Chapter 2

ZERO DIVISOR GRAPHS OF BOOLEAN POSETS

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Chapter 2

ZERO DIVISOR GRAPHS OF BOOLEAN POSETS

2.1 Introduction

In this Chapter, we study the zero divisor graph of a Boolean poset. We determine the diameter, radius, center, eccentricity and domination number of the zero divisor graph of a Boolean poset. It is proved that if $B$ is a Boolean poset and $S$ is a bounded pseudocomplemented poset such that $S \setminus Z(S) = \{1\}$ then $G(B) \cong G(S)$ if and only if $B \cong S$, which essentially extends the results of Mohammadian [43] and LaGrange [31]. Further, we characterize the graphs which can be realized as the zero divisor graphs of Boolean posets.

Throughout this Chapter, $P$ denotes a poset with the least element 0 and the greatest 1.
2.2 Zero divisor graphs of Boolean posets

We begin with necessary concepts and terminology in a poset $P$.

A poset $P$ is said to be \textbf{distributive} if, for all $a, b, c \in P$, $\{\{a, b\}^u, c\}^\ell = \{\{a, c\}^\ell, \{b, c\}^\ell\}^{u\ell}$ holds; see Larmerová and Rachůnek [39]. More details on distributive posets can be found in Waphare and Joshi [70].

An element $y \in P$ is said to be a \textbf{complement} of $x \in P$, if $\{\{x, y\}^u\}^\ell = \{\{x, y\}\}^\ell$ and $P$ is said to be \textbf{(uniquely) complemented} if every element of $P$ has a (unique) complement. The unique complement of $x$ is denoted by $x'$. A distributive complemented poset is called \textbf{Boolean}; see Halaš [24], Niederle [47]. A poset $P$ is called \textbf{atomic} if every non-zero element of $P$ contains an atom of $P$.

Let $P$ be a uniquely complemented poset and $A \subseteq P$. Denote $A' = \{a' : a \in A\}$. We say that $P$ satisfies the \textbf{De Morgan laws} if $\{\{x, y\}^u\}' = \{x', y'\}^\ell$ and $\{\{x, y\}^\ell\}' = \{x', y'\}^u$ for all $x, y \in P$; see Chajda [12], Waphare and Joshi [70].

An element $x^* \in P$ is said to be the \textbf{pseudocomplement} of $x \in P$ if $\{x, x^*\}^\ell = \{0\}$ and for $y \in P$, $\{x, y\}^\ell = \{0\}$ implies $y \leq x^*$. A poset $P$ is called \textbf{pseudocomplemented} if each element of $P$ has the pseudocomplement; see Halaš [22] and Venkatanarasimhan [67]. We use the notation $B(P) = \{x \in P : x = x^{**}\}$; see Halaš [22], Niederle [77].

An element $x \in P$ is called a \textbf{zero divisor} if $\{x, y\}^\ell = \{0\}$ for some $0 \neq y \in P$. Let $Z(P)$ denotes the set of all zero divisors of a poset $P$.

Recall that a poset $P$ with 0, we associate a simple undirected graph, called the \textbf{zero divisor graph of $P$}, denoted by $G_{(0)}(P)$ whose vertex set $V(G_{(0)}(P))$ consists of the non-zero zero divisors of $P$ and two
distinct vertices $x, y$ are adjacent if and only if $\{x, y\}^\ell = 0$; see Joshi[27].

Let $G$ be a graph and let $v \in V(G)$. A vertex $w \in V(G)$ is called a complement of $v$ in $G$, if $v$ is adjacent to $w$, and no vertex is adjacent to both $v$ and $w$, i.e., the edge $v - w$ is not an edge of any triangle in $G$. In such a case, we write $v \perp w$. Note that, if a vertex $w \in V(G)$ has a unique complement $v$ in $G$ and if $u - v - w$ is a path ($u \neq w$) then $u - v$ must be an edge of a triangle otherwise $u$ will be also a complement of $v$, a contradiction to the fact that $v$ has unique complement. Moreover, we say that $G$ is complemented if every vertex has a complement in $G$. An end is a vertex that is adjacent to precisely one other vertex.

The following result is due to Chajda [12].

**Lemma 2.2.1** (Chajda [12]). Let $P$ be a uniquely complemented poset. The following conditions are equivalent.

1. $x \leq y$ implies $y' \leq x'$ for $x, y \in P$.

2. $P$ satisfies De Morgan laws.

**Lemma 2.2.2.** Let $P$ be a Boolean poset. Then $(1 \neq) b$ is an atom if and only if its complement is the unique end adjacent to $b$ in $G_{\{0\}}(P)$.

**Proof.** Let $b$ be an atom and $b'$ be its complement in $P$. Let $(b' \neq) x \in V(G_{\{0\}}(P))$ be a vertex adjacent to $b$, that is, $\{x, b\}^\ell = \{0\}$. It is clear that $\{b, x\}^u \neq P$, otherwise $x = b'$. Hence, $\{b, x\}^u \neq \{1\}$. Therefore there exists $d \in \{b, x\}^u$ such that $d \neq 1$. Clearly, $d \neq 0$. Since $x, b \leq d$ and $P$ is Boolean, we have $x', b' \geq d'$. Therefore, $\{x, d'\}^\ell = \{0\}$ and $\{b, d'\}^\ell = \{0\}$. Hence, $d \notin \{0, x, b\}$, thus $x$ is not an end. We claim that if $b'$ is an end then it is the unique end adjacent to $b$. Indeed,
assume $y \neq 0$ with $\{y, b'\}^\ell = \{0\}$, then we have $y \leq b'' = b$, as every Boolean poset is pseudocomplemented. But $b$ is an atom, we have $y = b$.

