2 Preliminaries

2.1 Orbit Classes and $z$-Classes

In this section, we briefly recall the notion of $z$-class and state related important theorems; for details, see Kulkarni [29]. Let $G$ be a group acting on a set $X$. For $x \in X$, let $G(x)$ denote the $G$-orbit of $x$ and $G_x$ denote the stabilizer subgroup of $G$ at $x$. Then for $x, y \in X$, either $G(x) = G(y)$ or $G(x) \cap G(y) = \emptyset$. We write $x \sim y$ if $x, y$ are in the same orbit. This gives the first partition of $X$ by orbits:

$$X = \bigcup_{x \sim y} G(x).$$

(2.1)

Definition 2.1.1. We say that $x, y \in X$ are in the same orbit class, written $x \sim_o y$, if the stabilizers $G_x$ and $G_y$ are conjugate in $G$.

Clearly, $\sim_o$ is an equivalence relation on $X$. We denote the equivalence class of $x$ under $\sim_o$ by $R(x)$. Thus, we obtain the second partition of $X$ by orbit classes:

$$X = \bigcup_{x \sim_o y} R(x).$$

(2.2)

Since the elements of an orbit have conjugate stabilizers, an orbit class is a union of orbits in $X$.

An interesting case arises if $G = X$ and $G$ acts on $X$ by conjugation. Then, the stabilizer of any $x \in G$ is the centralizer of $x$:

$$Z_G(x) = \{y \in G : xy = yx\}.$$
Thus,
\[ x \sim y \text{ if and only if } x \text{ and } y \text{ are conjugate in } G, \]
\[ x \sim_o y \text{ if and only if } Z_G(x) \text{ and } Z_G(y) \text{ are conjugate in } G. \]

In this case, we call the orbit class of any \( x \in G \), the \textbf{z-class} of \( x \). Thus,
\[ z\text{-class of } x = \{ y \in G : Z_G(y) = gZ_G(x)g^{-1} \text{ for some } g \in G \}. \]

Coming back to the general case, we will see that, there is more refined structure on the orbit classes as described below. Let \( F_x \) denote the set of fixed points of \( G_x \):
\[ F_x = \{ y \in X : G_y \supseteq G_x \}. \]

Let \( F'_x \) denote the set of “generic” elements in \( X \), namely
\[ F'_x = \{ y \in X : G_y = G_x \}. \]

Finally, let
\[ N_x = \{ g \in G : gG_xg^{-1} = G_x \}, \] the normalizer of \( G_x \) in \( G \),
and \( W_x = N_x/G_x \).

We call \( W_x \), the \textbf{Weyl group} at \( x \). There are two natural actions of \( N_x \) on \( G/G_x \):
\[ N_x \times G/G_x \to G/G_x, \quad (n, gG_x) \mapsto ngn^{-1}G_x, \]
\[ N_x \times G/G_x \to G/G_x, \quad (n, gG_x) \mapsto gn^{-1}G_x. \]

Since, the subgroup \( G_x \) of \( N_x \) is in the kernel of both the actions, we have the following two actions of \( N_x/G_x = W_x \) on \( G/G_x \):
\[ \psi_1 : W_x \times G/G_x \to G/G_x, \quad (n, gG_x) \mapsto ngn^{-1}G_x \]
\[ \psi_2 : W_x \times G/G_x \to G/G_x, \quad (n, gG_x) \mapsto gn^{-1}G_x. \]

It is easy to see that the first action \( \psi_1 \) of \( W_x \) on \( G/G_x \) is non-necessarily free, but the second action is free. The Weyl group \( W_x \) also acts \textit{freely} on \( F'_x \):
\[ W_x \times F'_x \to F'_x, \quad (nG_x, y) \mapsto ny. \]
Consider the diagonal action of $W_x$ on $G/G_x \times F'_x$. The following theorem gives a parametrization of $W_x$-orbits of $G/G_x \times F'_x$ by the elements of the orbit class $R(x)$.

**Theorem 2.1.2** (Kulkarni, 2007). The map

$$\phi: G/G_x \times F'_x \to R(x), \quad \phi(gG_x, y) = g.y,$$

is well defined, and induces a bijection,

$$\bar{\phi}: \{G/G_x \times F'_x\}/W_x \to R(x).$$

**Proof.** See Kulkarni [29], Theorem 2.1, p. 41.

Suppose, both $G$ and $X$ are finite. Then $G/G_x \times F'_x$ is a union of finitely many $W_x$-orbits. Since the action of $W_x$ is free, $|R(x)|$ is the number of $W_x$-orbits of $G/G_x \times F'_x$, which is equal to $|G/G_x \times F'_x|/|W_x|:

$$|R(x)| = \frac{|G/G_x \times F'_x|}{|W_x|} = \frac{|G/G_x||F'_x|}{|N_x/G_x|} = [G:N_x]|F'_x|.$$  \hfill (2.3)

### 2.2 Commutator Calculus

We recall a part of this vast subject, which will be used in next chapters. For $x_1, x_2, \cdots, x_n \in G$, we define $[, , ]: = x_1^{-1}x_2^{-1}x_1x_2$, and

$$[, , ] = [[,,,]],$$

Also, for $g, h \in G$, we write $h^g := g^{-1}hg$. Then, the following commutator identities can be easily verified (see P. Hall [25], §2):

$$[b, a] = [a, b]^{-1},$$  \hfill (2.4)

$$[ab, c] = [a, c][a, c, b][b, c],$$  \hfill (2.5)

$$[a, bc] = [a, c][a, b][a, b, c],$$  \hfill (2.6)

(Hall-Witt Identity) \( [a, b, c^a] \cdot [b, c, a^b] \cdot [c, a, b^c] = 1, \quad \forall a, b, c \in G. \)  \hfill (2.7)

**Proposition 2.2.1.** Let $G$ be a group and $x, y \in G$.  


1. If \([x, y]\) commutes with \(x\), then \([x^n, y] = [x, y]^n\), \(\forall n \in \mathbb{Z}\).

2. If \([x, y]\) commutes with \(y\), then \([x, y^n] = [x, y]^n\), \(\forall n \in \mathbb{Z}\).

3. If \([x, y]\) commutes with both \(x\) and \(y\) then \((xy)^n = x^n y^n \cdot [y, x]^{n(n-1)/2}\), \(\forall n \in \mathbb{Z}\).

**Proof.** (1) We proceed by induction on \(n\):

\[
[x^{n+1}, y] = [x^n x, y]
\]

\[
= [x^n, y] \cdot [x^n, y] \cdot [x, y] \quad (\text{by identity 2.5})
\]

\[
= [x, y]^n \cdot [x, y]^n \cdot [x, y] \quad (\text{by induction})
\]

\[
= [x, y]^n \cdot 1 \cdot [x, y] = [x, y]^{n+1} \quad (\text{by hypothesis}).
\]

Proof of (2) is similar to that of (1).

