Chapter 3

An extension of a result of Zaharescu on irreducible polynomials

3.1 Origin of the problem and statements of results

A classical result concerning irreducible polynomials over a valued field $K$ which is complete with respect to a real valuation $v$ says that if $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ belonging to $K[x]$ is irreducible, then there exists a positive real number $\epsilon$ such that any monic polynomial $g(x) = x^d + b_1 x^{d-1} + \cdots + b_d$ belonging to $K[x]$ with $v(b_j - a_j) > \epsilon$, $1 \leq j \leq d$, is also irreducible over $K$ (see [Art, Chapter 2, Theorem 11]). Examples are known which show that the above result fails to hold when $K$ is not complete (cf. [Zah]). In 2004, Zaharescu [Zah] proved a similar result for valued fields which may not be complete but which are equipped with two valuations that satisfy a certain property which compensates for the lack of completeness of the given valued field. Precisely stated, he proved the following theorem.

**Theorem 3.1.A.** Let $K$ be a field of characteristic zero equipped with two non-archimedean valuations $v_1$ and $v_2$ having value groups $\Gamma_1$ and $\Gamma_2$ respectively. Let $A$ be a subring of $K$ with field of fractions $K$ which is integrally closed in $K$ and $\tilde{A}$ be the integral closure of $A$ in the algebraic closure $\tilde{K}$ of $K$. Let $\tilde{v}_1$ and $\tilde{v}_2$ be valuations
on $\tilde{K}$ whose restrictions to $K$ coincide with $v_1$ and $v_2$ respectively. Assume that for any $\beta \in \tilde{A} \setminus A$ and $\lambda_2 \in \Gamma_2$, there exists an element $\lambda_1 \in \Gamma_1$ such that

$$v_1(u - \beta) \leq \lambda_1 \quad \text{for all } u \in A \text{ with } v_2(u) \geq \lambda_2. \quad (3.1)$$

Let $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d \in A[x]$ be an irreducible polynomial over $K$. Then given $\lambda_2 \in \Gamma_2$, there exists $\lambda_1 \in \Gamma_1$ such that for any $b_1, b_2, \ldots, b_d \in A$ for which

$$v_1(b_i - a_i) \geq \lambda_1, \quad 1 \leq i \leq d, \quad (3.2)$$

and

$$v_2(b_i) \geq \lambda_2, \quad 1 \leq i \leq d, \quad (3.3)$$

the polynomial $g(x) = x^d + b_1 x^{d-1} + \cdots + b_d$ is irreducible over $K$.

In this chapter, the above result has been extended to irreducible polynomials with coefficients in arbitrary valued fields without any condition on the characteristic of $K$. Indeed we prove the following theorem.

**Theorem 3.1.1.** Let $K$ be a field equipped with two Krull valuations $v_1$ and $v_2$ of arbitrary rank. Let $A, \tilde{A}, \tilde{v}_1$ and $\tilde{v}_2$ be as in Theorem 3.1.A. Assume that for any $\beta \in \tilde{A} \setminus A$ and $\lambda_2 \in \Gamma_2$, there exists an element $\lambda_1 \in \Gamma_1$ such that (3.1) holds. Then for any polynomial $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d \in A[x]$ which is irreducible over $K$ and any $\lambda_2 \in \Gamma_2$, there corresponds $\lambda_1 \in \Gamma_1$ depending upon $f$ and $\lambda_2$ such that for any $b_1, b_2, \ldots, b_d \in A$ satisfying (3.2) and (3.3), the polynomial $g(x) = x^d + b_1 x^{d-1} + \cdots + b_d$ is irreducible over $K$.

As an application of Theorem 3.1.1, we shall deduce the result stated below.

**Theorem 3.1.2.** Let $K_0$ be a field complete with respect to a real valuation $v_0$. Let $f(x, y) = x^d + P_1(y)x^{d-1} + \cdots + P_d(y)$ be an irreducible polynomial in two variables over $K_0$. Let $v_0^g$ denote the Gaussian extension of $v_0$ to $K_0(y)$ defined by $v_0^g(\sum_i a_i y^i) = \min\{v_0(a_i) \mid a_i \in K_0\}$. Then given any integer $M$, there exists $N > 0$ (depending upon $f$ and $M$) such that whenever $Q_i(y), 1 \leq i \leq d$, are polynomials over $K_0$ satisfying (i) degree $Q_i(y) \leq M$, (ii) $v_0^g(Q_i(y) - P_i(y)) > N$, then $g(x, y) = \cdots$
\[ x^d + Q_1(y)x^{d-1} + \cdots + Q_d(y) \text{ is irreducible over } K_0. \]

An example has been given in Section 3.3 to show that the above theorem is not true in general if the polynomials \( Q_i(y) \) fail to satisfy condition (i) stated above even if each \( Q_i(y) \) is sufficiently close to \( P_i(y) \) with respect to \( v_q \).