Conversely, suppose $b'$ is the unique end adjacent to $b$. If $0 < x \leq b$ then $\{x, b'\}^\ell = \{0\}$. But by the hypothesis, $b'$ is the unique end adjacent to $b$. Thus $b$ is an atom.

\begin{align*}
\textbf{Theorem 2.2.3.} \quad &\text{Let } P \text{ be a Boolean poset with } |P| > 2. \text{ Then the atoms of } P \text{ are precisely the elements of } G_{\{0\}}(P) \text{ that are adjacent to an end.} \\
\text{Proof.} &\text{By Lemma 2.2.2, the atoms of } P \text{ are adjacent to an end. Let } x \text{ be an end and } b \neq 0 \text{ with } \{b, x\}^\ell = \{0\}. \text{ We claim that } b \text{ is an atom. Since } \{x, x'\}^\ell = \{0\} \text{ and } x \text{ is an end vertex, we have } x' = b, \text{i.e., } x = b'. \text{ Hence } b' \text{ is the unique end adjacent to } b. \text{ Thus } b \text{ is an atom, by Lemma 2.2.2.} \\
\end{align*}

We need the following two results due to Niederle [47], to prove that every vertex of a zero divisor graph of a Boolean poset has unique complement.

\begin{align*}
\textbf{Theorem 2.2.4} \quad &\text{(Niederle [47]). } A \text{ uniquely complemented pseudocomplemented poset is Boolean.} \\
\textbf{Theorem 2.2.5} \quad &\text{(Niederle [47]). } Let P \text{ be a pseudocomplemented poset. Then } B(P) \text{ is the Boolean poset with respect to the induced order.} \\
\textbf{Theorem 2.2.6.} \quad &\text{If } P \text{ is a Boolean poset then every vertex of } G_{\{0\}}(P) \text{ has the unique complement (in graph theoretic sense) in } G_{\{0\}}(P).}
\end{align*}
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Proof. Let $g$ be any vertex in $G_{\{0\}}(P)$. As $P$ is Boolean, $g$ has the complement $g'$ in $P$, hence $\{g, g'\}^\ell = \{0\}$ and $\{g, g'\}^u = \{1\}$. Clearly, $g$ and $g'$ are adjacent in $G_{\{0\}}(P)$.

First, we show that $g'$ is a complement of $g$ in $G_{\{0\}}(P)$. Assume on the contrary that there exists a vertex $t \in V(G_{\{0\}}(P))$ such that $g - t - g' - g$ is a triangle in $G_{\{0\}}(P)$. Hence $\{g, t\}^\ell = \{0\} = \{t, g'\}^\ell$. By the distributivity of $P$, we have $\{0\} = \{(g, t)^\ell, (t, g')^\ell\}^u = \{t, \{g, g'\}^u\}^\ell = \{t\}^\ell$, a contradiction to $t \neq 0$. Thus $g'$ is a complement of $g$ in $G_{\{0\}}(P)$.

Now, we show that $g'$ is the unique complement of $g$ in $G_{\{0\}}(P)$. Suppose that $s(\neq g')$ is another complement of $g$ in $G_{\{0\}}(P)$. This implies that $\{s, g\}^\ell = \{0\}$. Let $s'$ be the complement of $s$ in $P$. We claim that $\{s', g'\}^\ell = \{0\}$. Suppose on contrary that $\{s', g'\}^\ell \neq \{0\}$. Then there exists $(0 \neq) t \in \{s', g'\}^\ell$. But then $s - g$ is an edge of the triangle $t - s - g - t$, a contradiction to the fact that $s$ is a complement of $g$ in $G_{\{0\}}(P)$. Thus $\{s', g'\}^\ell = \{0\}$. By Lemma 2.2.1, we get $\{s', g'\}^u = \{s, g\}^{\ell'} = \{0\}^{\ell'} = \{1\}$. Therefore $\{s', g'\}^u = P$. Thus $\{s', g'\}^u = \{s', g'\}^{\ell u} = P$. Hence $s = s'' = g'$, a contradiction. Therefore $g'$ is the unique complement of $g$ in $G_{\{0\}}(P)$.

Now we prove that the converse of Theorem 2.2.6 when $P$ is a pseudocomplemented with certain condition.

**Theorem 2.2.7.** Let $P$ be a pseudocomplemented poset with $P \setminus Z(P) = \{1\}$ and let every vertex of $G_{\{0\}}(P)$ has the unique complement (in graph theoretic sense) in $G_{\{0\}}(P)$. Then $P$ is Boolean.