(3) We proceed by induction on \(n\):

\[
(xy)^{n+1} = (xy)^n xy
\]

\[
= x^n y^n \cdot [y, x]^{n(n-1)/2} \cdot xy \quad (\text{by induction})
\]

\[
= x^n y^n xy \cdot [y, x]^{n(n-1)/2} \quad (\text{by hypothesis})
\]

\[
= x^n \cdot xy^n \cdot y^{-n} x^{-1} \cdot y^n xy \cdot [y, x]^{n(n-1)/2}
\]

\[
= x^{n+1} y^n \cdot [y^n, x] \cdot [y, x]^{n(n-1)/2}
\]

\[
= x^{n+1} y^n \cdot [y^n, x] \cdot [y, x]^{n(n-1)/2} \quad (\text{by (1)})
\]

\[
= x^{n+1} y^{n+1} \cdot [y, x] \cdot [y, x]^{n(n-1)/2} \quad (\text{by hypothesis})
\]

\[
= x^{n+1} y^{n+1} [y, x]^{(n+1)n/2}.
\]

(4) It can be easily verified. \(\square\)

**Theorem 2.2.2** (Tuan H. F., 1950). Let \(G\) be a non-abelian finite \(p\)-group, having an abelian subgroup \(A\) of index \(p\). Then

\[
A/Z(G) \cong G', \text{ and hence } |G| = p.|Z(G)|.|G'|.
\]

**Proof.** Consider \(x \in G \setminus A\), and define \(\varphi_x: A \to A, a \mapsto [a, x]\). Since \([a, x] = a^{-1}(x^{-1}ax)\) \(\in A\), \(\varphi_x\) is well defined. Now, \(\varphi_x\) is a homomorphism: By identity (2.5), and since \(A\) is...
abelian, 
\[ \varphi_x(ab) = [ab, x] = [a, x][[a, x], b][b, x] = [a, x].1.[b, x] = \varphi_x(a)\varphi_x(b). \]

We show that \( \ker(\varphi_x) = Z(G) \) and \( \operatorname{Im}(\varphi_x) = G' \). Since \( A \) is a maximal abelian subgroup, \( Z(G) \subseteq A \). Then, it is clear that \( Z(G) \subseteq \ker(\varphi_x) \). If \( a \in A \) is in the \( \ker(\varphi_x) \), then \( a \) commutes with \( x \) as well as all elements of \( A \), hence \( a \in Z(G) \) (since \( G = \langle A, x \rangle \)).

Also, by definition of \( \varphi_x \), \( \operatorname{Im}(\varphi_x) \subseteq G' \). An arbitrary element of \( G \) is of the type \( ax^i \) (\( a \in A, 0 \leq i < p \)). For \( ax^i, bx^j \in G \), by identity (2.5) and (2.6) and since \( A \) is abelian, 
\[ [ax^i, bx^j] = [a, bx^j].[x^i, bx^j] \]
\[ = [a, b].[a, x^i].[x^i, b].[x^i, x^j] \]
\[ = 1.[a, x^i].[b, x^j]^{-1}.1 \in \operatorname{Im}(\varphi_x), \]

hence \( G' = \operatorname{Im}(\varphi_x) \). Therefore, \( A/\ker(\varphi_x) \cong \operatorname{Im}(\varphi_x) \), i.e. \( A/Z(G) \cong G' \).

**Remark 2.2.3.** In the proof of the Theorem 2.2.2, we can alternately prove \( \operatorname{Im}(\varphi_x) = G' \) as follows: let \( K = \operatorname{Im}(\varphi_x) \). By definition of \( \varphi_x \), \( K \subseteq G' \), and \( K \unlhd A \), since \( A \) is abelian.

Also, for any \([a, x] \in K \), \( x^{-1}[a, x]x = [x^{-1}ax, x] = [a', x] \in K \). Hence \( K \) is normalized by \( \langle A, x \rangle = G \). Since \( G = \langle A, x \rangle \), we have \( G/K = \langle A/K, xK \rangle \). Here \( A/K \) is abelian, and for any \( a \in A \), \([aK, xK] = [a, x]K = K \), i.e. \( xK \) commutes with all elements of \( A/K \), i.e. \( G/K \) is abelian. Therefore, \( G' \subseteq K = \operatorname{Im}(\varphi_x) \).

For \( H, K \leq G \), we define the subgroup 
\[ [H, K] = \langle [h, k] : h \in H, k \in K \rangle. \]

**Proposition 2.2.4.** Let \( G \) be a group, and \( N \) a normal subgroup such that \( G/N \) is cyclic. Then, \([G, N] = [G, G] \).

**Proof.** Clearly \([G, N] \subseteq [G, G] \). Let \( x \in G \) be such that \( G/N = \langle xN \rangle \). An arbitrary element of \( G \) is of the type \( x^kn \) (\( k \in \mathbb{Z}, n \in N \)). For \( n_1, n_2 \in N \), by the identity (2.5), 
\[ [x^in_1, x^jn_2] = [x^im_n_1, x^jm_n_2][x^im_n_1, x^jm_n_2]][x^im_n_1, x^jm_n_2][x^im_n_1, x^jm_n_2][x^im_n_1, x^jm_n_2]. \]
Since \( x^i, x^j \) commute, it is easy to see that
\[
[x^i n_1, x^j] = [n_1, x^j] = [x^j, n_1]^{-1} \in [G, N].
\]
Hence the right hand side of the equation (2.8) is in \([G, N]\), and the result follows.

### 2.3 Some Results on Centralizers

In any group \( G \), we denote the centralizer of \( x \) in \( G \) by \( Z_G(x) \), and by definition,
\[
Z_G(x) = \{ y \in G : xy = yx \}.
\]
Also, \( C_G(x) \) denotes the conjugacy class of \( x \) in \( G \). It is well known that, if \( G \) is a finite group, then \( [G : Z_G(x)] = |C_G(x)| \) for any \( x \in G \).

