5 Computations

In this chapter, we compute the number of $z$-classes in certain families of finite $p$-groups. The families of $p$-groups are chosen from the following observations: In a non-abelian group $G$ of order $p^3$, we always have $[G : Z(G)] = p^2$, hence $G$ has exactly $p + 2$ $z$-classes (by Theorem 4.2.3). However, as mentioned in Remark 4.2.5, any non-abelian group of order $p^4$ has exactly $p + 2$ $z$-classes. This important observation raises the following questions:

Question 1: What are the structural ingredients of non-abelian groups of order $p^4$ which force that they have exactly $p + 2$ $z$-classes?

Question 2: Does there exist a non-abelian group of order $p^5$ with more than $p + 2$ $z$-classes?

There are two important facts about the non-abelian groups $G$ of order $p^4$, which answer the question 1:

1. $[G : Z(G)]$ is $p^2$ or $p^3$.
2. $G$ always contains an abelian subgroup of index $p$ (see Theorem 2.4.1).

This suggests the following two families of $p$-groups:

**F1.** $p$-groups $G$ with $[G : Z(G)] = p^3$.

**F2.** $p$-groups containing an abelian subgroup of index $p$.

There are non-abelian groups of order $p^5$ with more than $p + 2$ $z$-classes, which answer question 2. In fact, we will explicitly determine the number of $z$-classes in groups of
order $p^5$. Also note that if $|G| = p^5$, since there exists an abelian normal subgroup $A$ of order $p^3$ (see Theorem 2.4.1), $G/A$ is abelian, hence $G' \subseteq A$ and $G'$ is abelian, i.e. $G$ is metabelian. Then, almost all groups of order $p^5$ fall in one of the following families:

1. $p$-groups with $[G: Z(G)] = p^3$,
2. $p$-groups containing an abelian subgroup of index $p$,
3. $p$-groups with commutator subgroup of order $p$,
4. metabelian $p$-groups of maximal class.

This suggest the following two new families of $p$-groups:

**F3.** $p$-groups $G$ with commutator subgroup of order $p$.

**F4.** metabelian $p$-groups of maximal (nilpotency) class.

We determine the $z$-classes in the families **F1-F4** of $p$-groups, mentioned above, and consequently, we will obtain the $z$-classes in all groups of order $p^5$.

### 5.1 $p$-Groups Containing Abelian Subgroup of Index $p$

**Theorem 5.1.1.** Let $G$ be a non-abelian $p$-group of order $p^n$. Suppose that $G$ has an abelian subgroup $A$ of index $p$.

1. If $G/Z(G)$ is abelian, then the number of $z$-classes of $G$ is $|G'| + 2$.
2. If $G/Z(G)$ is non-abelian, then $G$ has exactly $|Z(G/Z(G))| + 2$ $z$-classes.

**Proof.** By Theorem 3.2.5, $A$ is union of two $z$-classes. For $x \in G \setminus A$, by Theorem 2.3.5(i), $Z_G(x)Z(G) = \langle xZ(G) \rangle \cong C_p$, and hence

$$F'_x = \{ g \in G : Z_G(g) = Z_G(x) \} = Z_G(x) \setminus Z(G).$$

(1) If $G/Z(G)$ is abelian, then for $x \in G \setminus A$, $Z_G(x) \leq G$. Therefore, for any $x \in G \setminus A$,

$$|z\text{-class of } x| = |G : N_G(Z_G(x))|.|F'_x| = 1.(p - 1)|Z(G)|.$$
If $G \setminus A$ is union of $m$ $z$-classes, then

$$p^n - p^{n-1} = |G \setminus A| = m(p - 1)|Z(G)|.$$ 

Thus, $m = [G: Z(G)]/p = |G'|$ (by Theorem 2.2.2). Since $A$ is union of two $z$-classes, this proves (1).

(2) If $\overline{G} = G/Z(G)$ is non-abelian, then by Theorem 2.3.5(i) and (iii), for $x \in G \setminus A$,

$$[Z_G(x): Z(G)] = p$$

and $N_G(Z_G(x))/Z_G(x) \cong Z(\overline{G})$.

Let $[G: Z(G)] = p^k$. Then,

$$[N_G(Z_G(x)): Z(G)] = [N_G(Z_G(x)): Z_G(x)][Z_G(x): Z(G)] = |Z(\overline{G})|p$$


Hence $[G: N_G(Z_G(x))] = p^k/(p|Z(\overline{G})|)$, and for any $x \in G \setminus A$,

$$|z\text{-class of } x| = [G: N_G(Z_G(x))].|F'_x| = \frac{p^k}{p|Z(\overline{G})|}.(p - 1)|Z(G)| = \frac{p^n(p - 1)}{p|Z(\overline{G})|}.$$ 

If $G \setminus A$ is a union of $m$ $z$-classes, then

$$p^n - p^{n-1} = |G \setminus A| = m\frac{p^n(p - 1)}{p|Z(\overline{G})|},$$

hence $m = |Z(\overline{G})|$. Since $A$ is union of two $z$-classes, this proves (2). \qed

5.2 Groups with $[G: Z(G)] = p^3$

Let $G$ be any group. For $x \in G$, we call the subgroup $Z(Z_G(x))$, the critical abelian subgroup associated to the cyclic subgroup $\langle x \rangle$ (see Kulkarni [29], §3). Note that $Z(Z_G(x)) = Z_G(Z_G(x))$, the double centralizer of $x$ in $G$, and if $x \in Z(G)$, then $Z(Z_G(x)) = Z(G)$. Then we have,

**Lemma 5.2.1** (Kulkarni, 2007). *For any group $G$, there is a canonical bijection between the $z$-classes of $G$ and the conjugacy classes of critical abelian subgroups associated to cyclic subgroups of $G$, given by

$$z\text{-class of } x \mapsto \text{conjugacy class of } Z(Z_G(x)).$$*
Theorem 5.2.2. Let $G$ be a finite $p$-group with $[G: Z(G)] = p^3$. Suppose that $G$ has no abelian subgroup of index $p$. Then there is a canonical bijection between the $z$-classes of $G$ and the conjugacy classes of the cyclic subgroups of $G/Z(G)$. Hence,

1. If $G/Z(G)$ is abelian, then $G$ has $p^2 + p + 2$ $z$-classes.
2. If $G/Z(G)$ is non-abelian, then $G$ has $p + 3$ $z$-classes.