### 3.2 Preliminary results

We need the following theorem of Ershov proved in [Ers]. For ready reference, it is proved here.

**Theorem 3.2.B.** Let \( v \) be a valuation of arbitrary rank of an algebraically closed field \( K \) with value group \( \Gamma \) and \( v^x \) be the Gaussian extension of \( v \) to \( K(x) \) defined by (1.1). Let \( \epsilon > 0 \) be an element of \( \Gamma \). Let \( f(x), g(x) \) belonging to \( K[x] \) be monic polynomials of degree \( d \) such that \( v^x(f - g) > d\epsilon - (d+1)\epsilon v^x(f) \). If \( f(x) = \prod_{i=1}^{d} (x - \alpha_i) \) is a factorization of \( f(x) \) over \( K \), then we have a factorization \( \prod_{i=1}^{d} (x - \beta_i) \) of \( g(x) \) such that \( v(\alpha_i - \beta_i) > \epsilon \) for \( 1 \leq i \leq d \).

We first prove the following lemma needed for the proof of Theorem 3.2.B.

**Lemma 3.2.C.** Let \( K, v \) and \( \Gamma \) be as in Theorem 3.2.B and \( f(x), g(x) \in K[x] \) be monic polynomials of degree \( d \) such that \( v^x(f - g) > d\epsilon - 2d\epsilon v^x(f) \) for some positive \( \epsilon \) in \( \Gamma \). Then for each root \( \alpha \) of \( f(x) \), there corresponds a root \( \beta \) of \( g(x) \) such that \( v(\alpha - \beta) > \epsilon - v^x(f) \), \( v^x(f_0 - g_0) > \epsilon \), where \( f_0 = f/(x - \alpha) \), \( g_0 = g/(x - \beta) \).

**Proof.** Write \( f(x) = x^d + a_1x^{d-1} + \cdots + a_d \), \( g(x) = x^d + b_1x^{d-1} + \cdots + b_d \) and denote \( v^x(f) \) by \( -v(c) \), \( c \in K \). As \( f(x) \) is monic, \( v(c) \geq 0 \). Observe that for any root \( \alpha \) of \( f(x) \), \( \alpha c \) is a root of the polynomial \( x^d + a_1x^{d-1} + \cdots + a_d \) having coefficients in the valuation ring of \( v \) and hence \( v(\alpha c) \geq 0 \), which shows that

\[ v(\alpha) \geq -v(c) = v^x(f). \]  

(3.4)
Let $\alpha$ be a root of $f(x)$. Then it follows from the triangle law that

$$v(g(\alpha)) = v(g(\alpha) - f(\alpha)) \geq \min_{1 \leq i \leq d} \{v(b_i - a_i) + (d - i)v(\alpha)\}. \quad (3.5)$$

In view of the hypothesis $v^x(f - g) > d\epsilon - 2dv^x(f)$ and (3.4), we see that

$$v(b_i - a_i) + (d - i)v(\alpha) > d\epsilon - 2dv^x(f) + (d - i)v^x(f) \geq d\epsilon - dv^x(f).$$

It is clear from the above inequality and (3.5) that $v(g(\alpha)) > d\epsilon - dv^x(f)$, which immediately shows that for at least one root $\beta$ of $g(x)$

$$v(\alpha - \beta) > \epsilon - v^x(f). \quad (3.6)$$

To estimate $v^x(f_0 - g_0)$, write $f_0 - g_0 = \frac{f}{x - \alpha} - \frac{f}{x - \beta} + \frac{f}{x - \beta} - \frac{g}{x - \beta}$. Clearly

$$v^x\left(\frac{f}{x - \alpha} - \frac{f}{x - \beta}\right) = v^x(f) + v(\alpha - \beta) - v^x((x - \alpha)(x - \beta)) \geq v^x(f) + v(\alpha - \beta),$$

which in view of (3.6) gives

$$v^x\left(\frac{f}{x - \alpha} - \frac{f}{x - \beta}\right) > \epsilon. \quad (3.7)$$

Further by virtue of the hypothesis, we have

$$v^x\left(\frac{f}{x - \beta} - \frac{g}{x - \beta}\right) = v^x(f - g) - v^x(x - \beta) \geq v^x(f - g) > d\epsilon - 2dv^x(f) \geq \epsilon. \quad (3.8)$$

It is immediate from (3.7) and (3.8) that $v^x(f_0 - g_0) > \epsilon$ as desired.