**Proposition 2.3.1.** Let \( G \) be a finite group, \( N \) a normal subgroup. Then, for any \( x \in G \),
\[
|Z_G(x)| \geq |Z_{G/N}(xN)|.
\]

**Proof.** Let \( \varphi : G \to G/N \) be the natural homomorphism. For convenience, we write \( \overline{G} = G/N \) and \( \overline{x} = xN \). Let \( C_{\overline{G}}(\overline{x}) = \{\overline{x_1}, \cdots, \overline{x_r}\} \), \( r \geq 1 \). Since, by a homomorphism, conjugate elements are mapped to conjugate elements,
\[C_G(x) \subseteq \varphi^{-1}(C_{\overline{G}}(\overline{x})) = x_1N \cup x_2N \cup \cdots \cup x_rN.\]

Therefore,
\[
\frac{|G|}{|Z_G(x)|} = |C_G(x)| \leq r |N| = |C_{\overline{G}}(\overline{x})||N| = \frac{|G/N|}{|Z_{G/N}(xN)|} |N|,
\]
the result follows.

**Definition 2.3.2 (Supersolvable Group).** A group \( G \) is said to be supersolvable if there is a series of normal subgroups of \( G \),
\[
1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_m = G
\]
such that the chief factors \( G_i/G_{i-1} \) are cyclic (not necessarily finite), \( i = 1, 2, \cdots, m \).
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It is easy to see that the subgroups and the quotients of a supersolvable group are supersolvable.

**Proposition 2.3.3.** Let $G$ be a group such that $G/Z(G)$ is supersolvable and $A$ a maximal abelian normal subgroup (i.e. maximal among abelian normal subgroups) of $G$. Then $Z_G(A) = A$, i.e. $A$ is a maximal abelian subgroup of $G$.

**Proof.** It is clear that $Z(G) \subseteq A \subseteq Z_G(A)$. If $A \neq Z_G(A)$, consider the series of normal subgroups of $G$:

$$1 \subseteq A \subseteq Z_G(A) \subseteq G.$$ 

By hypothesis, the series has a refinement between $A$ and $Z_G(A)$,

$$1 \subseteq A \subseteq A_1 \subseteq \cdots \subseteq Z_G(A) \subseteq G,$$

where $A_1/A$ is cyclic. Consider $a_1 \in A_1 \setminus A$. Then $A_1 = \langle a_1, A \rangle$ is an abelian normal subgroup of $G$ with $A < A_1$, a contradiction to the maximality of $A$. Hence $Z_G(A) = A$. \qed

**Remark 2.3.4.** In the Proposition 2.3.3, the hypothesis ‘$G/Z(G)$ is supersolvable’ cannot be weakened to the hypothesis ‘$G/Z(G)$ is solvable’. In the group $SL(2, \mathbb{F}_3)$, the center is the only maximal abelian normal subgroup, whose centralizer is clearly the whole group. Here, $SL(2, \mathbb{F}_3)/Z(SL(2, \mathbb{F}_3)) \cong A_4$ is not supersolvable.

**Notation:** For any group $G$, $Z_2(G)$ denotes the subgroup of $G$ such that $Z_2(G)/Z(G) = Z(G/Z(G))$.

**Theorem 2.3.5.** Let $G$ be a non-abelian group with $[G: Z(G)] = p^k$, and $A$ an abelian subgroup of index $p$. Then for any $x \in G \setminus A$,

(i) $Z_G(x) = \langle x, Z(G) \rangle$ and $Z_G(x)/Z(G) = \langle xZ(G) \rangle \cong C_p$.

(ii) $N_G(Z_G(x)) = Z_G(x)Z_2(G) = \langle x, Z_2(G) \rangle$.

(iii) If $G/Z(G)$ is non-abelian, then $N_G(Z_G(x))/Z_G(x) \cong Z(G/Z(G))$. 


Proof. (i) Since $G/Z(G)$ is supersolvable, by Proposition 2.3.3, $Z_G(A) = A$. Consider $x \in G \setminus A$, hence

$$Z(G) = Z_G(x) \cap Z_G(A) = Z_G(x) \cap A$$

and $AZ_G(x) = G$. Therefore,

$$C_p \cong \frac{G}{A} = \frac{AZ_G(x)}{A} \cong \frac{Z_G(x)}{Z_G(x) \cap A} = Z_G(x)/Z(G).$$

(ii) Consider $x \in G \setminus A$, $g \in Z_2(G)$ and write $G/Z(G) = \overline{G}$. Then $[\overline{x}, \overline{g}] = 1$, hence $x^{-1}g^{-1}xg \in Z(G)$. Let $x^{-1}g^{-1}xg = t$. Then

$$g^{-1}Z_G(x)g = Z_G(g^{-1}xg) = Z_G(xt) = Z_G(x),$$

hence $Z_2(G) \leq N_G(Z_G(x))$. Since $G = \langle x, A \rangle$, $\overline{G} = \langle \overline{x}, \overline{A} \rangle$. If $a \in A$ normalizes $Z_G(x)$, then in $G/Z(G)$, $\overline{a}$ normalizes $Z_G(x)/Z(G) = \langle \overline{x} \rangle \cong C_p$ (by (i)), i.e. $\overline{a}$ centralizes $\overline{x}$. Since $\overline{G} = \langle \overline{x}, \overline{A} \rangle$ and $\overline{A}$ is abelian, we see that $\overline{a} \in Z(\overline{G})$, i.e. $a \in Z_2(G)$. Thus, $N_G(Z_G(x)) = Z_G(x)Z_2(G)$.

(iii) By hypothesis, $Z_2(G) \neq G$, and $A/Z(G)$ is an abelian subgroup of index $p$ in $G/Z(G)$. Hence $Z_2(G)/Z(G) \subseteq A/Z(G)$, i.e. $Z_2(G) \subseteq A$. Then

$$Z(G) \subseteq Z_G(x) \cap Z_2(G) \subseteq Z_G(x) \cap A = Z(G).$$

Hence,

$$\frac{N_G(Z_G(x))}{Z_G(x)} = \frac{Z_G(x)Z_2(G)}{Z_G(x)} \cong \frac{Z_2(G)}{Z_G(x) \cap Z_2(G)} = \frac{Z_2(G)}{Z(G)}.$$

Remark 2.3.6. If $G$ is any group, then clearly $Z(G)$ centralizes every subgroup of $G$. The proof of Theorem 2.3.5(ii) shows that $Z_2(G)$ normalizes every subgroup of $G$:

$$Z_2(G) \subseteq N_G(H)$$

for every subgroup $H$ of $G$.

2.4 Some Results on Finite $p$-Groups

The most basic fact about finite $p$-groups is that they have non-trivial center. As a consequence of this fact, it can be proved that a finite $p$-group contains normal subgroups
of every order which (necessarily) divides the order of the group (use induction on order of the group). In particular, if \( G \) is a finite \( p \)-group with \( |G| \geq p^2 \) then \( G \) contains a normal subgroup of order \( p^2 \), and groups of order \( p^2 \) are always abelian.