Proof. First we make two important observations.

1. $G/Z(G)$ has exponent $p$: if $L/Z(G)$ is a cyclic subgroup of order $p^2$ in $G/Z(G)$, then $L$ will be an abelian subgroup of index $p$, a contradiction to the hypothesis.
2. For $x \in G \setminus Z(G)$, clearly $Z_G(x) \supseteq \langle x, Z(G) \rangle$ and $[G: \langle x, Z(G) \rangle] = p^2$ (by (1)). If there is $y \notin \langle x, Z(G) \rangle$ such that $xy = yx$ then $\langle x, y, Z(G) \rangle$ will be an abelian subgroup of index $p$, a contradiction to the hypothesis. Therefore, for any $x \in G \setminus Z(G)$,

$$Z_G(x) = \langle x, Z(G) \rangle = Z(Z_G(x)),$$

which corresponds to cyclic subgroup of order $p$ in $G/Z(G)$ (under the natural homomorphism $G \rightarrow G/Z(G)$), and for $x \in Z(G), Z(Z_G(x)) = Z(G)$.

Since, by subgroup correspondence theorem, the conjugacy classes of subgroups of $G$ containing $Z(G)$ are in bijection with the conjugacy classes of subgroups of $G/Z(G)$, and by (1) the cyclic subgroups of $G/Z(G)$ are of order 1 or $p$, by Lemma 5.2.1, we have the following required bijection:

$$z\text{-class of } x \mapsto \text{conjugacy class of } Z(Z_G(x))/Z(G) \text{ (in } G/Z(G)).$$

Finally, by (1) and Corollary 2.4.4, $G/Z(G)$ is either elementary abelian or it is isomorphic to $U(3, \mathbb{F}_p)$ with $p > 2$. Now, (1) and (2) can be easily proved with above bijection.

Remark 5.2.3. From the Theorem 5.2.2, we see that, the groups satisfying the hypothesis of Theorem 5.2.2 attain the upper bound in the Corollary 3.3.7.
5.3 $p$-Groups with Commutator Subgroup of Order $p$

Let $G$ be a $p$-group with $|G'| = p$. By Theorem 2.7.3 of Hall, there is a group $H$, isoclinic to $G$, such that $Z(H) \subseteq H'$. It follows that $Z(H) = H' \cong C_p$, hence $H$ is extra-special (see Proposition 2.6.6). Thus, the groups with commutator subgroup of order $p$ are isoclinic to extra-special $p$-groups. Since isoclinic groups have same number of $z$-classes (see Corollary 3.4.2), it is sufficient to determine the $z$-classes in extra-special $p$-groups.

**Theorem 5.3.1.** If $G$ is a finite extra-special $p$-group, then there is a canonical bijection between the $z$-classes of $G$ and the cyclic subgroups of $G/Z(G)$ (including trivial subgroup). In particular, if $|G| = p^{2n+1}$, $n \geq 1$, then $G$ has exactly $\frac{p^{2n+1} - 1}{p-1} + 1$ $z$-classes.

**Proof.** $|Z(G)| = p$, and $G/Z(G)$ is elementary abelian. If $x \notin Z(G)$, then by Proposition 2.6.7, $Z(Z_G(x)) = \langle x, Z(G) \rangle$ is a (normal) subgroup of order $p^2$, which corresponds to a subgroup of order $p$ in $G/Z(G)$, and if $x \in Z(G)$, then $Z(Z_G(x)) = Z(G)$. The theorem follows by Lemma 5.2.1.

5.4 Metabelian $p$-Groups of Maximal Class

In this section, we determine the $z$-classes in $p$-groups of maximal class which are metabelian, i.e. commutator subgroup is abelian. Since non-abelian groups of order $p^3$ and $p^4$ have exactly $p + 2$ $z$-classes, we consider the groups of order greater than $p^4$.

Let $G$ be a $p$-group of maximal class, and assume that $|G| > p^4$. By Proposition 2.5.7, there are exactly $p + 1$ maximal subgroups of $G$. Let $M_0, M_1, \ldots, M_p$ be the maximal subgroups of $G$. By Proposition 2.5.9, there is unique maximal subgroup, say $M_0$, such that $Z(G) < Z(M_0)$. If $Z(M_0) = M_0$, then by Theorem 4.2.4, $G$ has $p + 2$ $z$-classes. The next possible case for $Z(M_0)$ is $[M_0: Z(M_0)] = p^2$.

**Theorem 5.4.1.** Let $G$ be a metabelian $p$-group of maximal class, of order $p^n$, $n > 4$. Assume that $G$ has no abelian subgroup of index $p$. If $[M_0: Z(M_0)] = p^2$, then $G$ has exactly $p + 4$ $z$-classes.
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Proof. Since 2-groups of maximal class contain a cyclic subgroup of index 2 (see Theorem 2.5.6), by hypothesis, we have \( p > 2 \).

\( M_0 \) is the maximal subgroup of \( G \) such that \( Z(G) < Z(M_0) \). Let \( M_1, M_2, \ldots, M_p \) be the other maximal subgroups of \( G \) (see Proposition 2.5.7). Now, \([M_0 : Z(M_0)] = p^2\), hence

\( [G : Z(M_0)] > [M_0 : Z(M_0)] = p^2 \) and \( |Z(M_0)| \geq p^{n-3} \geq p^2 \).

Since \( Z(M_0) \) is characteristic subgroup of \( M_0 \) and \( M_0 \lneq G \), hence \( Z(M_0) \leq G \). Thus, \( Z(M_0) \) is a normal subgroup of \( G \) of index at least \( p^3 \), and order at least \( p^2 \). By Proposition 2.5.7, \( Z(M_0) \) is one of

\[ \gamma_3(G), \gamma_4(G), \ldots, \gamma_{n-2}(G), \]

and since \( Z(G) = \gamma_{n-1}(G) \), we have

\( Z(G) < Z(M_0) < \gamma_2(G) < M_0 \).