Proof. If $G_{\{0\}}(P)$ is an empty graph then $Z(P) = \{0\}$. This together with $P \setminus Z(P) = \{1\}$ gives $P \cong C_2$, where $C_2$ is the two-element chain.
Thus $P$ is Boolean. So, we may assume that $G_{\{0\}}(P)$ is a non-empty graph. Hence, there exists a vertex $g \in V(G_{\{0\}}(P))$. As every vertex of $V(G_{\{0\}}(P))$ has the unique complement, $g$ has the unique complement $k \in V(G_{\{0\}}(P))$ in $G_{\{0\}}(P)$. Since $P$ is pseudocomplemented and $\{g, k\}_\ell = \{0\}$, we have $k \leq g^*$, where $g^*$ is the pseudocomplement of $g$ in $P$. If $k \neq g^*$ then $k - g - g^*$ is a path in $G_{\{0\}}(P)$. Since $g - g^*$ is an edge and $k$ is the unique complement of $g$ in $G_{\{0\}}(P)$ implies that, there exists $t \in V(G_{\{0\}}(P))$ such that $t$ is adjacent to both $g$ and $g^*$, otherwise $g^*$ will be a complement in $G_{\{0\}}(P)$. This proves that $g^*$ is also a complement of $g$ in $G_{\{0\}}(P)$. By uniqueness, $k = g^*$, i.e., $g^*$ is the unique complement of $g$ for every $g \in V(G_{\{0\}}(P))$. Similarly, $g^{**}$ is the unique complement of $g^*$ in $G_{\{0\}}(P)$. We claim that $g = g^{**}$. Clearly, $g \leq g^{**}$. If $g \neq g^{**}$ for some $g \in V(G_{\{0\}}(P))$ then $g - g^* - g^{**}$ is a path but $g^{**}$ is the unique complement of $g^*$ in $G_{\{0\}}(P)$, therefore $g - g^*$ is an edge of a triangle, a contradiction to the fact that $g^*$ is a complement of $g$. Hence $g = g^{**}$ for every $g \in V(G_{\{0\}}(P))$. This yields that $\{0, 1\} \cup V(G_{\{0\}}(P)) = B(P)$. By Theorem 2.2.5, $B(P)$ is the Boolean poset and consequently $V(G_{\{0\}}(P)) \cup \{0, 1\} = Z(P) \cup \{1\} = P$ is Boolean as well.

\[\Box\]

**Remark 2.2.8.** We show that both the assumptions of Theorem 2.2.7 are necessary. For the lattice $L$ depicted in Figure 2.2.1 (a) on page 43, we have $L \setminus Z(L) \neq \{1\}$ and every vertex of $G_{\{0\}}(L)$ has the unique complement in $G_{\{0\}}(L)$, but $L$ is not Boolean. Similarly, the non-modular
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lattice $N_5$ from Figure 2.2.1 (b) has the property that $N_5$ is a pseudo-complemented lattice such that $N_5 \setminus Z(N_5) = \{1\}$, but the vertex of zero divisor graph $G_{\{0\}}(N_5)$ has not unique complement. Clearly, $N_5$ is not Boolean. Also, note that for any Boolean poset $P$, $P \setminus Z(P) = \{1\}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{example_graphs.png}
\caption{Complemented zero divisor graphs.}
\end{figure}

Lemma 2.2.9. Let $P$ be a Boolean poset and let $S$ be a bounded pseudo-complemented poset such that $S \setminus Z(S) = \{1\}$. Then $G_{\{0\}}(P) \cong G_{\{0\}}(S)$ if and only if $P \cong S$.

Proof. Let $G_{\{0\}}(P) \cong G_{\{0\}}(S)$. Hence there is a graph isomorphism $f : V(G_{\{0\}}(P)) \to V(G_{\{0\}}(S))$. Define a map $\varphi : P \to S$ such that $\varphi(a) = f(a)$, for all $a \in V(G_{\{0\}}(P))$, $\varphi(0) = 0$ and $\varphi(1) = 1$. Since $f$ is an isomorphism, $\varphi$ is bijective.

To show that $\varphi$ is a poset isomorphism, we need to show that $\varphi$ satisfies the property that $a \leq b$ in $P$ if and only if $\varphi(a) \leq \varphi(b)$ in $S$.