**Theorem 2.4.1.** Any finite \( p \)-group of order at least \( p^4 \) contains an abelian normal subgroup of order \( p^3 \). In particular, any group of order \( p^4 \) contains an abelian (normal) subgroup of index \( p \).

**Proof.** Let \( G \) be a finite \( p \)-group, with \( |G| = p^n \geq p^4 \). The theorem is obvious if \( G \) is abelian, assume that \( G \) is non-abelian. As mentioned before this theorem, \( G \) contains an abelian normal subgroup of order \( p^2 \), say \( H \). Then, we obtain a homomorphism

\[
\varphi : G \rightarrow \text{Aut}(H), \ g \mapsto \{h \mapsto ghg^{-1}\}.
\]

Since \( H \) is isomorphic to either \( C_{p^2} \) or \( C_p \times C_p \), we have

either \( \text{Aut}(H) \cong C_p \times C_{p-1} \) or \( \text{Aut}(H) \cong \text{GL}(2, \mathbb{Z}/p\mathbb{Z}) \).

Hence, \( |\text{Aut}(H)| = p(p-1) \) or \( (p^2 - 1)(p^2 - p) \). Since \( G \) is a \( p \)-group, \( |\text{Im}(\varphi)| \) is a power of prime \( p \), which divides \( |\text{Aut}(G)| \); it is either 1 or \( p \), hence \( |\text{ker}(\varphi)| \) is at least \( p^{n-1} \geq p^3 \). Note that \( H \subseteq \ker(\varphi) \), since \( H \) is abelian. Consider the series of normal subgroups

\[
1 < H < \ker(\varphi) \leq G.
\]

This series has a refinement (consisting of normal subgroups of \( G \)),

\[
1 < K < H < H_1 \leq \cdots \leq \ker(\varphi) \leq G,
\]

where \( H_1/H \cong C_p \). For \( h_1 \in H_1 \setminus H \), \( H_1 = \langle h_1, H \rangle \). Now \( h_1 \in H_1 \subseteq \ker(\varphi) \), \( h_1 \) commutes with every element of \( H \), hence \( H_1 \) is abelian normal subgroup of order \( p^3 \). \( \square \)

**Theorem 2.4.2** (Miller, 1907). Let \( G \) be a non-abelian finite \( p \)-group.

(i) If \( [G : Z(G)] = p^2 \), then \( G \) has exactly \( p + 1 \) abelian subgroups of index \( p \).

(ii) If \( [G : Z(G)] \geq p^3 \), then \( G \) has at most one abelian subgroup of index \( p \).
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Proof. Let $G$ be non-abelian, and $H$ an abelian subgroup of index $p$. Then $Z(G) \leq H$ (otherwise $HZ(G) = G$ will be abelian).

(i) If $[G : Z(G)] = p^2$, then $G/Z(G) \cong C_p \times C_p$. By subgroup correspondence theorem, $G$ has exactly $p + 1$ subgroups of index $p$, containing $Z(G)$. Let $K$ be a subgroup of index $p$, containing $Z(G)$. Then, $K/Z(G)$ is cyclic (of order $p$), hence $K$ is abelian.

(ii) Let $[G : Z(G)] \geq p^3$, and $A, B$ be distinct abelian subgroups of index $p$. Then $AB = G$, and the formula $|G| = |A|, |B|/|A \cap B|$ shows that $A \cap B$ has index $p^2$ in $G$. Any $x \in A \cap B$ commutes with all elements of $A, B$, and since $A$ and $B$ generate $G$, we have $A \cap B \subseteq Z(G)$. Hence $[G : Z(G)] \leq p^2$, a contradiction.

There are five groups of order $p^3$, up to isomorphism (see Alperin-Bell [2], pp. 77-79):

$$C_{p^3}, \quad C_{p^2} \times C_p, \quad C_p \times C_p \times C_p, \quad G_1, \quad \text{and} \quad G_2,$$

where $G_1$ and $G_2$ are non-abelian groups, given by the presentation

$$G_1 = \langle x, y, z : x^p = y^p = z^p = 1, \; zx = xz, \; zy = yz, \; z = [x, y] \rangle,$$
$$G_2 = \langle x, y : x^{p^2} = 1, \; y^p = x^p, \; y^{-1}xy = x^{1+p} \rangle.$$

The group $G_1$ is isomorphic to the matrix group

$$U(3, \mathbb{F}_p) = \left\{ \begin{pmatrix} 1 & a & b \\ 1 & c & \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}.$$

Here $\mathbb{F}_p$ is the finite field (Galois field) of order $p$. Note that $U(3, \mathbb{F}_2)$ is isomorphic to the dihedral group of order 8 and for $p > 2$, $U(3, \mathbb{F}_p)$ has exponent $p$.

Theorem 2.4.3 (Miller G. A., 1900). Let $G$ be a group and $S$ a generating subset of $G$. If there exists $x \in G$, $x \neq 1$, such that $x \in \langle s \rangle$ for every $s \in S$, then there is no group $H$ such that $H/Z(H) \cong G$.

Proof. Let $S = \{s_\alpha : \alpha \in \Lambda\}$ be a set of generators, and let $x \in G$, $x \neq 1$, be such that $x \in \langle s_\alpha \rangle$ for all $\alpha \in \Lambda$. Suppose, there is a group $H$ such that $H/Z(H) \cong G$. For
simplicity, we write $G = H/Z(H)$. Choose $h$ and $h_\alpha$ in $H$, such that $h_\alpha Z(H) = s_\alpha$ for all $\alpha \in \Lambda$ and $hZ(H) = x$. Note that $h \notin Z(H)$, since $x \neq 1$. Let $H_\alpha = \langle h_\alpha, Z(H) \rangle$. Then $H_\alpha$ is abelian for every $\alpha \in \Lambda$. By hypothesis, since $G = \langle s_\alpha : \alpha \in \Lambda \rangle$, and $x \in \langle s_\alpha \rangle$ for all $\alpha \in \Lambda$, this implies that,

$$H = \langle H_\alpha : \alpha \in \Lambda \rangle \text{ and } h \in H_\alpha \text{ for all } \alpha.$$ 

As $H_\alpha$ is abelian for all $\alpha$, the last equation implies that $h \in Z(H)$, a contradiction. □

**Corollary 2.4.4.** Let $G$ be a $p$-group, with $[G : Z(G)] = p^3$. Then $G/Z(G)$ is either elementary abelian or isomorphic to $U(3, \mathbb{F}_p)$.  