**Proof of Theorem 3.2.B.** The theorem will be proved by induction on the degree $d$ of $f(x)$. When $d = 1$, $f = x - \alpha$, $g = x - \beta$ and we see that

$$v(\beta_1 - \alpha_1) = v^x(f - g) > \epsilon - 2v^x(f) \geq \epsilon.$$ 

Consider now the case when $d = 2$. Then by the hypothesis,

$$v^x(f - g) > 2\epsilon - 3v^x(f) = 2(\epsilon - v^x(f)) - 4v^x(f).$$

Applying Lemma 3.2.C (with $\epsilon$ replaced by $\epsilon - v^x(f)$), we see that there exists a root $\beta_1$ of $g(x)$ satisfying
\[ v(\alpha_1 - \beta_1) > (\epsilon - v^e(f)) - v^e(f) = \epsilon - 2v^e(f) \geq \epsilon \]

and
\[ v^e(f_0 - g_0) > \epsilon - v^e(f) \geq \epsilon \]

where \( f_0 = f/(x - \alpha_1), \) \( g_0 = g/(x - \beta_1) \). On writing \( f_0 = x - \alpha_2, \) \( g_0 = x - \beta_2 \), the last inequality shows that \( v(\alpha_2 - \beta_2) = v^e(f_0 - g_0) > \epsilon \) proving the theorem in the case \( d = 2 \).

Assume now that \( f(x), g(x) \) have degree \( d \geq 3 \). Then \( dl \geq d(d - 1) \geq 2d \). In view of the hypothesis,
\[ v^e(f - g) > dl\epsilon - (d + 1)!v^e(f) = d[(d - 1)!\epsilon - dlv^e(f)] - dlv^e(f) \geq d[(d - 1)!\epsilon - dlv^e(f)] - 2dv^e(f). \]

By Lemma 3.2.C (applied with \( \epsilon \) replaced by \( (d - 1)!\epsilon - dlv^e(f) \)) given a root \( \alpha_1 \) of \( f(x) \), there exists a root \( \beta_1 \) of \( g(x) \) such that
\[ v(\alpha_1 - \beta_1) > (d - 1)!\epsilon - dlv^e(f) - v^e(f) > \epsilon \]

and
\[ v^e(f_0 - g_0) > (d - 1)!\epsilon - dlv^e(f) \geq (d - 1)!\epsilon - dlv^e(f_0), \]

where \( f_0 = f/(x - \alpha_1), \) \( g_0 = g/(x - \beta_1) \). The theorem now follows by the induction hypothesis applied to \( f_0, g_0 \).

**Notations.** Let \( (K, v) \) be as in Theorem 3.2.B. The constant \( dl\epsilon - (d + 1)!v^e(f) \) occurring in this theorem will be denoted by \( \epsilon_f \) and will be referred to as Ershov’s constant with respect to \( v \), associated to a polynomial \( f(x) \in K[x] \), corresponding to \( \epsilon \) belonging to the value group of \( v \). For any polynomial \( f(x) \in K[x] \), we shall denote by \( \omega_f \) the constant defined by
\[ \omega_f = \max\{v(\alpha), v(\alpha - \alpha') \mid \alpha \neq \alpha', \alpha, \alpha' \text{ run over the roots of } f(x)\}, \]

which will be referred to as the Generalized Krasner’s constant associated to \( f \) with respect to \( v \). Note that in case \( f(x) \) has a single root \( \alpha \), then \( \omega_f = v(\alpha) \).
3.3 Proof of Theorems 3.1.1 and 3.1.2

