) = \min_i \{\tilde{v}(A_i(\alpha)) + iw_{\alpha, \delta}(f(x))\}.$$

Let $(\alpha, \delta)$ and $w_{\alpha, \delta}$ be as in Theorem 4.2.A. For any non-zero polynomial $F(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum_i A_i(x)f(x)^i$, we shall denote by $I_{\alpha, \delta}(F(x))$, $S_{\alpha, \delta}(F(x))$ respectively the minimum and the maximum integers belonging to the set

$$\{i \mid w_{\alpha, \delta}(F(x)) = \tilde{v}(A_i(\alpha)) + iw_{\alpha, \delta}(f(x))\}.$$

The following already known result will be used in the proof of the theorem (cf. [Kh-Ku, Lemma 2.1]). Its proof is omitted.
Theorem 4.2.B. Let \((\alpha, \delta)\) be a \((K, v)\)-minimal pair. For any non-zero polynomials \(F(x), G(x)\) in \(K[x]\), one has

(a) \(I_{\alpha, \delta}(F(x)G(x)) = I_{\alpha, \delta}(F(x)) + I_{\alpha, \delta}(G(x))\),

(b) \(S_{\alpha, \delta}(F(x)G(x)) = S_{\alpha, \delta}(F(x)) + S_{\alpha, \delta}(G(x))\).

We now prove a lemma to be used in the sequel.

Lemma 4.2.1. Let \(\alpha\) be a root of a monic polynomial \(f(x)\) belonging to \(R_v[x]\) such that \(\bar{f}(x)\) is irreducible over the residue field of \(v\). Let \((\alpha, \delta)\) be a \((K, v)\)-minimal pair with \(\delta > 0\). Then for any polynomial \(\psi(x)\) belonging to \(R_v[x]\) with \(f(x)\)-expansion \(\sum_i D_i(x)f(x)\), one has

\[ w_{\alpha, \delta}(\psi(x)) = \min_i \{\psi(D_i(x)) + iw_{\alpha, \delta}(f(x))\} \geq 0. \]

Proof. We first show that for any polynomial \(A(x) = \sum a_i x^i\) belonging to \(K[x]\) of degree less than \(\deg \alpha\), we have

\[ \tilde{\nu}(A(\alpha)) = \nu^\delta(A(x)). \quad (4.2) \]

Clearly (4.2) needs to be verified when \(m = \deg \alpha > 1\). Now \(\bar{\alpha}\) being a root of the irreducible polynomial \(\bar{f}(x)\) is non-zero and so \(\nu(\bar{\alpha}) = 0\). If (4.2) does not hold, then the triangle inequality would imply \(\tilde{\nu}(A(\alpha)) > \min\{\tilde{\nu}(a_i\alpha^i)\} = \nu(a_i)\) (say), which gives

\[ \sum_{i=0}^{m-1} \left(\frac{a_i}{a_j}\right) \alpha^i = 0 \]

contradicting the fact that the minimal polynomial of \(\alpha\) over the residue field of \(v\) has degree \(m\).

Denote \(w_{\alpha, \delta}(f(x))\) by \(\lambda\). Write \(f(x) = \sum_{i=1}^m c_i(x - \alpha)^i\), \(c_i \in R_v[\alpha]\). Then \(\bar{\nu}(c_i) \geq 0\).

Using (4.1) and the fact that \(\delta > 0\), we get

\[ \lambda = w_{\alpha, \delta}(f(x)) = \min_{1 \leq i \leq m} \{\tilde{\nu}(c_i) + i\delta\} \geq 0. \quad (4.3) \]

Keeping in mind that \(\psi(x)\) and hence each \(D_i(x)\) belongs to \(R_v[x]\), it follows immediately from Theorem 4.2.A, (4.2) and (4.3) that

\[ w_{\alpha, \delta}(\psi(x)) = \min_i \{\tilde{\nu}(D_i(\alpha)) + i\lambda\} = \min_i \{\nu^\delta(D_i(x)) + i\lambda\} \geq 0. \]
4.3 Proof of Theorem 4.1.1

Denote \( \nu_\alpha(\phi(x)) \) by \( \lambda \). Fix a root \( \alpha \) of \( f(x) \). Write \( f(x) = \sum_{i=1}^{m} c_i(x - \alpha)^i, \ c_i \in \bar{K} \).