**Proof.** There are five groups of order $p^3$, up to isomorphism, given by the list (2.9). Since $G/Z(G)$ is never cyclic unless $G$ is abelian, we have $G/Z(G) \not\cong C_{p^3}$. Consider $C_{p^2} \times C_p$ and $G_2$

$$C_{p^2} \times C_p = \langle x, y : x^{p^2} = y^p = 1, xy = yx \rangle,$$

$$G_2 = \langle x, y : x^{p^2} = 1, y^p = x^{p}, y^{-1}xy = x^{1+p} \rangle.$$ 

In the group $C_{p^2} \times C_p$, clearly $(xy)^p = x^p$. therefore

$$x^p \in \langle xy \rangle \cap \langle x \rangle.$$

Thus, by Miller’s Theorem 2.4.3, there is no group $H$ such that $H/Z(H) \cong G_1$.  

For $G_2$, we consider two cases: for $p = 2$, 

$$(xy)^2 = xy.xy = yx.y^{-1}xy = xx.x^{-1} = x^2.$$ 

Thus, $(xy)^2 \in \langle xy \rangle \cap \langle x \rangle$, and by Miller’s Theorem 2.4.3, there is no group $H$ such that $H/Z(H) \cong G_2$. For $p > 2$, since $Z(G) = G' = \langle x^p \rangle \cong C_p$, by Proposition 2.2.1(3), it is easy to see that 

$$(xy)^p = x^p \in \langle xy \rangle \cap \langle x \rangle,$$

thus by Theorem 2.4.3, there is no group $H$ such that $H/Z(H) \cong G_2$. □
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The following theorem is a key theorem in one of our main results. We give a proof of the theorem for convenience. For more general versions of this theorem, see Haber, Rosenfeld [23], Corollary, p. 493 and Cohn [18], Theorem 1, p. 44.

**Theorem 2.4.5.** Suppose that a non-cyclic \( p \)-group \( G \) of order \( p^m \) is covered by \( n \) proper subgroups \( H_1, H_2, \ldots, H_n \), i.e. \( G = \bigcup_{i=1}^{n} H_i \). Then

1. (Haber, Rosenfeld, 1959) \( n \geq p + 1 \).
2. (Cohn, 1994) If \( n = p + 1 \), then all \( H_i \)'s are maximal and \( |G: \cap_{i=1}^{p+1} H_i| = p^2 \).

**Proof.**

1. If \( n \leq p \),

\[
|G| \leq \sum_{i=1}^{n} |H_i| \leq n(p^{m-1} - 1) + 1 \leq p(p^{m-1} - 1) + 1 < |G|,
\]

a contradiction.

2. Let \( G = \bigcup_{i=1}^{p+1} H_i \). Then,

\[
G = H_{p+1} \cup \left( \bigcup_{i=1}^{p} H_i \setminus H_{p+1} \right) .
\]  \hspace{1cm} (2.12)

If \( H_k \) is not maximal for some \( k \leq p \), then \( |H_k| \leq p^{m-2} \) and \( |H_k \cap H_{p+1}| \leq p^{m-3} \) (since \( H_k \not\subseteq H_{p+1} \) by (1)). Hence,

\[
|H_k \setminus H_{p+1}| \leq p^{m-2} - p^{m-3} < p^{m-1} - p^{m-2}
\]

and

\[
|G| \leq p^{m-1} + \sum_{i=1}^{p} |H_i \setminus H_{p+1}| < p^{m-1} + p(p^{m-1} - p^{m-2}) = |G|,
\]

a contradiction. In a similar way, if \( H_{p+1} \) is not maximal, we will arrive at a contradiction for \( |G| \). Hence, all \( H_i \)'s are maximal subgroups of \( G \). This implies that for \( i \neq j \), \( H_i H_j = G \). Therefore,

\[
p^m = |H_i H_j| = \frac{|H_i| \cdot |H_j|}{|H_i \cap H_j|} = \frac{p^{m-1} \cdot p^{m-1}}{|H_i \cap H_j|}, \text{ and } |H_i \cap H_j| = p^{m-2}.
\]

Also, \( H_i \setminus H_j = H_i \setminus (H_i \cap H_j) \). By equation (2.12)

\[
|G| \leq p^{m-1} + p(p^{m-1} - p^{m-2}) = p^m = |G|.
\]
Hence the union in the equation (2.12) is disjoint:

\[(H_i \setminus H_{p+1}) \cap (H_j \setminus H_{p+1}) = \phi,\]

i.e. \(H_i \cap H_j \cap H_{p+1}^c = \phi,\)

i.e. \(H_i \cap H_j \subseteq H_{p+1}.\)

By similar arguments for \(H_k\) where \(k \neq i, j\), we see that \(H_i \cap H_j \subseteq H_k\). Therefore,

\[H_i \cap H_j \subseteq \cap_{k=1}^{p+1} H_k \subseteq H_i \cap H_j,\]

and \(|H_i \cap H_j| = p^{m-2}\). This proves (2).

\[\square\]

### 2.5 p-Groups of Maximal Class

**Definition 2.5.1** (Lower Central Series). The lower central series of a group \(G\) is the series

\[G \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots\]

of subgroups of \(G\), where \(\gamma_2(G) = [G, G]\), and \(\gamma_i(G) = [\gamma_{i-1}(G), G]\) for \(i \geq 3\).

**Definition 2.5.2** (Upper Central Series). The upper central series of a group \(G\) is the series

\[1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots\]

of subgroups, where \(Z_1(G) = Z(G)\), and for \(i \geq 2\), \(Z_i(G)\) is the subgroup of \(G\) such that

\[Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G)).\]

If one of the upper and lower central series contains finitely many (distinct) terms, the number of terms, excluding 1 is called the length of the series. One of the important differences between these two series is that the subgroups in the upper central series are characteristic subgroups (i.e. they are invariant under any automorphism of \(G\)), whereas the subgroups in the lower central series are not only characteristic but they are fully invariant subgroups (i.e. for any homomorphism \(\varphi: G \rightarrow G, \varphi(\gamma_i(G)) \subseteq \gamma_i(G)\)). However, there is a numerical invariant for both the series: If one of these two series has
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finite length, so does the other; moreover the lengths are equal, and the common length is called the nilpotency class of the group (see Robinson [8], 5.1.9, p. 125).

**Definition 2.5.3 (Nilpotent Group).** A group $G$ is said to be nilpotent if its upper or lower central series has finite length, and the length of the series is called the nilpotency class of $G$, which we denote by $c(G)$.