First, we prove that $f(x^*) = [f(x)]^*$, where $x^*$ is the pseudocomplement of $x$ in $P$. Suppose on the contrary that there exists $x \in P$ such that $f(x^*) \neq [f(x)]^*$ in $G_{\{0\}}(S)$. Since $x$ and $x^*$ are adjacent
and \( f \) is a graph isomorphism, we have \( f(x) \) and \( f(x^*) \) are adjacent. Further, \( f(x) \in S \) and \([f(x)]^* \) is the pseudocomplement of \( f(x) \) in \( S \), \( f(x) \) and \([f(x)]^* \) are adjacent. Thus \( f(x^*) - f(x) - [f(x)]^* \) is a path in \( G_{\{0\}}(S) \). Since \( P \) is a Boolean poset, therefore by Theorem 2.2.9, every vertex of \( G_{\{0\}}(P) \) has the unique complement in \( G_{\{0\}}(P) \). Further, \( G_{\{0\}}(P) \cong G_{\{0\}}(S) \) gives every vertex of \( G_{\{0\}}(S) \) has the unique complement. Hence \( f(x) \) has the unique complement in \( G_{\{0\}}(S) \). Thus \( f(x^*) - f(x) \) or \( f(x) - [f(x)]^* \) is an edge of a triangle. If \( f(x^*) - f(x) - f(y) - f(x^*) \) is a triangle in \( G_{\{0\}}(S) \), then \( x^* - x - y - x^* \) is a triangle in \( G_{\{0\}}(P) \). This gives \( \{x, y\}^\ell = \{0\} \). But then \( y \leq x^* \), a contradiction to \( \{y, x^*\}^\ell = \{0\} \). If \([f(x)]^* - f(x) - f(y) - [f(x)]^* \) is a triangle in \( G_{\{0\}}(S) \) then \( \{f(x), f(y)\}^\ell = \{0\} \) gives \( f(y) \leq [f(x)]^* \) contradiction to \( \{f(y), [f(x)]^*\}^\ell = \{0\} \). Hence \( f(x^*) = [f(x)]^* \) for all \( x \in P \). Thus \( f(x) \) has the unique complement \([f(x)]^* \) in \( G_{\{0\}}(S) \). Note that \([f(x)]^{**} = f(x)\).

Now, we show that \( \varphi \) satisfies the property that \( a \leq b \) in \( P \) if and only if \( \varphi(a) \leq \varphi(b) \) in \( S \). Clearly, \( \varphi \) satisfies the property for 0 and 1. So we consider \( a, b \in P \setminus \{0, 1\} = V(G_{\{0\}}(P)) \) such that \( a \leq b \). If \( a = b \) then clearly \( \varphi(a) = \varphi(b) \). Suppose \( a < b \) then \( a - b^* \) is an edge in \( G_{\{0\}}(P) \) and hence \( f(a) - f(b^*) \) is an edge in \( G_{\{0\}}(S) \), i.e., \( \{f(a), [f(b)]^*\}^\ell = \{0\} \) which gives \( f(a) \leq [f(b)]^{**} = f(b) \). Since \( f \) is injective, we have \( f(a) < f(b) \). Hence \( \varphi(a) < \varphi(b) \). Similarly, we can prove that \( \varphi(a) \leq \varphi(b) \) in \( G_{\{0\}}(S) \) which implies that \( a \leq b \) in \( G_{\{0\}}(P) \). Hence \( \varphi \) is a poset isomorphism from \( P \) to \( S \).

**Remark 2.2.10.** If the poset \( P \) in Lemma 2.2.9 is pseudocomplemented
but not Boolean then the assertion of the Lemma 2.2.9 need not be true. The posets \( P_1 \) and \( P_2 \) depicted in Figure 2.2.2 are bounded pseudocomplemented posets with \( P_1 \setminus Z(P_1) = \{1\} = P_2 \setminus Z(P_2) \) such that \( G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \) but \( P_1 \not\cong P_2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2_2_2.png}
\caption{Non-isomorphic posets with isomorphic zero divisor graphs.}
\end{figure}

**Theorem 2.2.11.** Let \( P_1 \) and \( P_2 \) be Boolean posets. Then \( G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \) if and only if \( P_1 \cong P_2 \).

**Proof.** Let \( P_1 \) and \( P_2 \) be Boolean posets and \( G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \). Since \( P_2 \) is a Boolean poset, we have \( P_2 \) is bounded pseudocomplemented poset with \( P_2 \setminus Z(P_2) = \{1\} \) and hence by Lemma 2.2.9, \( P_1 \cong P_2 \). \( \square \)

As an immediate consequence of the above theorem is the following result which is due to LaGrange [31].

**Corollary 2.2.12** (LaGrange [31, Theorem 4.1]). Let \( P_1 \) and \( P_2 \) be Boolean algebras. Then \( G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2) \) if and only if \( P_1 \cong P_2 \).

It is well known that there is a 1–1 correspondence between Boolean algebras and Boolean rings, see Grätzer [18]. Thus as a consequence of Theorem 2.2.11, the following result due to Mohammadian [43] can be deduced.
Corollary 2.2.13 (Mohammadian [13, Theorem 7]). Let $P_1$ and $P_2$ be Boolean rings. Then $G_{\{0\}}(P_1) \cong G_{\{0\}}(P_2)$ if and only if $P_1 \cong P_2$.

Lemma 2.2.14. A complemented poset is Boolean if and only if the condition $(\star)$ holds:

$$(\star) : \{a, b\}^\ell = \{0\} \text{ if and only if } a \leq b', \text{ where } b' \text{ is a complement of } b.$$ 

Proof. Let $P$ be a Boolean poset and $\{a, b\}^\ell = \{0\}$. Then $a \leq b^* = b'$. Now, if $a \leq b'$ then clearly $\{a, b\}^\ell = \{0\}$.

Conversely, let $P$ be a complemented poset satisfying the condition $(\star)$. In view of Theorem 2.2.3, it is enough to show that $P$ is uniquely complemented pseudocomplemented poset. Suppose $(0 \neq) b \in P$ has two complements in $P$, say $b'$ and $b''$. Hence $\{b, b'\}^\ell = \{0\}$ and $\{b, b''\}^\ell = \{0\}$. By the condition $(\star)$, we get $b \leq b''$ and $b'' \leq b'$. Therefore by transitivity, we have $b \leq b'$, a contradiction. Thus $b$ has a unique complement in $P$ and therefore $P$ is a uniquely complemented poset. By the condition $(\star)$, it follows that $P$ is pseudocomplemented.