Let \( p \geq 1 \) denote the multiplicity of each root of \( f(x) \), \( p \) being the characteristic of \( K \) or 1. Let \( \alpha_1, \alpha_2, \ldots, \alpha_d \) be the roots of \( f(x) \) in \( \bar{K} \) not necessarily distinct. For any non-empty proper subset \( S \) of \( \{1, 2, \ldots, d\} \) having \( r \) elements, we shall denote the elementary symmetric sums by

\[
\sigma_{S,1} = \sum_{j \in S} \alpha_j, \quad \sigma_{S,2} = \sum_{i \in S, i < j} \alpha_i \alpha_j, \ldots, \quad \sigma_{S,r} = \prod_{j \in S} \alpha_j.
\]

Also \( f_S(x) \) will stand for the polynomial \( \prod_{j \in S} (x - \alpha_j) \), i.e.,

\[
f_S(x) = x^r - \sigma_{S,1} x^{r-1} + \cdots + (-1)^r \sigma_{S,r}.
\]

Since \( f(x) \) is irreducible over \( K \) and \( A \) is integrally closed, at least one of the coefficients of \( f_S(x) \) belongs to \( \bar{A} \setminus A \). For any such \( S \), we shall denote by \( j_S \) the smallest positive integer for which \( \sigma_{S,j_S} \) belongs to \( \bar{A} \setminus A \). Set

\[
\delta = \min\{v_1(\alpha_i) \mid 1 \leq i \leq d\} \quad \text{and} \quad \Delta = \max\{v_1(\alpha_i) \mid 1 \leq i \leq d\}.
\]

Define

\[
\delta' = \min\{0, (d-1)\delta\} \quad \text{and} \quad \mu_f = \frac{\delta'}{p^d} + \left( \frac{p^d - d}{p^d} \right) \omega_f,
\]

where \( \omega_f \) is the Generalized Krasner’s constant associated to \( f \) with respect to \( v_1 \). We shall denote by \( \epsilon_f \) the Ershov’s constant associated to \( f(x) \) corresponding to a fixed positive element \( \epsilon \geq \omega_f \) belonging to the value group \( \bar{\Gamma}_1 \) of \( v_1 \). We divide the proof into two steps.

**Step I.** In this step, we show that with \( \lambda_1 > \epsilon_f \), if \( g(x) = x^d + b_1 x^{d-1} + \cdots + b_d \) belonging to \( A[x] \) is any polynomial satisfying \( (3.2) \), then there exists a factorization \( \prod_{i=1}^{d} (x - \theta_i) \) of \( g(x) \) over \( \bar{K} \) such that

\[
\tilde{v}_1(\theta_i - \alpha_i) \geq \frac{\lambda_1}{p^d} + \mu_f \quad \text{for} \quad 1 \leq i \leq d.
\]  

(3.9)
Since $\lambda_1 > \epsilon_f$, by Theorem 3.2.B the roots $\theta_1, \theta_2, \ldots, \theta_d$ of $g(x)$ can be arranged so that
\[ \hat{v}_1(\theta_i - \alpha_i) > \omega_f \text{ for } 1 \leq i \leq d. \] (3.10)
We are going to prove that (3.10) implies (3.9). Fix one $i$ and denote $\theta_i, \alpha_i$ by $\theta, \alpha$ respectively. Since $\omega_f \geq \hat{v}_1(\alpha)$, it is immediate from (3.10) and the strong triangle law that $\hat{v}_1(\theta) = \hat{v}_1(\alpha) \geq \delta$. To prove (3.9), consider first the case when $f(x)$ has at least two distinct roots. For any root $\alpha' \neq \alpha$ of $f(x)$, in view of (3.10) and the strong triangle law, it follows that
\[ \hat{v}_1(\theta - \alpha') = \min \{ \hat{v}_1(\theta - \alpha), \hat{v}_1(\alpha - \alpha') \} = \hat{v}_1(\alpha - \alpha'). \] (3.11)
Keeping in mind that $v_1(a_j - b_j) \geq \lambda_1$ and $\hat{v}_1(\theta) = \hat{v}_1(\alpha) \geq \delta$, we obtain
\[
\hat{v}_1(f(\theta)) = \hat{v}_1(f(\theta) - g(\theta)) \geq \min_{1 \leq j \leq d} \{ \hat{v}_1(a_j - b_j) + (d - j)\hat{v}_1(\theta) \} \geq \lambda_1 + \min \{ (d - j)\delta \} = \lambda_1 + \delta'.
\]
Therefore on substituting $f(\theta) = \prod_{\alpha' \neq \alpha} (\theta - \alpha')^{p'}$, where $\alpha'$ runs over distinct roots of $f(x)$, the above inequality shows that
\[
\hat{v}_1(f(\theta)) = p'\hat{v}_1(\theta - \alpha) + p'\sum_{\alpha' \neq \alpha} \hat{v}_1(\theta - \alpha') \geq \lambda_1 + \delta'.
\]
Using (3.11), the above inequality can be rewritten as
\[
\hat{v}_1(\theta - \alpha) + \sum_{\alpha' \neq \alpha} \hat{v}_1(\alpha - \alpha') \geq (\lambda_1 + \delta')/p',
\]
which in view of $\hat{v}_1(\alpha - \alpha') \leq \omega_f$ implies that
\[
\hat{v}_1(\theta - \alpha) \geq \frac{\lambda_1 + \delta'}{p'} + \left( \frac{p' - d}{p'} \right) \omega_f = \frac{\lambda_1}{p'} + \mu_f.
\]
This proves (3.9) when $f(x)$ has at least two distinct roots.