Thus, for $n \geq 2$,

\[
c(G) = n - 1 \iff n \text{ is the smallest positive integer such that } \gamma_n(G) = 1
\]

\[
\iff n \text{ is the smallest positive integer such that } Z_{n-1}(G) = G
\]

**Proposition 2.5.4.** For any group $G$, if $N \trianglelefteq G$, then $\gamma_i(G/N) = \gamma_i(G)/N$.  

**Proof.** Let $\varphi: G \to G/N$ be the natural homomorphism. It is enough to prove that

\[
\gamma_i(\varphi(G)) = \varphi(\gamma_i(G)).
\]

We proceed by induction on $n$. Since for any subgroups $H, K$ of $G$, clearly $[\varphi[H, K] = [\varphi(H), \varphi(K)]$, the result if obvious for $i = 2$. For $i > 2$, and by induction hypothesis,

\[
\gamma_i(\varphi(G)) = \gamma_i(G/N) = [\gamma_{i-1}(G/N), G/N] = [\gamma_{i-1}(\varphi(G), \varphi(G)]
\]

\[
= [\varphi(\gamma_{i-1}(G)), \varphi(G)] = \varphi(\gamma_{i-1}(G), G] = \varphi(\gamma_i(G)).
\]

\[\square\]

By Proposition 2.5.4 and induction on $n$, it is easy to see that if $|G| = p^n$ ($n \geq 2$) then $\gamma_n(G) = 1$, hence $G$ is nilpotent, with nilpotency class at most $n - 1$.

**Definition 2.5.5 (Group of Maximal Class).** A group of order $p^n$ ($n \geq 3$) is said to be of maximal class if its nilpotency class is $n - 1$.

Let $G$ be a non-abelian group of order $p^n$. Then $[G: \gamma_2(G)] \geq p^2$, and if $\gamma_i(G) \neq 1$ then $[\gamma_i(G): \gamma_{i+1}(G)] \geq p$. Therefore, if $G$ is of maximal class, then we must have

\[
[G: \gamma_2(G)] = p^2, \text{ and } [\gamma_i(G): \gamma_{i+1}(G)] = p \text{ for } 2 \leq i \leq n - 1.
\]
Similarly, we can show that, if $G$ is of maximal class and $|G| = p^n$, $n \geq 3$, then

$$|Z_i(G)| = p^i \text{ if } Z_i(G) \neq G, \ 0 \leq i \leq n-2.$$ 

Let $G$ be a non-abelian finite $p$-group. If $G$ is of maximal class, then $[G: G'] = p^2$. The converse is not true, in general, but it is true for $p = 2$.

**Theorem 2.5.6** (Olga Taussky, 1937). If $G$ is a finite non-abelian $2$-group with $[G: G'] = 4$, then $G$ is of maximal class, and it is isomorphic to one of the following:

- **Dihedral group:** $D_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$ ($n \geq 3$),
- **Quaternion group:** $Q_{2^n} = \langle x, y : x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \rangle$ ($n \geq 3$),
- **Semi-dihedral group:** $SD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$ ($n \geq 4$).

**Proof.** See Taussky [34], pp. 83-85 or Berkovich [3], Proposition 1.6, p. 26.

**Proposition 2.5.7.** Let $G$ be a $p$-group of maximal class. Then

1. $G$ has exactly $p + 1$ normal subgroups of index $p$ (i.e. maximal subgroups).
2. For $i \geq 2$, the only normal subgroup of index $p^i$ is $\gamma_i(G)$.
3. The upper and lower central series coincide: $\gamma_{n-k}(G) = Z_k(G)$, $0 \leq k \leq n-2$.

**Proof.** (1) $G/\gamma_2(G) \cong C_p \times C_p$. If $M$ is a maximal subgroup, then $G/M \cong C_p$, hence $\gamma_2(G) \leq M$. By correspondence theorem, $G$ has exactly $p + 1$ maximal subgroups.

(2) If $N$ is a normal subgroup of index $p^i$ ($i \geq 2$), then $|G/N| = p^i$, hence $\gamma_i(G/N) = 1$, i.e. $\gamma_i(G)N/N = 1$ (by Proposition 2.5.4), i.e. $\gamma_i(G) \subseteq N$. Since these subgroups have same index in $G$, we must have $N = \gamma_i(G)$.

(3) This follows from (2).

**Remark 2.5.8.** If $G$ is a $p$-group of maximal class and if $|G| = p^n$, then, by Proposition 2.5.7, the lattice of normal subgroups of $G$ is given by...
Conversely, if the lattice of normal subgroups of a non-abelian $p$-group is isomorphic to the above lattice, then by a theorem of H. Bender (1924), the group is of maximal class (see Bender [15], p. 428).

**Proposition 2.5.9.** Let $G$ be a $p$-group of maximal class with $|G| > p^3$. Then there exists a unique maximal subgroup $M$ such that $Z(G) < Z(M)$, and the centers of the other maximal subgroups coincide with the center of $G$.

**Proof.** Consider $x \in Z_2(G) \setminus Z(G)$. Then $xZ(G)$ is central in $G/Z(G)$. By Proposition 2.3.1,

$$p^{n-1} = |Z_{G/Z(G)}(xZ(G))| \leq |Z_G(x)| < |G| = p^n.$$ 

Hence $Z_G(x)$ is a maximal subgroup, and its center contains the subgroup $\langle x, Z(G) \rangle$, which properly contains $Z(G)$ (since $x \notin Z(G)$).

Let $M$ be any maximal subgroup. Since $|Z(G)| = p$ and $Z(G)$ intersects every normal subgroup of $G$ non-trivially, it follows that $Z(G) \subseteq M$, hence $Z(G) \subseteq Z(M)$. Suppose that $M_1, M_2$ are distinct maximal subgroups such that

$$Z(G) < Z(M_1) \text{ and } Z(G) < Z(M_2).$$

For $i = 1, 2$, since $Z(M_i)$ is characteristic in $M_i$ and $M_i \leq G$, hence $Z(M_i)$ is a normal subgroup of $G$ of order at least $p^2$ (since $Z(G) < Z(M_i)$). By Proposition 2.5.7, $Z_2(G) \subseteq Z(M_1) \cap Z(M_2)$, i.e. $Z_2(G)$ is central in $\langle M_1, M_2 \rangle = G$, a contradiction. This proves the theorem. \qed

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Remark 2.5.10. The first part of the Proposition 2.5.9 can also be proved in the following way: consider \( x \in Z_2(G) \setminus Z(G) \), and let \( C(x) \) denote the conjugacy class of \( x \) in \( G \). Since \( Z_2(G) \setminus Z(G) \) is invariant under conjugation, \( |C(x)| \) is at most \( |Z_2(G) \setminus Z(G)| = p^2 - p \) and it is a power of \( p \). Hence \( |C(x)| = 1 \) or \( p \). Since \( x \notin Z(G) \), we have \( |C(x)| = p \). Hence \( [G : Z_G(x)] = p \), and \( Z_G(x) \) is a maximal subgroup with the required property.