We recall the following Theorem due to Joshi [27].

Theorem 2.2.15 (Joshi [27, Theorem 2.4]). Let $P$ be a poset. Then $G_{\{0\}}(P)$ is connected with $\text{diam}(G_{\{0\}}(P)) \leq 3$. Furthermore, if $G_{\{0\}}(P)$ contains a cycle then $\text{gr}(G_{\{0\}}(P)) \leq 7$.

In [9], Alizadeh et. al. characterized the diameter and girth of a zero divisor graph of a poset. In the same paper they raised the following problem.

Problem 1: (Alizadeh et. al. [9]). Characterize those posets $P$ for which $(\text{diam}(G_{\{0\}}(P)), \text{gr}(G_{\{0\}}(P))) = (2, 3)$ or $(3, 3)$. 
In the following result, we have shown that, atomic Boolean posets and hence finite Boolean posets is a class of posets for which \((\text{diam}(G_{\{0\}}(P)), gr(G_{\{0\}}(P))) = (3, 3)\).

Before we proceed, we recall the following definitions.

**Definition 2.2.16.** The **eccentricity** \(e(a)\) of a vertex \(a\) of a graph \(G\) is defined to be \(e(a) = \max\{d(a, b) \mid a \neq b, b \in G\}\). The set of vertices of \(G\) with minimal eccentricity is called the **center** of the graph and the minimum eccentric value is called the **radius** of \(G\) and is denoted by \(r(G)\).

**Theorem 2.2.17.** Let \(P\) be an atomic Boolean poset with \(|P| > 4\). Then the following hold.

1. \((\text{diam}(G_{\{0\}}(P)), gr(G_{\{0\}}(P))) = (3, 3)\). In fact, \(G_{\{0\}}(P)\) is a complemented graph.

2. \(2 \leq e(x) \leq 3\) for all \(x \in V(G_{\{0\}}(P))\).

3. \(r(G_{\{0\}}(P)) = 2\) and center of \(G_{\{0\}}(P)\) is the set of all atoms in \(P\).

**Proof.** 1. Let \(P\) be an atomic Boolean poset with \(|P| > 4\). Clearly, there exist at least two distinct atoms in \(P\), say \(a\) and \(b\). Hence \(\{a, b\}^\ell = \{0\}\). Using Lemma 2.2.2, we have \(a'\) and \(b'\) are the unique end adjacent to \(a\) and \(b\) in \(G_{\{0\}}(P)\) respectively. It is easy to verify that \(a' - a - b - b'\) is a path of length 3 in \(G_{\{0\}}(P)\). This together with the Theorem 2.2.13 gives \(\text{diam}(G_{\{0\}}(P)) = 3\).

Since \(P\) is an atomic Boolean poset with \(|P| > 4\), it is easy to see that, \(P\) contains at least three distinct atoms, say \(a\), \(b\) and \(c\).
Thus \( \{a, b\}^\ell = \{b, c\}^\ell = \{a, c\}^\ell = \{0\} \) and hence \( gr(G_{\{0\}}(P)) = 3 \). Hence \( (\text{diam}(G_{\{0\}}(P)), gr(G_{\{0\}}(P))) = (3, 3) \). From Theorem 2.2.6, it is cleared that \( G_{\{0\}}(P) \) is complemented graph.

2. If possible, \( e(a) = 1 \) for some \( a \in V(G_{\{0\}}(P)) \), then \( \text{diam}(G_{\{0\}}(P)) \leq 2 \), a contradiction. Thus \( 2 \leq e(x) \leq 3 \) for all \( x \in V(G_{\{0\}}(P)) \).

Let \( a \in V(G_{\{0\}}(P)) \). There are only two possibilities: either \( a \) is an atom in \( P \) or \( a \) is not an atom in \( P \).

Case (1): Let \( a \) is an atom in \( P \). Now we claim that \( e(a) = 2 \). Suppose on contrary that, \( e(a) = 3 \). Thus there exists \( b \in V(G_{\{0\}}(P)) \) such that \( a - u - v - b \) is a path of length 3 in \( G_{\{0\}}(P) \). As \( \{0\} \neq \{a, v\}^\ell, \{a, b\}^\ell \subseteq \{a\}^\ell \) and \( a \) is an atom in \( P \), we have \( \{a, v\}^\ell = \{a, b\}^\ell = \{0, a\} \). Hence \( a \leq v, b \). But then \( a \in \{v, b\}^\ell = \{0\} \), a contradiction. Thus \( e(a) = 2 \) for all atoms \( a \in V(G_{\{0\}}(P)) \).