Claim 1: \( Z(M_0) \setminus Z(G) \) and \( \gamma_2(G) \setminus Z(M_0) \) are \( z \)-class.

Clearly, all the elements of \( Z(M_0) \setminus Z(G) \) have centralizer equal to \( M_0 \leq G \), and if \( g \in G \) has centralizer equal to \( M_0 \), then \( g \notin Z(G) \) and \( g \) will be central in \( M_0 \), i.e. \( g \in Z(M_0) \setminus Z(G) \). Hence \( Z(M_0) \setminus Z(G) \) is a \( z \)-class. By similar arguments, \( \gamma_2(G) \setminus Z(M_0) \) is a \( z \)-class.

Claim 2: \( M_i \setminus \gamma_2(G) \) is a \( z \)-class for \( 1 \leq i \leq p \).

Fix \( i \) (\( 1 \leq i \leq p \)). Then \( Z(M_i) = Z(G) \) (by Proposition 2.5.9). Consider \( x \in M_i \setminus \gamma_2(G) \). Then \( Z_G(x) < G \). If \( Z_G(x) \subseteq M_k \) for some \( k \neq i \), then \( x \in M_i \cap M_k = \gamma_2(G) \), a contradiction. If \( Z_G(x) = M_i \), then \( Z(M_i) \supseteq \langle x, Z(G) \rangle > Z(G) \), a contradiction. Thus, \( Z_G(x) < M_i \). In particular, \( Z_G(x) = Z_{M_i}(x) \). Also, if \( Z_{M_i}(x) \subseteq G \), then \( Z_{M_i}(x) \) is a normal subgroup of index at least \( p^2 \) in \( G \). By Proposition 2.5.7, \( Z_{M_i}(x) \leq \gamma_2(G) \), and hence \( x \in \gamma_2(G) \), a contradiction. Therefore \( Z_{M_i}(x) \notin G \), i.e. \( N_G(Z_{M_i}(x)) < G \). Again, it is easy to see that \( N_G(Z_{M_i}(x)) \subseteq M_i \). Therefore,

\[ N_G(Z_G(x)) = N_G(Z_{M_i}(x)) = N_{M_i}(Z_{M_i}(x)) \]

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Applying Theorem 2.3.5 to $M_i$, we obtain

$$(e_1) \quad Z_G(x) = Z_{M_i}(x) = \langle x, Z(M_i) \rangle = \langle x, Z(G) \rangle,$$

$$(e_2) \quad Z_G(x)/Z(G) = Z_{M_i}(x)/Z(M_i) \cong C_p,$$ and

$$(e_3) \quad N_G(Z_G(x))/Z_G(x) = N_{M_i}(Z_{M_i}(x))/Z_{M_i}(x) = Z(M_i/Z(M_i)) \cong C_p.$$ 

Then, by $(e_1)$ and $(e_2)$,

$$F'_x = \{ y \in G : Z_G(y) = Z_G(x) \} = Z_G(x) \setminus Z(G) \text{ and } |F'_x| = (p-1)|Z(G)|,$$

and by $(e_2)$ and $(e_3)$, $[G : N_G(Z_G(x))] = p^{n-3}$. Therefore,

$$|z\text{-class of } x| = |R(x)| = [G : N_G(Z_G(x))]|F'_x| = p^{n-3}p(p-1).$$

Now, by Claim 1, $\gamma_2(G)$ is union of the following $z$-classes:

$$Z(G) \cup (Z(M_0) \setminus Z(G)) \cup \gamma_2(G) \setminus Z(M_0).$$

For any $x \in M_i \setminus \gamma_2(G)$, as seen above, $Z_G(x) \subseteq M_i$. Hence by Proposition 3.2.2(1), $R(x) \subseteq M_i$ and $R(x) \cap \gamma_2(G) = \phi$ since $x$ is not in $\gamma_2(G)$ which is a union of $z$-classes, i.e. $R(x) \subseteq M_i \setminus \gamma_2(G)$, and since both the sets have same size,

$$z\text{-class of } x = M_i \setminus \gamma_2(G) \quad (1 \leq i \leq p).$$

This proves Claim 2.

Note that $G = \gamma_2(G) \cup (\cup_{i=0}^{p-1} M_i \setminus \gamma_2(G))$. By Claim 1, $\gamma_2(G)$ is a union of, and for $i \geq 1$, $M_i \setminus \gamma_2(G)$ is a $z$-class. Hence $M_0 \setminus \gamma_2(G)$ is a union of $z$-classes.

**Claim 3:** $M_0 \setminus \gamma_2(G)$ is a $z$-class:

As seen above, $[M_0 : Z(M_0)] = p^2$. Consider $x \in M_0 \setminus \gamma_2(G)$. It is easy to see that

$$Z_G(x) = \langle x, Z(M_0) \rangle, \text{ and } Z_G(x)/Z(M_0) \cong C_p.$$ 

Since $Z(M_0)$ is a union of $z$-classes different from $R(x)$, we have

$$F'_x = \langle x, Z(M_0) \rangle \setminus Z(M_0).$$
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Since, \([G: Z_G(x)] = p^2\), we have \(Z_G(x) \not\trianglelefteq G\) (otherwise by Proposition 2.5.7, \(Z_G(x) \subseteq \gamma_2(G), x \in \gamma_2(G)\)). Hence \(Z_G(x) < N_G(Z_G(x)) < G\), and \([G: N_G(Z_G(x))]) = p\). Thus,

\[
|z\text{-class of } x| = |G: N_G(Z_G(x))| |F'_{x}| = p \cdot p^{n-2}(p - 1) = |M_0 \setminus \gamma_2(G)|.
\]

Since \(R(x) \subseteq M_0 \setminus \gamma_2(G)\) and both the sets have same size, it follows that \(M_0 \setminus \gamma_2(G)\) is az-class.