### 2.6 Extra-special \( p \)-Groups

**Definition 2.6.1 (Frattini Subgroup).** The Frattini subgroup of a group \( G \neq 1 \), denoted by \( \Phi(G) \), is defined as the intersection of all maximal subgroups of \( G \), and if \( G \) has no maximal subgroup, then we define \( \Phi(G) = G \).

For finite \( p \)-groups, the following theorem characterizes the Frattini subgroup.

**Theorem 2.6.2.** If \( G \neq 1 \) is a finite \( p \)-group, then \( \Phi(G) \) is the smallest normal subgroup of \( G \) such that \( G/\Phi(G) \) is elementary abelian \( p \)-group.

**Definition 2.6.3 (Special \( p \)-Group).** A \( p \)-group \( G \) is called a special \( p \)-group if \( Z(G) = G' = \Phi(G) \).

**Definition 2.6.4 (Extra-special \( p \)-Group).** A \( p \)-group \( G \) is said to be extra-special if \( Z(G) = G' = \Phi(G) \cong C_p \).

An interesting example of extra-special \( p \)-group is the group of \( n \times n \) upper unitriangular matrices over the field \( \mathbb{Z}/p\mathbb{Z} \), with arbitrary entries in first row and last column, and other entries 0.

\[
\begin{bmatrix}
1 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\
1 & 0 & \cdots & 0 & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & 0 & a_{n-2n} \\
1 & a_{n-1n} & \cdots & 0 & a_{n1} \\
\end{bmatrix} : a_{ij} \in \mathbb{Z}/p\mathbb{Z}
\]  

(2.13)
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Theorem 2.6.5. (1) A finite extra-special $p$-group has order $p^{2n+1}$, for some $n \geq 1$. 
(2) For every $n \geq 1$, there are exactly two extra-special $p$-groups of order $p^{2n+1}$, up to isomorphism:

$$E_1 = \langle x_1, y_1, \ldots, x_n, y_n, z : x_i^2 = y_i^2 = z = 1, [x_i, x_j] = [y_i, y_j] = 1, [x_i, y_j] = z^\delta_{i,j} \rangle.$$ $$E_2 = \langle x_1, y_1, \ldots, x_n, y_n, z : x_i^2 = 1, x_i^p = y_i^p = z, [x_i, x_j] = [y_i, y_j] = 1, [x_i, y_j] = z^\delta_{i,j} \rangle.$$  

where $1 \leq i \leq n$, $1 \leq j \leq n$, $\delta_{i,j} = 0$ if $i \neq j$, and $\delta_{i,i} = 1$.

Proof. (1) Let $G$ be any finite extra-special $p$-group. Now, $V = G/Z(G)$, being elementary abelian $p$-group, can be considered as a vector space over the finite field $\mathbb{F}_p$ of order $p$, and $\dim_{\mathbb{F}_p}(V) = k$ if $|G/Z(G)| = p^k$. Let $G' = \langle x \rangle$. Then for any $g, h \in G$, $[g, h] = x^i$ for some $i$, $0 \leq i < p$, and it is easy to see that the map

$$V \times V \rightarrow \mathbb{F}_p, \quad (gZ(G), hZ(G)) \mapsto i$$

is a non-degenerate alternating form on $V$. Hence $\dim_{\mathbb{F}_p}(V)$ must be even, say $2n$, $n \geq 1$. Therefore, $|G| = p, |V| = p^{2n+1}$.

(2) See Gorenstein [6], Theorem 5.2 or Leedham-Green, McKay [7], Lemma 2.2.9 and Theorem 2.2.10. \qed

For $p > 2$, the group $E_1$ has exponent $p$ and the group $E_2$ has exponent $p^2$. Also, the group $E_1$ is isomorphic to the matrix group in (2.13).

Proposition 2.6.6. A $p$-group $G$ is extra-special if and only if $G' = Z(G) \cong C_p$.

Proof. Let $G' = Z(G) \cong C_p$. We show that $G' = \Phi(G)$. Since $G' \subseteq Z(G)$, for any $x, y \in G$, $[x^p, y] = [x, y]^p = 1$ (by Proposition 2.2.1). Therefore $x^p \in Z(G) = G'$, and $G/G'$ is elementary abelian $p$-group. Hence $\Phi(G) \subseteq G'$ and for finite $p$-groups, we always have $G' \subseteq \Phi(G)$ (see Theorem 2.6.2), and the result follows. \qed

Proposition 2.6.7. Let $G$ be an extra-special group of order $p^{2n+1}$. Then,

1. for all $x \in G \setminus Z(G)$, $Z_G(x)$ is a maximal subgroup of $G$;
2. every maximal subgroup of $G$ is centralizer of an element;
3. for every $x \in G \setminus Z(G)$, $Z(Z_G(x)) = \langle x, Z(G) \rangle$.

Proof. (1) For $x \in G \setminus Z(G)$, since $G$ has nilpotency class 2, by identity (2.5), the map

$$\varphi_x: G \to G', \ g \mapsto [g, x].$$

is a homomorphism and $\ker(\varphi_x) = Z_G(x)$. Since $x \notin Z(G)$, there exists $g \in G$ such that $[g, x] \neq 1$. Therefore, $\operatorname{Im}(\varphi_x) = G' \cong C_p$ and $\ker(\varphi_x) = Z_G(x)$ is maximal subgroup.

(2) If $n = 1$, then $G$ is non-abelian group of order $p^3$, and every maximal subgroup is of the type $\langle x, Z(G) \rangle$ for some $x \notin Z(G)$. It is easy to see that $\langle x, Z(G) \rangle = Z_G(x)$. Suppose $n > 1$. Let $H$ be a maximal subgroup of $G$. Since $|Z(G)| = p$, we have $Z(G) \subseteq H$, hence $Z(G) \subseteq Z(H)$. Clearly, $H' \subseteq G' \cong C_p$. If $H' = 1$ (i.e. $H$ is abelian subgroup of index $p$), then by Theorem 2.2.2, $|G| = p, |Z(G)|, |G'| = p^3$, a contradiction. Hence, we must have $H' = G'$. If $Z(G) = Z(H)$, then

$$H' = G' = Z(G) = Z(H) \cong C_p.$$ 

By Proposition 2.6.6, $H$ is extra-special $p$-group of order $p^{2n}$, a contradiction (see Theorem 2.6.5). Hence, we must have, $Z(G) < Z(H)$. Consider $y \in Z(H) \setminus Z(G)$. Then $Z_G(y)$ is a maximal subgroup of $G$ (by (1)) and it contains $H$, hence $H = Z_G(y)$.