Determine \( \delta \) in \( \bar{\Gamma} \) so that
\[
\lambda = \min_{i \geq 1} \{ i \delta \}, \ \text{i.e.,} \ \delta = \max_{i \geq 1} \left\{ \frac{\lambda - \delta(c_i)}{i} \right\}.
\]

Note that \( \delta > 0 \) in view of the fact that \( c_m = 1 \) and \( \lambda > 0 \) by hypothesis. So \( (\alpha, \delta) \) is a \((K, \nu)\)-minimal pair and \( w_{\alpha, \delta}(f(x)) = \lambda \) by virtue of (4.1) and the choice of \( \lambda \). Therefore keeping in mind assumptions (i) and (ii) of the theorem, it follows from Lemma 4.2.1 that
\[
w_{\alpha, \delta}(\phi(x)) = \min_{i} \{ \nu^\ast(A_i(x)) + i \lambda \} = s \lambda = \nu^\ast(A_0(x)). \quad (4.4)
\]

It is immediate from (4.2) and (4.4) that
\[
I_{\alpha, \delta}(\phi(x)) = 0 \ \text{and} \ S_{\alpha, \delta}(\phi(x)) = s. \quad (4.5)
\]

Write \( \phi(x) \) as a product \( \phi_1(x)\phi_2(x) \cdots \phi_r(x) \) of monic, irreducible polynomials over \( R_\nu \). We split the proof into two main steps.

**Step I.** In this step, it will be shown that there exists \( j, \ 1 \leq j \leq r \), such that
\[
w_{\alpha, \delta}(\phi_j(x)) = w_{\alpha, \delta}(\phi(x)) \ \text{and} \ w_{\alpha, \delta}(\phi_i(x)) = 0 \ \text{for each} \ i \neq j.
\]

Applying Theorem 4.2.B, we have
\[
S_{\alpha, \delta}(\phi(x)) = \sum_{i=1}^{r} S_{\alpha, \delta}(\phi_i(x)), \ I_{\alpha, \delta}(\phi(x)) = \sum_{i=1}^{r} I_{\alpha, \delta}(\phi_i(x)). \quad (4.6)
\]

So it follows from (4.5) and (4.6) that there exists \( j \) such that
\[
S_{\alpha, \delta}(\phi_j(x)) > 0, \ I_{\alpha, \delta}(\phi_i(x)) = 0 \ \text{for} \ 1 \leq i \leq r, \quad (4.7)
\]

which in view of Lemma 4.2.1 implies that
\[
0 \leq w_{\alpha, \delta}(\phi_i(x)) \in \Gamma, \ 1 \leq i \leq r. \quad (4.8)
\]
We now show that
\[ S_{\alpha,\delta}(\phi_j(x)) = s. \] (4.9)
Let \[ \sum_{i=0}^t B_i(x)f(x)^i \] be the \( f(x) \)-expansion of \( \phi_j(x) \). Denote \( S_{\alpha,\delta}(\phi_j(x)) \) by \( s_1 \). In view of (4.5) and (4.6), \( s_1 \leq s \). Using the equality \( \tilde{v}(B_i(\alpha)) = v^r(B_i(x)) \) proved in (4.2), we have
\[ w_{\alpha,\delta}(\phi_j(x)) = v^r(B_{s_1}(x)) + s_1 \lambda. \] (4.10)
As shown in (4.8), \( w_{\alpha,\delta}(\phi_j(x)) \) belongs to \( \Gamma \) and hence by (4.10), \( s_1 \lambda \in \Gamma \). The desired equality \( s_1 = s \) now follows on recalling that \( \lambda = v^r(\phi_j(x)) \) and that \( s \) is the smallest positive integer for which \( s\lambda \) belongs to \( \Gamma \) by assumption (iii) of the theorem.

Keeping in mind that \( \phi_j(x) \) and hence \( B_i(x) \) belongs to \( R_v[x] \), (4.10) together with (4.9) implies that
\[ w_{\alpha,\delta}(\phi_j(x)) = v^r(B_s(x)) + s_1 \lambda. \] (4.11)
Since \( w_{\alpha,\delta}(\phi_i(x)) \geq 0 \) for each \( i \) by (4.8) and \( w_{\alpha,\delta}(\phi(x)) = s \lambda \) by (4.4), it follows from (4.11) that
\[ w_{\alpha,\delta}(\phi_j(x)) = s \lambda, \quad v^r(B_s(x)) = 0 \] (4.12)
and hence the assertion of Step I is proved.

**Step II.** In this step, it will be shown that \( \phi_j(x) \) has degree \( sm \) and is a Generalized Schönenmann polynomial with respect to \( v \) and \( f(x) \) which will complete the proof of the theorem.