Case (2): Suppose \( a \) is not an atom in \( P \). Since \( a \neq 0 \) and \( P \) is an atomic poset, there exists an atom \( p \) such that \( p < a \). Using Lemma 2.2.2, we have \( p' \) is the unique end adjacent to \( p \) in \( G_{\{0\}}(P) \). It is easy to see that \( d(a, p') \neq 1 \). Now we suppose that \( d(a, p') = 2 \), then there exists \( b \in V(G_{\{0\}}(P)) \) such that \( a - b - p' \) is a path of length 2 in \( G_{\{0\}}(P) \). Therefore \( b \in N(p') = \{p\} \). This implies that \( b = p \) and hence \( a - p \) is an edge in \( G_{\{0\}}(P) \), a contradiction to the fact that \( p < a \). Thus \( d(a, p') = 3 \) and hence \( e(a) = 3 \).

3. From the above two cases, it is cleared that \( r(G_{\{0\}}(P)) = 2 \) and center of \( G_{\{0\}}(P) \) is the set of all atoms in \( P \).

\( \square \)
**Theorem 2.2.18.** Let $P$ an atomic Boolean poset $P$. Then for any atom $a \in P$, neighborhood $N(a)$ of $a$ in $G_{\{0\}}(P)$ is a dominating set.

**Proof.** Let $a$ be an atom in $P$. From Theorem 2.2.17, we have $e(a) = 2$. Now, we claimed that neighborhood $N(a)$ of $a$ in $G_{\{0\}}(P)$ is a dominating set. For this, let $b \in V(G_{\{0\}}(P)) \setminus N(a)$. Clearly, $b$ is not adjacent to $a$ in $G_{\{0\}}(P)$. This together with $e(a) = 2$, there exists $c \in V(G_{\{0\}}(P))$ such that $a - c - b$ is a path of length 2 in $G_{\{0\}}(P)$. Thus $c \in N(a)$ with $b - c$ is an edge in $G_{\{0\}}(P)$ and hence $N(a)$ is a dominating set in $G_{\{0\}}(P)$.

The following theorem is due to Joshi [27].

**Theorem 2.2.19** (Joshi [27]). For an atomic poset $P$, $\chi(G_{\{0\}}(P)) = \omega(G_{\{0\}}(P)) = \text{number of atoms in } P$.

In the following result, we calculate the domination number $\gamma$, clique number $\omega$ and the vertex chromatic number $\chi$ of the zero divisor graph of an atomic Boolean poset.

**Theorem 2.2.20.** Let $P$ be an atomic Boolean poset with $|P| > 4$. Then $\gamma(G_{\{0\}}(P)) = \omega(G_{\{0\}}(P)) = \chi(G_{\{0\}}(P)) = |\text{Center}(G_{\{0\}}(P))| = \text{number of atoms in } P$.

**Proof.** Let $P$ be an atomic Boolean poset with $|P| > 4$. Let $D = \{a \in P \mid a \text{ is an atom in } P\}$. Clearly, $D \subseteq V(G_{\{0\}}(P))$. Let $x \in V(G_{\{0\}}(P)) \setminus D$. Since $P$ is atomic and $x' \neq 0$, therefore there exists an atom $p \in D$ such that $p \leq x'$. Thus for $x \in V(G_{\{0\}}(P)) \setminus D$, there exists an atom $p \in D$ such that $x - p$ is an edge in $G_{\{0\}}(P)$. Hence
$D$ is a dominating set. Let $D_1$ be any dominating set in $G_{(0)}(P)$. Now we prove that either $x \in D_1$ or $x' \in D_1$, for every $x \in D$. Suppose there exists $q \in D$ such that $q, q' \not\in D_1$. Since $q$ is an atom in $P$ and hence by Lemma 2.2.2, $q'$ is the unique end adjacent to $q$ in $G_{(0)}(P)$, i.e., $q$ is the only vertex which is adjacent to $q'$ in $G_{(0)}(P)$. Since $q' \not\in D_1$ and $D_1$ is a dominating set in $G_{(0)}(P)$, there exists $x \in D_1$ such that $q' - x$ is an edge in $G_{(0)}(P)$. Clearly, $q = x \in D_1$, a contradiction to the fact that $q \not\in D_1$. Hence either $x \in D_1$ or $x' \in D_1$, for every $x \in D$. Thus $|D| \leq |D_1|$, for every dominating set $D_1$ in $G_{(0)}(P)$. Therefore $\gamma(G_{(0)}(P)) = |D| = \text{number of atoms in } P$. Using Theorem 2.2.17 and Theorem 2.2.19, we have $\gamma(G_{(0)}(P)) = \omega(G_{(0)}(P)) = \chi(G_{(0)}(P)) = |\text{Center}(G_{(0)}(P))| = \text{number of atoms in } P$. 

2.3 Realization of zero divisor graphs

One of the main problems in the theory of zero divisor graphs is the realization problem of zero divisor graphs, that is, which graphs are zero divisor graphs. Nimbhorkar, Wasadikar and Pawar [49] considered the realization problem for complete zero divisor graphs of lattices.

Corollary 2.3.1 (Wasadikar and Nimbhorkar [68, Theorem 3.2]). Let $G$ be the complete graph on $n$ vertices. Then $G = G_{(0)}(P)$ for some poset $P$ if and only if $V(G_{(0)}(P)) = A(P) \cup \{0\}$, where $A(P)$ is the set of all atoms of $P$ and $|A(P)| = n$.