Thus, \(G\) has \(p + 4\) \(z\)-classes:

\[
Z(G), Z(M_0) \setminus Z(G), \gamma_2(G) \setminus Z(M_0), \text{ and } M_i \setminus \gamma_2(G) \quad (0 \leq i \leq p).\]

\[\square\]

Example 5.4.2. We give examples of \(p\)-groups of maximal class having \(p + 4\) \(z\)-classes.

For \(2 < r < p\), consider the group

\[
G = \langle x_1, \ldots, x_r, y, z : x_i^p = y^p = 1 = z^p = [x_i, x_j], (1 \leq i, j \leq r) \quad y^{-1}x_1y = x_1, \quad y^{-1}x_iy = x_{i-1}x_i (2 \leq i \leq r) \quad z^{-1}x_1z = x_1, \quad z^{-1}x_i = x_{i-1}x_i (2 \leq i < r) \quad z^{-1}x_rz = x_1x_{r-1}x_r, \quad z^{-1}yz = yx_r \rangle.
\]

Then \(|G| = p^{r+2} < p^{p+2}\). Note that, if \(r = 2\), then \(|G| = p^4\) and \(G\) will have \(p + 2\) \(z\)-classes. The lower central series of \(G\) is

\[
G > \langle x_1, \ldots, x_r \rangle > \langle x_1, \ldots, x_{r-1} \rangle > \cdots > \langle x_1 \rangle > 1 = \gamma_{r+2}(G),
\]

hence \(G\) is of maximal class. Also, \(\gamma_2(G)\) is elementary abelian. There are exactly \(p + 1\) maximal subgroups of \(G\), given by

\[
\langle \gamma_2(G), y \rangle, \langle \gamma_2(G), z \rangle, \langle \gamma_2(G), yz \rangle, \cdots, \langle \gamma_2(G), y^{p-1}z \rangle.
\]

Among these maximal subgroups, first \(p\) subgroups are of maximal class, and \(M = \langle \gamma_2(G), y^{p-1}z \rangle = \langle x_1, x_2, \ldots, x_r \rangle \cong \langle y^{p-1}z \rangle\) is not of maximal class; here the action of \(y^{p-1}z\) on the elementary abelian subgroup (considered as a vector space over \(\mathbb{F}_p\)) is given
by the linear transformation

\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_{r-1} \\
x_r
\end{pmatrix} \mapsto \begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
\vdots \\
x_{r-1} \\
x_r
\end{pmatrix} = \begin{pmatrix}
x_1 \\
\vdots \\
x_{r-1} \\
x_1 + x_r
\end{pmatrix}.
\]

Since, \(y^{p-1}z\) fixes \(x_1, x_2, \ldots, x_{r-1}\), \(Z(M) = \langle x_1, x_2, \ldots, x_{r-1}\rangle > Z(G)\) (since \(r > 2\)), and \([M : Z(M)] = p^2\). By Theorem 5.4.1, \(G\) has \(p + 4\) \(z\)-classes.

**Remark 5.4.3.** In the above theorem, the conclusion that

\[Z(G), Z(M_0) \setminus Z(G), \gamma_2(G) \setminus Z(M_0), \text{ and } M_i \setminus \gamma_2(G) \ (1 \leq i \leq p)\]

are \(z\)-classes in \(G\), is independent of the condition \([M_0 : Z(M_0)] = p^2\).

With the same notations as in Theorem 5.4.1, we consider the general case

\([M_0 : Z(M_0)] = p^i, 2 \leq i \leq n - 3.\)

Note that, \(M_0\) is the unique maximal subgroup of \(G\) such that \(Z(G) < Z(M_0)\). Therefore, if \([M_0 : Z(M_0)] = p^{n-2}\) then \(|Z(M_0)| = p\), a contradiction.

Since \(Z(M_0)\) is a normal subgroup of \(G\) of index \(p^{i+1} \geq p^3\), and \(Z(M_0) \neq Z(G) = \gamma_{n-1}(G)\). By Proposition 2.5.7, \(Z(M_0)\) is one of

\[\gamma_3(G), \gamma_4(G), \ldots, \gamma_{n-2}(G).\]

Similarly, \(\gamma_2(M_0)\) is one of the

\[\gamma_3(G), \gamma_4(G), \ldots, \gamma_{n-1}(G).\]

In particular, either \(Z(M_0) < \gamma_2(M_0)\) or \(\gamma_2(M_0) \leq Z(M_0)\).

Since \([M_0 : Z(M_0)] = p^i, 2 \leq i \leq n - 3\), we have \(|Z(M_0)| = p^{n-i-1}\). Since \(M_0\) has an abelian subgroup of index \(p\), namely \(\gamma_2(G)\) (by hypothesis), by Theorem 2.2.2,

\[p^{n-1} = |M_0| = p.|Z(M_0)|.\gamma_2(M_0)| = p^{n-i}.\gamma_2(M_0)|.\]
Thus, \(|\gamma_2(M_0)| = p^{i-1}\), and \(Z(M_0) = p^{n-i-1}\), \(2 \leq i \leq n - 3\). Then,

\[
Z(M_0) < \gamma_2(M_0) \iff n < 2i
\]

and \(\gamma_2(M_0) \leq Z(M_0) \iff 2i \leq n\).

**Theorem 5.4.4.** Let \(G\) be a metabelian \(p\)-group of maximal class, of order \(p^n\). If \(M_0\) is the maximal subgroup of \(G\) such that \([M_0: Z(M_0)] = p^i\), \(2 \leq i \leq n - 3\), then the number of \(z\)-classes of \(G\) is

1. \(p^{i-2} + p + 3\) if \(\gamma_2(M_0) \leq Z(M_0)\)

2. \(\frac{|Z(M_0/Z(M_0))|}{p} + p + 3\) if \(Z(M_0) < \gamma_2(M_0)\).