In view of (4.9), \( \deg \phi_j(x) \geq sm \). We shall prove that \( \deg \phi_j(x) = sm \). Suppose to the contrary that \( \deg \phi_j(x) > sm \). Define polynomials \( G(x) \) and \( H(x) \) by
\[ G(x) = f(x)^s, \quad H(x) = B_t(x)f(x)^{t-s} + B_{t-1}(x)f(x)^{t-s-1} + \cdots + B_s(x). \] (4.13)
It will be shown that the polynomials \( \phi_j(x), G(x) \) and \( H(x) \) satisfy the conditions of Rychlik's Lemma. These polynomials clearly satisfy condition (i) of this lemma. We first show that
\[ v^r(\phi_j - GH) > 0. \] (4.14)
In view of (4.12) and Lemma 4.2.1, we have
\[ s\lambda = w_{\alpha, \delta}(\phi_j(x)) = \min_{0 \leq i \leq s-1} \{ v^\tau(B_i(x)) + i\lambda \}. \]
Consequently
\[ v^\tau(B_i(x)) + i\lambda \geq s\lambda, \quad 0 \leq i \leq s - 1. \] (4.15)
Since \( I_{\alpha, \delta}(\phi_j(x)) = 0 \) by (4.7), using (4.2), we see that
\[ w_{\alpha, \delta}(\phi_j(x)) = \hat{\nu}(B_0(\alpha)) = v^\tau(B_0(x)). \]
The above equation together with (4.12) implies that \( v^\tau(B_0(x)) = s\lambda \). Thus (4.15) can be rewritten as
\[ \frac{v^\tau(B_i(x))}{s-i} \geq \frac{v^\tau(B_0(x))}{s} = \frac{v^\tau(A_0(x))}{s} > 0 \quad \text{for} \quad 0 \leq i \leq s - 1 \] (4.16)
which implies that \( v^\tau(B_i(x)) > 0 \) for \( 0 \leq i \leq s - 1 \); consequently
\[ v^\tau(\phi_j - GH) = v^\tau(B_{s-1}(x)f(x)^{s-1} + \cdots + B_0(x)) \geq \min_{0 \leq i \leq s-1} \{ v^\tau(B_i(x)) \} > 0 \]
and hence (4.14) is proved. Keeping in mind (4.14), conditions (ii) and (iii) of Rychlik’s Lemma are verified once we show that
\[ v(R(G, H)) = 0. \] (4.17)
Let \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_m \) denote the roots of \( f(x) \). Keeping in mind (4.13) and the fact that \((K, \nu)\) is henselian, it is clear that
\[ v(R(G, H)) = s\nu\left( \prod_{i=1}^{m} H(\alpha_i) \right) = s\nu\left( \prod_{i=1}^{m} B_\delta(\alpha_i) \right) = ms\nu(B_\delta(\alpha)). \]
Using (4.2) and (4.12), it now follows from the above equation that \( v(R(G, H)) = msv^\tau(B_\delta(x)) = 0 \) which proves (4.17). Applying Rychlik’s Lemma, we see that \( \phi_j(x) \) has a factor \( g(x) \) over \( K \) of degree equal to that of \( G(x) \), i.e., \( sm \), which contradicts the irreducibility of \( \phi_j(x) \). This contradiction proves that the degree of \( \phi_j(x) \) is \( sm \). Moreover \( \phi_j(x) \) is a Generalized Schônenmann polynomial with respect to \( \nu \) and \( f(x) \) in view of (4.16) and hence the theorem.
4.4 Proof of Corollaries 4.1.2 and 4.1.3

Proof of Corollary 4.1.2. The hypothesis implies that \( v^s(A_i(x)) > 0 \) for \( 0 \leq i \leq s - 1 \), \( v^s(A_s(x)) = 0 \) and \( v^s(A_0(x)) = 1 \). Therefore by Theorem 4.1.1, \( F(x) \) has an irreducible factor \( g(x) \) of degree \( sm \) over \( \hat{K} \) which is a Generalized Schönhemann polynomial with respect to \( \hat{v} \) and \( f(x) \). The desired result now follows from the last assertion of the theorem and the fact that \( v^s(A_0(x)) = 1 \).

Proof of Corollary 4.1.3. Applying Corollary 4.1.2 with \( f(x) = x \), we see that \( F(x) \) has an irreducible factor \( g(x) \) of degree \( s \) over the completion \( (\hat{K}, \hat{v}) \) of \( (K, v) \), which is an Eisenstein polynomial with respect to \( \hat{v} \). Write \( F(x) = g(x)h(x) \), where \( g(x) = x^s + b_{s-1}x^{s-1} + \cdots + b_0 \), \( h(x) = x^n - x^{n-s} - 1 + \cdots + c_0 \). In view of the hypothesis, \( \tilde{F}(x) = x^s(\bar{a}_{n-s}x^{n-s} - 1 + \cdots + \bar{a}_s) \), so \( \tilde{h}(x) = x^n - \bar{a}_{n-s}x^{n-s} - 1 + \cdots + \bar{a}_s \), which is given to be irreducible over the residue field of \( v \). Hence \( h(x) \) is also irreducible over \( \hat{K} \). Note that \( \bar{e}_0 = \bar{a}_s \neq 0 \) by hypothesis. If \( F(x) \) were reducible over \( K \), then \( g(x) \) and \( h(x) \) being irreducible over \( \hat{K} \), would belong to \( K[x] \) and consequently the equality \( a_0 = b_0c_0 \) would contradict assumption (iii) of the corollary for the divisor \( c_0 \) belonging to \( R_v \) of \( a_0 \).

It may be pointed out that our method of proof can be easily carried over (using henselisation of \( (K, v) \) instead of its completion) to prove the analogue of Corollary 4.1.3 when \( v \) is a Krull valuation of \( K \) of arbitrary rank having value group \( \Gamma \) with \( v(\pi) \) as the smallest positive element of \( \Gamma \).