Similar type of results are also proved by Afkhami et. al. [8, Proposition 2.4], see also Alizadeh [9, Theorem 3.4].
The realization problem for zero divisor graphs of posets is completely solved by Lu and Wu [37] by using the tool of compact graphs.

**Definition 2.3.2.** A simple graph $G$ is called a **compact graph** if $G$ contains no isolated vertices and for each pair $x, y$ of non-adjacent vertices of $G$, there is a vertex $z$ with $N(x) \cup N(y) \subseteq N(z)$, where $N(a)$ denotes the neighborhood of $a$, i.e., the set of all vertices adjacent to $a$.

**Example 2.3.3.** Now we consider the graph $G_1$ as shown in Figure 2.3.1, it is easy to observed that there is no vertex $v \in V(G_1)$ such that $N(a) \cup N(e) \subseteq N(v)$. Thus $G_1$ is not a compact graph, where as it is easy to observed that, the graph $G$ as shown in Figure 2.3.2 is an example of a compact graph.

![Figure 2.3.1: Example of a non-compact graph](image)

![Figure 2.3.2: A compact graph which is not a zero divisor graph of any lattice](image)
Theorem 2.3.4 (Lu and Wu [37, Theorem 3.1]). A simple graph $G$ is the zero divisor graph of a poset if and only if $G$ is a compact graph.

Remark 2.3.5. From the above theorem, it is cleared that the compact graphs are the zero divisor graph of posets. But it should be noted that every compact graph need not be the zero divisor graph of a lattice. In Figure 2.3.2 on page 53, the graph $G$ is a compact graph where we can not associate any lattice to this graph whose zero divisor graph is isomorphic to the given compact graph. For this, suppose the given compact graph $G$ as shown in Figure 2.3.2, is isomorphic to $G_{\{0\}}(L)$, for some lattice $L$, i.e., $G \cong G_{\{0\}}(L)$. Since $a - x$ and $a - y$ are not edges in $G$, we have $a \land x \neq 0$ and $a \land y \neq 0$. Clearly, $a \land x \land b = 0$ and $a \land y \land b = 0$. Hence $a \land x, a \land y \in V(G_{\{0\}}(L))$. Thus we have the following six possibilities.

Case (1): If $a \land x = b$, then $b \leq a, x$, a contradiction to $a \land b = 0$.

Case (2): If $a \land x = c$, then $c \leq a, x$, a contradiction to $a \land c = 0$.

Case (3): If $a \land x = d$, then $d \leq a, x$, a contradiction to $a \land d = 0$.

Case (4): If $a \land x = x$, then $x \leq a$. This gives $x \land b = 0$, a contradiction.

Case (5): If $a \land x = y$, then we have $y \land b = 0$, a contradiction. From the above discussion, we have the only possibility that $a \land x = a$, i.e., $a \leq x$. Similarly, we can show that $a \land y = a$, and hence $a \leq y$. Further, it is easy to see that $b \leq x, y$. Thus $a \lor b \leq x \land y$. Since $x \land y \land c = 0$, we have $(a \lor b) \land c = 0$ and hence $a \lor b \in V(G_{\{0\}}(L))$. Thus we have the following five possibilities in this case.

Subcase (1): If either $a \lor b = a$ or $a \lor b = b$, then we get a contradiction to $a \land b = 0$. 
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Subcase (2): If \( a \lor b = c \), then we get a contradiction to \( a \land c = b \land c = 0 \).

Subcase (3): If \( a \lor b = d \), then we get a contradiction to \( a \land d = b \land d = 0 \).

Subcase (4): If \( a \lor b = x \), then \( x \leq y \). This gives \( x \land c = 0 \), a contradiction.

Subcase (5): If \( a \lor b = y \), then \( y \leq x \). This gives \( y \land d = 0 \), a contradiction. Hence in all cases we get contradiction. Thus \( G \not\cong G_{\{0\}}(L) \), for any lattice \( L \). But it has corresponding poset \( P \) as shown in Figure 2.3.2 on page 53.

In [32], LaGrange characterized the zero divisor graphs of a Boolean ring. A closure look at the proof of Theorem 3.1 of [32], shows that the condition 1 and 3 essentially characterizes the zero divisor graphs of lattices.

Before we proceed, we need the following definitions.

**Definition 2.3.6.** Let \( G \) be an (undirected) graph. For any \( \emptyset \neq A \subseteq V(G) \), let \( N(A) \) denote the set of all vertices of \( G \) which are adjacent to every element of \( A \). When \( A = \{v_1, \ldots, v_n\} \) is finite, we shall write \( N(A) = N(v_1, \ldots, v_n) \). Clearly, distinct \( v, w \in V(G) \) satisfy \( v \in N(w) \) if and only if \( w \in N(v) \).

Let \( G \) be a non-empty (undirected) graph and \( \varphi : V(G) \to V(G) \) be a bijection. Define \( \leq_\varphi \) on \( V(G) \) by \( r \leq_\varphi s \) if and only if \( r \in N(\varphi(s)) \). It is easy to prove that \( \leq_\varphi \) is a partial order on \( V(G) \) if and only if \( \varphi \) satisfies the following properties.