(3) Consider $x \in G \setminus Z(G)$. By (1), $Z_G(x)$ is a maximal subgroup. Consider $y \in G \setminus Z_G(x)$. Then $G = \langle Z_G(x), y \rangle$. Hence,

$$Z(G) = Z_G(G) = Z_G(\langle Z_G(x), y \rangle) = Z_G(Z_G(x)) \cap Z_G(y) = Z(Z_G(x)) \cap Z_G(y).$$

Now $Z_G(y)$ is a maximal subgroup and $Z(Z_G(x)) \nsubseteq Z_G(y)$ (since $x \notin Z_G(y)$). Hence $G = Z(Z_G(x))Z_G(y)$. By isomorphism theorem,

$$\frac{Z(Z_G(x))}{Z(G)} = \frac{Z(Z_G(x))}{Z(Z_G(x)) \cap Z_G(y)} \cong C_p$$

and hence $Z(Z_G(x)) = \langle x, Z(G) \rangle$. 

$\square$
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2.7 Isoclinic Groups

For a group $G$, consider the map $G \times G \rightarrow G'$, $(a, b) \mapsto [a, b] = a^{-1}b^{-1}ab$. If $s, t \in Z(G)$, then $[as, bt] = [a, b]$ for all $a, b \in G$. Hence, we have a natural commutator map

$$\eta: G/Z(G) \times G/Z(G) \rightarrow G', \quad (aZ(G), bZ(G)) \mapsto [a, b].$$

**Definition 2.7.1** (P. Hall, 1939). Let $G_1$ and $G_2$ be any groups. Let

$$\eta_1: G_1/Z(G_1) \times G_1/Z(G_1) \rightarrow G'_1 \quad \text{and} \quad \eta_2: G_2/Z(G_2) \times G_2/Z(G_2) \rightarrow G'_2$$

be the natural commutator maps as defined above. Then $G_1$ and $G_2$ are said to be isoclinic if

1. there is an isomorphism $\varphi: G_1/Z(G_1) \rightarrow G_2/Z(G_2),$
2. there is an isomorphism $\psi: G'_1 \rightarrow G'_2,$

such that the following diagram commutes:

$$\begin{array}{ccc}
G_1/Z(G_1) \times G_1/Z(G_1) & \xrightarrow{\varphi \times \varphi} & G_2/Z(G_2) \times G_2/Z(G_2) \\
\eta_1 \downarrow & & \eta_2 \downarrow \\
G'_1 & \xrightarrow{\psi} & G'_2
\end{array}$$

In this case, we call the pair $(\varphi, \psi)$ an isoclinism between $G_1$ and $G_2$.

It is easy to see that ‘being isoclinic’ is an equivalence relation on any family of groups. Also, any abelian group is isoclinic to the trivial group 1.

**Remark 2.7.2.** An isoclinism between two groups need not be induced by any homomorphism between the groups. As an example, the non-abelian groups of order 8 - dihedral group and quaternion group - are isoclinic, but there is no homomorphism between these two groups which induces an isoclinism between them.
From the definition of isoclinism, we see that isoclinic groups need not have the same order. But, there are many numerical as well as structural invariants associated to a family of isoclinic groups. We mention a few of them.

**Notation:** If $G$ is nilpotent, let $c(G)$ denote its nilpotency class. If $G$ is solvable, let $d(G)$ denote its derived class. Note that $c(G)$ is the length of the upper as well as the lower central series, and $d(G)$ is the length of the derived series of $G$.

1. Since $c(G) = 1 + c(G/Z(G))$, if $G_1$ and $G_2$ are isoclinic and $G_1$ is nilpotent, then $G_2$ is also nilpotent; moreover $c(G_1) = c(G_2)$.

2. Since $d(G) = 1 + d(G')$, if $G_1$ and $G_2$ are isoclinic and $G_1$ is solvable, then so is $G_2$ and $d(G_1) = d(G_2)$.

3. Let $G_1$ and $G_2$ be finite isoclinic groups, and $d$ a positive integer. For $i = 1, 2$, let $m_i$ be the number of irreducible linear representations of $G_i$ of degree $d$, over $\mathbb{C}$. Then,

$$m_1 |G_2| = m_2 |G_1| .$$

In particular, if $G_1$ and $G_2$ have the same order, then they have precisely the same degrees of irreducible representations (see P. Hall [24], p. 135 - 136).

4. Let $G_1$ and $G_2$ be finite isoclinic groups, and $c$ a positive integer. For $i = 1, 2$, denote by $n_i$ the number of conjugacy classes of $G_i$ of size $c$. Then,

$$n_1 |G_2| = n_2 |G_1| .$$

In particular, if $G_1$ and $G_2$ have the same order, then they have precisely the same conjugacy class sizes with multiplicities (see P. Hall [24], p. 135 - 136).

**Theorem 2.7.3** (P. Hall, 1939). *For any group $G$, there exists a group $G_1$ such that $G$ is isoclinic to $G_1$ and $Z(G_1) \subseteq [G_1, G_1].$*

*Proof.* For proof, see P. Hall [24], p. 135. \qed

**Proposition 2.7.4.** *If $G$ is a group with $[G: Z(G)] < \infty$, then $G$ is isoclinic to a finite group. In particular, if $[G: Z(G)] = p^k$, then $G$ is isoclinic to a finite $p$-group.*
Proof. Since $[G : Z(G)]$ is finite, by Schur’s theorem, $[G, G]$ is finite (see Suzuki [12], Theorem 9.8, p. 250). By Theorem 2.7.3, there exists a group $G_1$ isoclinic to $G$ such that $Z(G_1) \leq [G_1, G_1]$. Now,

$$|Z(G_1)| \leq |[G_1, G_1]| = |[G, G]| < \infty \text{ and } [G_1 : Z(G_1)] = [G : Z(G)] < \infty.$$  

It follows that $G_1$ is a finite group.

Now assume that $[G : Z(G)] = p^k$, and let $G_1$ a finite group isoclinic to $G$ (as seen above). Let $\varphi : G_1 \to G_1/Z(G_1)$ be the natural homomorphism. If $x \in G_1$ is an element of order relative prime to $p$, then, since $G_1/Z(G_1) \cong G/Z(G)$ is a $p$-group, it follows that $x \in \ker(\varphi) = Z(G_1)$. Thus every Sylow subgroup of $G_1$ of order prime to $p$ is central, and hence a direct factor of $G_1$. Thus $G_1 = P \times A$ where $P$ is a $p$-subgroup of $G_1$ and $A$ is an abelian subgroup of $G_1$ of order prime to $p$. Since $Z(G_1) \subseteq [G_1, G_1]$, we must have $A = 1$. Thus, $G$ is isoclinic to $G_1$ which is a finite $p$-group.  

\[ \square \]