Consider now the case when $f(x)$ has a single root $\alpha$ repeated $p$ times. In this situation $\omega_f = \hat{v}_1(\alpha)$. For any root $\theta$ of $g(x)$, arguing as in the first case, we see that
\[
\hat{v}_1(f(\theta) - g(\theta)) \geq \min \{ \hat{v}_1(a_j - b_j) + (d - j)\hat{v}_1(\theta) \} \geq \lambda_1 + \delta'.
\]
As $f(\theta) = (\theta - \alpha)^p$, the above inequality gives
\[
\hat{v}_1(\theta - \alpha) \geq \frac{\lambda_1 + \delta'}{p'} = \frac{\lambda_1}{p'} + \mu_f
\]
as desired. This completes the proof of Step I.
Step II. In this step, we prove the irreducibility of \( g(x) \) giving the final choice of \( \lambda_1 \).

Fix an element \( \lambda_2 \) in \( \Gamma_2 \). Define \( \lambda_2' = \min\{\lambda_2/d, \lambda_2\} \). Let \( g(x) = x^d + b_1 x^{d-1} + \cdots + b_d \) belonging to \( A[x] \) be any monic polynomial satisfying (3.2) and (3.3) with \( \lambda_1 > \varepsilon \).

We first show that for each root \( \theta \) of \( g(x) \),

\[
\tilde{v}_2(\theta) \geq \lambda_2'.
\] (3.12)

On writing \( \theta^d = -(b_1 \theta^{d-1} + \cdots + b_d) \) and using the triangle law, we have

\[
d\tilde{v}_2(\theta) \geq \min_{1 \leq j \leq d} \{v_2(b_j) + (d-j)\tilde{v}_2(\theta)\} \geq \lambda_2 + \min_{1 \leq j \leq d} \{(d-j)\tilde{v}_2(\theta)\} = \lambda_2 + (d-i)\tilde{v}_2(\theta) \text{ (say)}
\]

which implies \( \tilde{v}_2(\theta) \geq \lambda_2/i \geq \lambda_2' \) proving (3.12).

Recall that for any non-empty proper subset \( S \) of \( \{1, 2, \ldots, d\} \), the coefficient \( \sigma_{S,j}\bar{s} \) of \( f_S(x) = \prod_{j \in S} (x - \alpha_j) \) belongs to \( \bar{A} \setminus A \). On applying (3.1) with \( \beta \) replaced by \( \sigma_{S,j}\bar{s} \) and \( \lambda_2 \) by \( j_S \lambda_2' \), there exists an element \( \lambda_{1,S} \) belonging to \( \Gamma_1 \) such that

\[
\tilde{v}_1(u - \sigma_{S,j}\bar{s}) \leq \lambda_{1,S}
\] (3.13)

for all \( u \in A \) with \( v_2(u) \geq j_S \lambda_2' \).

Suppose that \( g(x) \) is reducible over \( K \). It will be shown that this will give an upper bound (depending upon \( \lambda_2 \) and \( f(x) \)) on \( \lambda_1 \). Write \( g(x) = G(x)H(x) \), with \( G(x), H(x) \) monic, non-constant polynomials belonging to \( K[x] \cap \bar{A}[x] = A[x] \). Denote \( G(x) \) by \( x^m + c_1 x^{m-1} + \cdots + c_m \). It is immediate from (3.12) that

\[
v_2(c_i) \geq i \lambda_2', \quad 1 \leq i \leq m.
\] (3.14)