1. The containment \( r \in N(\varphi(r)) \) holds for all \( r \in V(G) \).

2. If \( r, s \in V(G) \) are distinct and \( r \in N(\varphi(s)) \), then \( s \not\in N(\varphi(r)) \).
3. If \( r, s, x \in V(G) \) with \( r \in N(\varphi(s)) \) and \( s \in N(\varphi(x)) \), then \( r \in N(\varphi(x)) \).

Thus, we say that the bijection \( \varphi \) is **order-inducing** when the above three properties are satisfied.

**Theorem 2.3.7** (LaGrange [32, Theorem 3.1]). Let \( \Gamma \neq \emptyset \) be a graph. Then \( \Gamma = \Gamma(R) \) for some Boolean ring \( R \) if and only if there exists an order-inducing bijection \( \varphi : V(\Gamma) \to V(\Gamma) \) which satisfies the following properties.

1. The map \( \varphi^2 \) is the identity on \( V(\Gamma) \) (that is, \( \varphi \) can be defined by partitioning \( V(\Gamma) \) into sets of order 2);

2. For all \( r, s \in V(\Gamma) \), either \( N(r, s) = \emptyset \) or there exists an \( x \in N(r, s) \) such that \( N(r, s) \subseteq N(\varphi(x)) \);

3. If \( r, s \in V(\Gamma) \), then \( r \in N(s) \) if and only if \( N(\varphi(r), \varphi(s)) = \emptyset \).

Now, we characterize graphs which can be realized as zero divisor graphs of Boolean posets.

**Theorem 2.3.8.** Let \( G \) be a simple graph. Then \( G = G_{[0]}(P) \) for some Boolean poset \( P \) if and only if there exists an order-inducing bijection \( \varphi : V(G) \to V(G) \) which satisfies the following properties.

1. The map \( \varphi^2 \) is the identity on \( V(G) \).

2. If \( r, s \in V(G) \), then \( r \in N(s) \) if and only if \( N(\varphi(r), \varphi(s)) = \emptyset \).

**Proof.** Let \( G = G_{[0]}(P) \) for some Boolean poset \( P \). Define \( \varphi : V(G) \to V(G) \) by \( \varphi(r) = r' \), where \( r' \) is the complement of \( r \) in \( P \). It is routine
to show that $\varphi$ is well defined and bijective. Let $r, s \in V(G)$. In a Boolean poset $P$, $r \leq s$ if and only if $\{r, s\}^\ell = \{0\}$ if and only if $r \in N(s') = N(\varphi(s))$ if and only if $r \leq \varphi s$. Thus $\varphi$ is order-inducing. Clearly (1) holds. For (2), $t \in \{r, s\}^\ell \setminus \{0\}$ if and only if $(0 \neq) t \leq r, s$ if and only if $t \neq 0$ and $\{t, r'\}^\ell = \{0\} = \{t, s'\}^\ell$ if and only if $t \in N(r', s') = N(\varphi(r), \varphi(s))$. Thus we have $\{r, s\}^\ell \setminus \{0\} = N(\varphi(r), \varphi(s))$ and hence (2) holds.

To prove the converse, let $G$ be a simple graph with an order inducing bijection $\varphi : V(G) \to V(G)$ which satisfies (1) and (2). Let $0, 1$ be any two distinct elements which do not belong to $V(G)$ and set $P = V(G) \cup \{0, 1\}$. Extend the map $\varphi : P \to P$ by letting $\varphi(0) = 1$ and $\varphi(1) = 0$. Define the relation $\leq$ on $P$ by declaring $0 \leq r \leq 1$ for all $r \in V(G)$, and $r \leq s$ for all $r, s \in V(G)$ if and only if $r \leq \varphi s$ (see Definition 2.3.6). It follows that $\leq$ is a partial order on $P$.

As $r \in N(\varphi(r))$ for all $r \in V(G)$ and $\varphi$ satisfies the properties (1) and (2) we have $N(r, \varphi(r)) = \emptyset$, for every $r \in V(G)$. First, we show that $\{r, \varphi(r)\}^{ul} = P = \{r, \varphi(r)\}^{lu}$ for every $r \in V(G)$. For, let $(1 \neq) t \in \{r, \varphi(r)\}^u$. Then $t \in V(G)$ and $r, \varphi(r) \in N(\varphi(t))$ and hence $\varphi(t) \in N(r, \varphi(r))$, a contradiction. Therefore $\{r, \varphi(r)\}^u = \{1\}$. So $\{r, \varphi(r)\}^{ul} = P$. Similarly, we can show that $\{r, \varphi(r)\}^{lu} = P$. This gives $\varphi(r)$ is a complement of $r$ in $P$. We put $\varphi(r) = r'$. Thus $P$ is complemented. Now, we claim that the condition $(\ast)$ of Lemma 2.2.14 holds. Let $r, s \in \{0, 1\}$. Then the condition $(\ast)$ holds. Therefore assume that $r, s \in V(G)$ such that $\{r, s\}^\ell = \{0\}$, i.e., $r$ and $s$ are adjacent in $G$. Thus $r \in N(s) = N(\varphi^2(s))$ and hence $r \leq \varphi(s) = s'$. \[\Box\]