**Proof.** Note that \(M_0\) is the unique maximal subgroup of \(G\) such that \(Z(G) < Z(M_0)\). If \(A\) is an abelian subgroup of index \(p\) in \(G\), then \(Z(A) = A > Z(G)\), hence \(A = M_0\). This contradicts the hypothesis that \([M_0: Z(M_0)] \geq p^2\). Thus, \(G\) has no abelian subgroup of index \(p\). Since groups of order \(p^4\) contain an abelian subgroup of index \(p\) (see Theorem 2.4.1), we must have \(n \geq 5\). By Remark 5.4.3, we already have obtained the following \(p + 3\) \(z\)-classes of \(G\):

\[
Z(G), Z(M_0) \setminus Z(G), \gamma_2(G) \setminus Z(M_0) \text{ and } M_i \setminus \gamma_2(G) \quad (1 \leq i \leq p) \quad (*)
\]

It remains to determine \(z\)-classes in \(M_0 \setminus \gamma_2(G)\).

Fix \(x \in M_0 \setminus \gamma_2(G)\). It is easy to see that \(Z_G(x) \subseteq M_o\), hence \(Z_G(x) = Z_{M_0}(x)\). By Theorem 2.3.5(i),

\[
Z_G(x) = Z_{M_0}(x) = \langle x, Z(M_0) \rangle.
\]

It follows that

\[
F'_x = \langle x, Z(M_0) \rangle \setminus Z(M_0), \text{ and } |F'_x| = (p - 1)|Z(M_0)|.
\]

**Case 1.** \(\gamma_2(M_0) \leq Z(M_0)\).

Then \(Z_G(x) = Z_{M_0}(x) \subseteq M_0\). As \(x \notin \gamma_2(G)\) and \(Z(M_0) \subseteq \gamma_2(G)\), \(x\) is non-central in \(M_0\), hence \(Z_{M_0}(x) < M_0\). If \(Z_G(x) \trianglelefteq G\), then it will be a normal subgroup of index at
least \( p^2 \) in \( G \), and by Proposition 2.5.7, it will be a subgroup of \( \gamma_2(G) \), a contradiction to the choice of \( x \). Hence, \( N_G(Z_G(x)) = M_0 \), and
\[
|R(x)| = |G : N_G(Z_G(x))|, |F'_x| = p.(p - 1)|Z(M_0)| = p(p - 1)p^{n-1-1}.
\]

If \( M_0 \setminus \gamma_2(G) \) is a union \( m \) \( z \)-classes, then
\[
p^{n-1} - p^{n-2} = m.p^{n-i}(p - 1), \text{ hence } m = p^{i-2}.
\]

Thus \( G \) has \( p^{i-2} + p + 3 \) \( z \)-classes.

**Case 2.** \( Z(M_0) < \gamma_2(M_0) \).

As in the proof of Claim 2, Theorem 5.4.1, we can show that
\[
Z_G(x) = Z_{M_0}(x) \quad \text{and} \quad N_G(Z_G(x)) = N_{M_0}(Z_{M_0}(x))
\]

Since \( M_0 \) has an abelian subgroup of index \( p \) (namely \( \gamma_2(G) \)) and \( M_0/Z(M_0) \) is non-abelian, by Theorem 2.3.5(iii),
\[
[Z_{M_0}(x) : Z(M_0)] = p \quad \text{and} \quad N_{M_0}(Z_{M_0}(x))/Z_{M_0}(x) \cong Z(M_0/Z(M_0)).
\]

Thus, \( |N_G(Z_G(x))| = |N_{M_0}(Z_{M_0}(x))| = |Z(M_0/Z(M_0)|.p.|Z(M_0)| \), and
\[
|R(x)| = |G : N_G(Z_G(x))|, |F'_x|
= \frac{|G|}{|Z(M_0/Z(M_0)|.p.|Z(M_0)|}.(p - 1)|Z(M_0)|
\]

If \( M_0 \setminus \gamma_2(G) \) is union of \( m \) \( z \)-classes, then
\[
p^{n-1} - p^{n-2} = m.\frac{(p - 1)|G|}{p.|Z(M_0/Z(M_0))|}
\]

Hence \( m = \frac{1}{p}.|Z(M_0/Z(M_0))| \), and these \( z \)-classes together with the \( p + 3 \) \( z \)-classes in (*) give all the \( z \)-classes of \( G \). \( \square \)

## 5.5 \( z \)-Classes in Groups of Order \( p^5 \)

As a consequence of the results in Section 5.1-5.4, we obtain the number of \( z \)-classes in groups of order \( p^5 \), as given in the following table.
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Theorem 5.5.1. The number of z-classes in non-abelian groups of order \( p^5 \) is given by the following table.

<table>
<thead>
<tr>
<th></th>
<th>([G : Z(G)])</th>
<th>(c(G))</th>
<th>Additional Hypothesis</th>
<th>no. of z-classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1</td>
<td>1</td>
<td>–</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>(p^2)</td>
<td>2</td>
<td>–</td>
<td>(p + 2)</td>
</tr>
<tr>
<td>3.</td>
<td>(p^3)</td>
<td>2</td>
<td>(G) has abelian subgroup of index (p)</td>
<td>(p^2 + 2)</td>
</tr>
<tr>
<td>4.</td>
<td>(p^3)</td>
<td>3</td>
<td>(G) has no abelian subgroup of index (p)</td>
<td>(p + 3)</td>
</tr>
<tr>
<td>6.</td>
<td>(p^4)</td>
<td>2</td>
<td>–</td>
<td>(\frac{p^4-1}{p-1} + 1)</td>
</tr>
<tr>
<td>7.</td>
<td>(p^4)</td>
<td>3</td>
<td>(G) has abelian subgroup of index (p)</td>
<td>(3p + 2)</td>
</tr>
<tr>
<td>8.</td>
<td>(p^4)</td>
<td>4</td>
<td>(G) has no abelian subgroup of index (p)</td>
<td>(p + 2)</td>
</tr>
<tr>
<td>9.</td>
<td>(p^4)</td>
<td>4</td>
<td>(G) has abelian subgroup of index (p)</td>
<td>(p + 4)</td>
</tr>
</tbody>
</table>

Before starting the proof, we make important remarks.