Recall that by Step 1, \( \theta_1, \theta_2, \ldots, \theta_d \) is an arrangement of roots of \( g(x) \) satisfying (3.10). Write \( G(x) = \prod_{j \in S} (x - \theta_j) \) where \( S \) is a proper subset of \( \{1, 2, \ldots, d\} \). Consider

\[
f_S(x) = \prod_{j \in S} (x - \alpha_j) \text{. One of the coefficients of } f_S(x) \text{ is } \]

\[
(-1)^{j_S} \sigma_{S,j}\bar{s} = (-1)^{j_S} \sum_{i_1, i_2, \ldots, i_{j_S} \in S} c_{i_1} c_{i_2} \cdots c_{i_{j_S}},
\]
and the corresponding coefficient in $G(x)$, say $u_0$ is given by:

$$(-1)^{js} u_0 = (-1)^{js} \sum_{i_1, i_2, \ldots, i_{js} \in S} \prod_{i_1 < i_2 < \cdots < i_{js}} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_{js}}.$$  

We are going to prove that

$$v_i(u_0 - \sigma_{S, js}) > \frac{\lambda_1}{p^f} + \mu_f - \Delta - \rho_S,$$

where $\Delta = \max_{1 \leq i \leq d} \{v_i(\alpha_i)\}$ and $\rho_S$ is an element of $\tilde{\Gamma}_1$ depending upon $f$ and $S$, to be specified later. For any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, d\}$, we can write

$$\theta_{i_1} \theta_{i_2} \cdots \theta_{i_k} - \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} = \theta_{i_1} \cdots \theta_{i_{k-1}} (\theta_{i_k} - \alpha_{i_k}) + \theta_{i_1} \cdots \theta_{i_{k-2}} (\theta_{i_{k-1}} - \alpha_{i_{k-1}}) \alpha_{i_k}$$

$$+ \cdots + (\theta_{i_1} - \alpha_{i_1}) \alpha_{i_2} \cdots \alpha_{i_k}$$ (3.15)

where $\theta'_{i_k}$ and $\alpha'_{i_k}$ may not all be distinct. Recall that $v_i(\theta_i) = \tilde{v}_i(\alpha_i)$. Using (3.9), it follows that

$$\tilde{v}_1(\theta_{i_1} \cdots \theta_{i_{k-1}} (\theta_{i_k} - \alpha_{i_k})) = \tilde{v}_1(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}) + \tilde{v}_1(\theta_{i_k} - \alpha_{i_k})$$

$$\geq \tilde{v}_1(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}) - \Delta + \frac{\lambda_1}{p^f} + \mu_f.$$

Arguing similarly for other summands on the right hand side of (3.15), we see that

$$\tilde{v}_1(\theta_{i_1} \theta_{i_2} \cdots \theta_{i_k} - \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}) \geq \tilde{v}_1(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}) - \Delta + \frac{\lambda_1}{p^f} + \mu_f.$$

It immediately follows from the above inequality and the triangle law that

$$\tilde{v}_1(u_0 - \sigma_{S, js}) \geq \frac{\lambda_1}{p^f} + \mu_f - \Delta + \min_{i_1, i_2, \ldots, i_{js} \in S} \{\tilde{v}_1(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{js}})\},$$

i.e.,

$$\tilde{v}_1(u_0 - \sigma_{S, js}) \geq \frac{\lambda_1}{p^f} + \mu_f - \Delta + \rho_S,$$ (3.16)

where $\rho_S = \min_{i_1, \ldots, i_{js} \in S} \{\tilde{v}_1(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{js}})\}$ is in $\tilde{\Gamma}_1$. Recall that by virtue of (3.14), the coefficient $u_0$ of $x^{m-j}$ in $G(x)$ satisfies $v_2(u_0) \geq j_2 \lambda'_2$. Therefore it follows from
(3.13) that

\[ v_1(u_n - \sigma_{S,jb}) \leq \lambda_{1,S}. \]  

(3.17)

The inequalities (3.16) and (3.17) imply that \( \lambda_1 \leq p'(\lambda_{1,S} + \Delta - \mu_f - \rho_S) \).

Thus it follows that if we start with an element \( \lambda_1 \) greater than \( \epsilon_f \) and \( \lambda_1 > p'(\Delta - \mu_f) + p'\max_S\{\lambda_{1,S} - \rho_S\} \) where \( S \) runs over all non-empty proper subsets of \{1, 2, \ldots, d\}, then each polynomial \( g(x) \) satisfying (3.2) and (3.3) must be irreducible over \( K \).