(1) In Cases 1, 2 and 3 in the table, from the corresponding values of \([G : Z(G)]\) and \(c(G)\), it turns out that \(G\) contains an abelian subgroup of index \(p\). Similarly, in Cases 6 and 7, it turns out that \(G\) has no abelian subgroup of index \(p\) (see the proof of the Theorem).

(2) Since, isoclinic groups have the same number of z-classes, it is sufficient to determine the number of z-classes in a suitable representative a family of isoclinic groups. Philip Hall has been completely classified the groups of order \(p^5\), up to isoclinism (see Hall [24], §4, p. 136 - 140). According to his classification,

1. There are eight families of groups of order \(2^5\), up to isoclinism;

2. For \(p > 2\), there are ten families of groups of order \(p^5\), up to isoclinism.

However, we do not use this classification. We consider the cases according to

(a) \([G : Z(G)]\),

(b) the nilpotency class of \(G\),

(c) the presence or absence of abelian subgroup of index \(p\) (if necessary).
We will arrive at nine different families of groups of order $p^5$ as shown in the table, and in fact, the groups in different families are non-isoclinic, since the above three parameters, (a), (b), (c), are invariant for any family of isoclinic groups.

But as mentioned above, for $p = 2$, there are eight families of groups of order $2^5$, up to isoclinism, whereas, we have nine families of groups, according to the three parameters. Which case do not occur for $p = 2$? It is well known that the 2-groups of maximal class always contain a cyclic subgroup of index 2 (see Theorem 2.5.6). Thus, Case 9 will not occur for $p = 2$.

In particular, a non-abelian group $G$ of order $p^5$ is uniquely determined, up to isoclinism, by the three parameters: $[G : Z(G)]$, the nilpotency class $c(G)$, and the presence or absence of abelian subgroup of index $p$.

This is a joint work with Vikas Jadhav.

**Proof of Theorem 5.5.1:**

**Case 1:** $[G : Z(G)] = 1$, i.e. $G$ is abelian.

Clearly, $G$ has only one $z$-class.

**Case 2:** $[G : Z(G)] = p^2$.

By Theorem 4.2.3, $G$ has exactly $p + 2$ $z$-classes.

**Case 3:** $[G : Z(G)] = p^3$, and $c(G) = 2$, i.e. $G/Z(G)$ is abelian.

We show that $G$ contains an abelian subgroup of index $p$. By Corollary 2.4.4, $G/Z(G)$ is elementary abelian.

**Claim:** $G' = Z(G)$.

If $G' < Z(G)$, then $|G'| = p$. Consider $x \in G \setminus Z(G)$. Since $c(G) = 2$, it follows that

$$
\varphi_x : G \rightarrow G', \ g \mapsto [g, x]
$$

is a homomorphism. Since $x \notin Z(G)$, there exists $g \in G$ such that $[g, x] \neq 1$, and since $|G'| = p$, it follows that $\varphi_x$ is surjective. Therefore, $\ker(\varphi_x) = Z_G(x)$ is a maximal subgroup. Now, $|Z_G(x)/Z(G)| = p^2$. If $Z_G(x)/Z(G) \cong C_{p^2}$, then $Z_G(x)$ is abelian, and by Theorem 2.2.2,

$$
$$

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a contradiction. If $Z_G(x)/Z(G) \cong C_p \times C_p$, then $Z_G(x)/Z(G) = \langle xZ(G), yZ(G) \rangle$ for some $y \in Z_G(x) \setminus Z(G)$. Then $Z_G(x) = \langle x, y, Z(G) \rangle$, which is an abelian subgroup of index $p$, leading to a contradiction as before. Therefore, $G' = Z(G)$, as required.

Consider a subgroup $K$ of $G' = Z(G)$, with $|K| = p$. Then $G/K$ is a non-abelian group of order $p^4$ and $\gamma_2(G/K) = \gamma_2(G)/K \cong C_p$, hence $Z(G/K)$ has order $p^2$ (by Theorem 2.2.2). Let $H \leq G$ be such that $H/K = Z(G/K)$. Then $|H| = p^3$ and $Z(G) < H$. For $h \in H \setminus Z(G)$, $hK \in Z(G/K)$, and by Proposition 2.3.1

$$p^5 > |Z_G(h)| \geq |Z_{G/K}(hK)| = p^4.$$ 

Hence, $Z_G(h)$ is a maximal subgroup, and as in the proof of the above claim, it is abelian. By Theorem 5.1.1(1), $G$ has $|G'| + 2 = p^2 + 2 \ z$-classes.

**Case 4:** $[G': Z(G)] = p^3$, $c(G) = 3$, and $G$ has an abelian subgroup of index $p$.

Since $G/Z(G)$ is non-abelian group of order $p^3$, by Theorem 4.2.4, $G$ has $p + 2 \ z$-classes.

**Case 5:** $[G': Z(G)] = p^3$, $c(G) = 3$, and $G$ has an abelian subgroup of index $p$.

By Theorem 5.2.2(2), $G$ has $p + 3 \ z$-classes.

**Case 6:** $[G': Z(G)] = p^4$, $c(G) = 2$.

Here $|Z(G)| = p$, and $G/Z(G)$ is abelian. Hence $G' \subseteq Z(G)$. It follows that, $G' = Z(G)$, hence $G$ is extra-special $p$-group (see Proposition 2.6.6). By Theorem 5.3.1, $G$ has $\frac{p^4 - 1}{p - 1} + 1 \ z$-classes.

**Case 7:** $[G': Z(G)] = p^4$, $c(G) = 3$.

Here $\bigbar{G} = G/Z(G)$ is non-abelian group of order $p^4$ and of class 2. By Theorem 2.4.1, $\bigbar{G}$ contains an abelian subgroup of index $p$. By Theorem 2.2.2 to $\bigbar{G}$,

either $|\gamma_2(\bigbar{G})| = p$, $|Z(\bigbar{G})| = p^2$ or $|\gamma_2(\bigbar{G})| = p^2$, $|Z(\bigbar{G})| = p$.

Since in any $p$-group, the center intersects any normal subgroup non-trivially, we have either $\gamma_2(\bigbar{G}) < Z(\bigbar{G})$ or $Z(\bigbar{G}) < \gamma_2(\bigbar{G})$. Only in the first case, $\bigbar{G}$ is of class 2. Let $\varphi: G \to \bigbar{G}$ be the natural homomorphism. Then

$$|Z_2(G)/Z(G)| = |Z(\bigbar{G})| = p^2 \text{ and } |\gamma_2(G)/Z(G)| = |\gamma_2(G/Z(G))| = |\gamma_2(\bigbar{G})| = p.$$
Since $|Z(G)| = p$, we obtain $|Z_2(G)| = p^3$ and $|\gamma_2(G)| = p^2$. In particular, by Theorem 2.2.2, $G$ has no abelian subgroup of index $p$. Also, since $\gamma_2(G) < Z(G)$, we have $Z(G) < \gamma_2(G) < Z_2(G)$. Now, $[Z_2(G), \gamma_2(G)] = 1$ (Suzuki [13], Corollary 2, p. 20), therefore, $Z_2(G)$ is a group of order $p^3$ with center containing $\gamma_2(G)$, which has order $p^2$. It follows that $Z_2(G)$ is abelian. Summarizing the discussion,

(i) $|Z(G)| = p$, $|\gamma_2(G)| = p^2$, $|Z_2(G)| = p^3$.

(ii) $Z(G) < \gamma_2(G) < Z_2(G)$.

(iii) $Z_2(G)$ is an abelian subgroup (of order $p^3$).

(iv) $G$ has no abelian subgroup of index $p$.

Consider $y_0 \in \gamma_2(G) \setminus Z(G)$. Since $Z(G), \gamma_2(G) \trianglelefteq G$ and $[\gamma_2(G): Z(G)] = p$, $G$ acts by conjugation on the $p$ cosets of $Z(G)$ in $\gamma_2(G)$:

$$G \times \gamma_2(G)/Z(G) \to \gamma_2(G)/Z(G), \quad (g, xZ(G)) \mapsto gxg^{-1}Z(G).$$

Since the trivial coset is fixed, and $G$ is a $p$-group, the action must be trivial. Hence $gy_0g^{-1}Z(G) = y_0Z(G)$ for any $g \in G$. It follows that the conjugacy class of $y_0$ is $y_0Z(G)$, which has order $p$, hence $Z_G(y_0)$ is maximal subgroup. By (iv), $Z_G(y_0)$ is non-abelian group of order $p^4$, and its center necessarily contains $\langle y_0, Z(G) \rangle$. In fact,

$$Z(Z_G(y_0)) = \langle y_0, Z(G) \rangle = \gamma_2(G),$$

otherwise, $[Z_G(y_0): Z(Z_G(y_0))] \leq p$ and $Z_G(y_0)$ will be abelian, a contradiction. Also, note that $Z_G(y_0) \supseteq Z_2(G)$, since $Z_2(G)$ is abelian and $y_0 \in Z_2(G)$.

Next, we consider $y_1 \in Z_2(G) \setminus \gamma_2(G)$. As above, $Z_G(y_1)$ is a maximal subgroup, containing $Z_2(G)$, and $Z(Z_G(y_1)) = \langle y_1, Z(G) \rangle$. In particular, $Z_G(y_1) \neq Z_G(y_0)$. Continuing this process, we obtain $p + 1$ distinct centralizers $Z_G(y_i)$, $i = 0, 1, \cdots, p$, which are maximal subgroups of $G$, containing $Z_2(G)$, such that their centers are distinct subgroups of order $p^2$ in $Z_2(G)$. Moreover,

$$Z_2(G) = \bigcup_{i=0}^{p} Z(Z_G(y_i)).$$

The above discussion shows that
(v) $Z_2(G)/Z(G) \cong C_p \times C_p$, hence $G$ has exactly $p+1$ normal subgroups of order $p^2$, given by $Z(Z_G(y_i)) = \langle y_i, Z(G) \rangle$, $0 \leq i \leq p$.

Now $Z_G(y_0)/Z(Z_G(y_0)) \cong C_p \times C_p$, hence $Z_G(y_0)$ has $p+1$ subgroups of index $p$ containing $Z(Z_G(y_0)) = \gamma_2(G)$, and one of them is $Z_2(G)$, let $A_1, \cdots, A_p$ be the remaining subgroups. If $A$ is any abelian normal subgroup of $G$ of order $p^3$, then $G/A$ is abelian, hence $\gamma_2(G) \subseteq A$. Since $y_0 \in \gamma_2(G) = Z(Z_G(y_0))$, $Z_G(y_0) \supseteq A$, and hence $A$ is one of $Z_2(G)$, $A_1, \cdots, A_p$. Thus,

(vi) $G$ has exactly $p+1$ abelian normal subgroups of order $p^3$: $Z_2(G), A_1, \cdots, A_p$.

We have the following sub-lattice of subgroups of $G$. 

For $0 \leq i \leq p$, $Z_G(y_i)$’s are normal in $G$, hence

$$\text{z-class of } y_i = \{ g \in G : Z_G(g) = Z_G(y_i) \}$$

$$\subseteq Z(Z_G(y_i)) \setminus Z(G)$$

$$= \bigcup_{j=1}^{p-1} y_j^i Z(G) \subseteq \text{z-class of } y_i$$

Thus $Z(Z_G(y_i)) \setminus Z(G)$ is a z-class and $Z_2(G)$ is union of $p+2$ z-classes:

$$Z_2(G) = Z(G) \cup \bigcup_{i=0}^{p} \text{z-class of } y_i.$$ 

For $1 \leq i \leq p$, consider $a_i \in A_i \setminus Z_2(G) = A_i \setminus \gamma_2(G)$.