Proof of Theorem 3.1.2. Denote \( K_0(y) \) by \( K \) and the Gaussian valuation \( v^g_0 \) of \( K \) by \( v_1 \). Let \( v_2 \) stand for the valuation of \( K \) defined for any polynomial \( h(y) \) by \( v_2(h(y)) = -\deg h(y) \); here and elsewhere \( \deg \) stands for the degree. Let \( \tilde{v}_1 \) and \( \tilde{v}_2 \) be any prolongations of \( v_1, v_2 \) to the algebraic closure \( \tilde{K} \) of \( K \). Set \( A = K_0[y] \) and denote by \( \tilde{A} \) the integral closure of \( A \) in \( \tilde{K} \).

In view of Theorem 3.1.1, the desired result is proved once we verify that for any \( \beta \in \tilde{A} \setminus A \) and any integer \( \lambda_2 \), there exists an integer \( \lambda_1 \) satisfying (3.1). Suppose to the contrary that (3.1) is not satisfied for some \( \beta \in \tilde{A} \setminus A \) and some integer \( \lambda_2 \). Then there exists a sequence \( \{u_n\} \) in \( A \) such that

\[ \tilde{v}_1(u_n - \beta) \geq n, \quad v_2(u_n) \geq \lambda_2 \quad \text{for} \quad n \geq 1. \]  

(3.18)

The first inequality above shows that \( \{u_n\} \) will be a Cauchy sequence in \( K \) with respect to \( v_1 \) and the second says that the sequence \( \{\deg u_n\} \) is bounded, say by \( D \).

Write \( u_n = \sum_{i=0}^{D} c_i y^i \). Note that \( \{c_n\}_n \) is a Cauchy sequence with respect to \( v_0 \) and hence converges to an element \( c_i \) of the complete field \( K_0 \). Therefore \( \{u_n\} \) converges to an element \( \sum_{i=0}^{D} c_i y^i \) of \( A \) with respect to \( v_1 \). But the first inequality of (3.18) implies that \( \{u_n\} \) converges to \( \beta \) which does not belong to \( A \). This contradiction shows that the hypothesis of Theorem 3.1.1 is satisfied.

The following example shows that condition (i) of Theorem 3.1.2 cannot be dispensed with.
Example 3.3.1. Let $K_0$ be a field of characteristic zero which is complete with respect to a non-trivial real valuation $v_0$. Fix an element $\alpha$ in $K_0$ satisfying $v_0(\alpha) > 0$ if the characteristic of the residue field of $v_0$ is zero and $v_0(\alpha) > 2v_0(p)$ if the characteristic of the residue field of $v_0$ is $p > 0$. Set $f(x, y) = x^2 - (1 + \alpha y)$ and for any $m \geq 1$, define $g_m(x, y) = x^2 - (A_m(y))^2$ where $A_m(y) = 1 + \sum_{k=1}^{m} \left( \frac{1}{k} \right) \alpha^k y^k \in K_0[y]$ and $\left( \frac{1}{k} \right) = \frac{(1/2 - 1 - (1/2 - 1 + k + 1)}{1/2}$. Note that $A_m(y)$ are the partial sums of Taylor series expansion of $\sqrt{1 + \alpha y}$. Hence $(A_m(y))^2 - (1 + \alpha y)$ as a polynomial in $y$ has only terms of degree $> m$. For $k > m$, the coefficient $c_k$ of $y^k$ in $(A_m(y))^2 - (1 + \alpha y)$ is $\sum_{i+j=k, i,j \geq 0} \left( \frac{1}{2i} \right) \left( \frac{1}{2j} \right) \alpha^k$. Keeping in mind the fact that $v_0(\alpha) \leq \frac{1}{p-1}v_0(p)$, it can be easily seen that $v_0(c_k) \geq kv_0(\alpha) - \frac{k}{p-1}v_0(p) - v_0(2^k) \geq k[v_0(\alpha) - 2v_0(p)]$ which tends to infinity as $k$ approaches infinity in view of the choice of $\alpha$. Hence $(A_m(y))^2$ converges to $(1 + \alpha y)$ with respect to $v_0$. Therefore the coefficients of $f(x, y)$ and $g_m(x, y)$ are sufficiently close for large $m$, but each $g_m(x, y)$ is reducible over $K_0$ while $f(x, y)$ is irreducible.