**Claim 1:** $A_i \setminus \gamma_2(G)$ is a z-class:

Since $a_i \in A_i$ and $A_i$ is abelian, $|Z_G(a_i)| \geq |A_i| \geq p^3$. If $|Z_G(a_i)| = p^4$. By (iv),
$Z_G(a_i)$ is a non-abelian group (of order $p^4$), hence, $Z(Z_G(a_i)) = \langle a_i, Z(G) \rangle$. By (v), $Z(Z_G(a_i)) = Z(Z_G(y_j))$ for some $j$, $0 \leq j \leq p$. But then, $a_i \in Z_2(G) \cap A_i = \gamma_2(G)$, a contradiction. Therefore, $|Z_G(a_i)| = p^3 = |A_i|$, and

$$Z_G(a_i) = A_i, \ 1 \leq i \leq p.$$ 

$A_i$ is an abelian normal subgroup (of order $p^3$), and by (iv), it is a maximal abelian normal subgroup. By Theorem 3.2.4, $A_i$ is a union of $z$-classes. The elements of $A_i \setminus \gamma_2(G)$ have centralizer equal to $A_i$ which is normal in $G$, whereas the elements of $\gamma_2(G)$ have centralizer either $Z_G(y_0)$ or $G$, it follows that $A_i \setminus \gamma_2(G)$ is a $z$-class, of size $p^3 - p^2$ ($1 \leq i \leq p$).

For $1 \leq i \leq p$, consider $h_i \in Z_G(y_i) \setminus Z_2(G)$.

**Claim 2:** $Z_G(y_i) \setminus Z_2(G)$ is a $z$-class of $h_i$ ($1 \leq i \leq p$):

Since $|Z_G(y_i)| = p^4$ and $|A_j| = p^3$ ($1 \leq i, j \leq p$), hence $Z_G(y_i) A_j = G$ and $Z_G(y_i) \cap A_j$ is a normal subgroup of $G$ of order $p^2$; by (v), it is one of $\langle y_k, Z(G) \rangle$, $0 \leq k \leq p$. In particular $Z_G(y_i) \cap A_j \subseteq Z_2(G)$.

Now, $Z_G(h_i) \supseteq \langle h_i, y_i, Z(G) \rangle$. As in the proof of the Claim 1 above, we can show that $|Z_G(h_i)| = p^3$, hence $Z_G(h_i) = \langle h_i, y_i, Z(G) \rangle$, which is abelian. If $Z_G(h_i) \subseteq G$, then by (vi), $Z_G(h_i)$ is one of $Z_2(G), A_1, \cdots, A_p$. In this case,

$$h_i \in Z_2(G) \text{ or } h_i \in A_j \cap Z_G(y_i) \subseteq Z_2(G),$$

which is a contradiction to the choice of $h_i$. Thus, $Z_G(h_i) \not\subseteq G$, for $1 \leq i \leq p$. Since $[G: Z_G(h_i)] = p^2$, and $G$ is a $p$-group, by above discussion, we must have

$$Z_G(h_i) \not\subseteq N_G(Z_G(h_i)) \setminus G,$$


Now, $Z_G(h_i) = \langle h_i, y_i, Z(G) \rangle$. Any element of $Z_G(h_i) \setminus \langle y_i, Z(G) \rangle$ has centralizer of order $p^3$ (by similar arguments as in the proof of Claim 1), and it contains $Z_G(h_i)$ (since $Z_G(h_i)$ is abelian). Hence, the elements of $Z_G(h_i) \setminus \langle y_i, Z(G) \rangle$ have centralizer equal to $Z_G(h_i)$. On the other hand, the elements of $\langle y_i, Z(G) \rangle$ have centralizer of order $\geq p^4$, as seen before Claim 1. Therefore,

$$F_{h_i}' = \{ g \in G: Z_G(g) = Z_G(h_i) \}$$

$$= \langle h_i, y_i, Z(G) \rangle \setminus \langle y_i, Z(G) \rangle$$

and $|F_{h_i}'| = p^3 - p^2$. 

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Therefore, for any $h_i \in Z_G(y_i) \setminus Z_2(G) \ (1 \leq i \leq p)$, we have

$$|\text{z-class of } h_i| = [G: N_G(Z_G(h_i))], |F_{h_i}| = p(p^3 - p^2).$$

Further, $Z_G(h_i) \subseteq Z_G(y_i) \trianglelefteq G$, by Proposition 3.2.2, the z-class of $h_i$ is contained in $Z_G(y_i)$. Since $h_i \notin Z_2(G)$ and $Z_2(G)$ is a union of z-classes, it follows that

$$\text{z-class of } h_i \subseteq Z_G(y_i) \setminus Z_2(G).$$

Since these sets have same size, the z-class of $h_i$ is $Z_G(y_i) \setminus Z_2(G)$, proving the Claim 2.

Hence $G$ has $3p + 2$ z-classes:

1. $Z(G), \gamma_2(G) \setminus Z(G),$
2. $Z(Z_G(y_i)) \setminus Z(G), (1 \leq i \leq p),$
3. $A_i \setminus \gamma_2(G), (1 \leq i \leq p),$
4. $Z_G(y_i) \setminus Z_2(G), (1 \leq i \leq p).$

**Case 8:** $[G: Z(G)] = p^4, c(G) = 4$, and $G$ contains an abelian subgroup of index $p$.

By first two conditions in this hypothesis, $G$ is a $p$-group of maximal class. By Theorem 4.2.4, $G$ has $p + 2$ z-classes.

**Case 9:** $[G: Z(G)] = p^4, c(G) = 4$, and $G$ has no an abelian subgroup of index $p$.

Again by first two conditions in this hypothesis, $G$ is a $p$-group of maximal class. Let $M_0$ be the unique maximal subgroup of $G$ such that $Z(G) < Z(M_0)$ (see Proposition 2.5.9). Then $|M_0| = p^4$, and $[Z(M_0)] > p$. If $|Z(M_0)| = p^4$ i.e. $M_0 = Z(M_0)$, then $M_0$ will be abelian subgroup of index $p$, contradicting the hypothesis. If $|Z(M_0)| = p^3$, then $M_0/Z(M_0)$ will be cyclic, a contradiction. Thus, we must have $[M_0: Z(M_0)] = p^2$. By Theorem 5.4.1, $G$ has $p + 4$ z-classes. □