4 \textbf{z}-Classes in \textit{p}-Groups

Let \( G \) be a group such that \([G: Z(G)]\) is a prime power, say \( p^k \). By Proposition 2.7.4, the group \( G \) is isoclinic to a finite \( p \)-group. Since, isoclinic groups have the same number of \( z \)-classes, we can assume that group \( G \) is a finite \( p \)-group, while studying \( z \)-classes in \( G \) such that \([G: Z(G)]\) is a power of prime \( p \).

In this chapter, we study \( z \)-classes in finite \( p \)-groups. As mentioned above, the results obtained in this chapter are valid for groups with center of prime power index. We obtain better upper and lower bounds on the number of \( z \)-classes. We give a characterization of finite \( p \)-groups attaining the lower bound. We also obtain necessary conditions on \( p \)-groups attaining upper bounds.

4.1 Bounds on the Number of \textit{z}-Classes

\textbf{Proposition 4.1.1} (Upper Bound 2). Let \( G \) be a finite \( p \)-group. If \([G: Z(G)] = p^k \) \((k \geq 2)\), then \( G \) has at most \( \frac{p^k - 1}{p - 1} + 1 \) \( z \)-classes.

\textit{Proof}. The number of \( z \)-classes in \( G \) is at most the number of rational conjugacy classes of \( G/Z(G) \) (by Proposition 3.3.3). Since \(|G/Z(G)| = p^k\), the rational conjugacy class of any non-trivial element \( xZ(G) \) in \( G/Z(G) \) contains at least \( p - 1 \) elements

\[ xZ(G), x^2Z(G), \ldots, x^{p-1}Z(G). \]

Thus, \( G/Z(G) \) has at most \( \frac{p^k - 1}{p - 1} + 1 \) rational conjugacy classes, and the result follows. \( \square \)
Example 4.1.2. Consider the extra-special $p$-group

$$G = \langle x_1, \cdots, x_r, y_1, \cdots, y_r, t : \ x_i^p = y_i^p = 1, \ t = [x_i, y_i], \ * \rangle \ (r \geq 1),$$

where $*$ denotes that the other commutators of the generators are trivial. Then $G$ has exactly $\frac{p^{2r-1} - 1}{p-1} + 1$ $z$-classes, and it is equal to the number of irreducible rational representations of $G/Z(G)$ (see Theorem 5.3.1).

Proposition 4.1.3 (Lower Bound 2). If $G$ is a non-abelian finite $p$-group, then $G$ has at least $p + 2$ $z$-classes.

Proof. By hypothesis, $Z(G) < G$. Also, $Z(G)$ is the $z$-class of $1 \in G$. Let

$$\{1 = x_0, \ x_1, \ x_2, \ \cdots, \ x_l\}$$

be a set of representatives of the $z$-classes of $G$. We show that $l > p$. For $1 \leq i \leq l$, since $x_i \notin Z(G)$, hence $Z(G) < Z_G(x_i) < G$. Therefore, there is a maximal subgroup $H_i$ containing $Z(G)$ such that $Z_G(x_i) \leq H_i$. Since $H_i \leq G$, by Proposition 3.2.2,

$$\text{z-class of } x_i = R(x_i) \subseteq \bigcup_{g \in G} gZ_G(x_i)g^{-1} \subseteq H_i.$$ 

Then $G = Z(G) \cup (\bigcup_{i=1}^l R(x_i)) \subseteq \bigcup_{i=1}^l H_i$. If $l \leq p$ and $|G| = p^n$, then

$$p^n - 1 = |G \setminus \{1\}| \leq \Sigma_{i=1}^l |H_i \setminus \{1\}| \leq p(p^{n-1} - 1),$$

a contradiction, hence $l > p$. 

Example 4.1.4. For $n \geq 2$, consider the group

$$G = \langle x, y : x^{p^n} = y^p = 1, y^{-1}xy = x^{1+p^{n-1}} \rangle = \langle x_1, \cdots, x_n, y : y^p = x_1^p = 1, x_{i+1}^p = x_1, \cdots, x_n^p = x_{n+1}, [y, x_n] = x_1 \rangle.$$ 

Now, $Z(G) = \langle x^p \rangle$ and $G/Z(G) = \langle \overline{x}, \overline{y} \rangle \cong C_p \times C_p$. Hence $G$ has $p + 1$ maximal subgroups containing $Z(G)$, say $H_1, H_2, \cdots, H_{p+1}$. For each $i$, since $H_i/Z(G) \cong C_p$, $H_i$ is abelian. By Theorem 3.2.5, $H_i \setminus Z(G)$ is a $z$-class, and these together with $Z(G)$ are the $p + 2$ $z$-classes of $G$. 

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4.2 Groups Attaining Bounds

For a group $G$ with $[G: Z(G)] < \infty$, by Corollary 3.3.7, the number of $z$-classes in $G$ is at most the number of rational irreducible representations of $G/Z(G)$. The following theorem gives a necessary condition on $p$-groups which attain this upper bound. Recall that $x, y \in G$ are said to be rationally conjugate if $\langle x \rangle$ and $\langle y \rangle$ are conjugate.

**Theorem 4.2.1** (cf. [28]). Let $G$ be a non-abelian finite $p$-group. Suppose that the number of $z$-classes in $G$ is equal to the number of rational irreducible representations of $G/Z(G)$. Then either $G/Z(G) \cong C_p \times C_p$ or $G$ has no abelian subgroup of index $p$.

**Proof.** Assume that $G/Z(G) \not\cong C_p \times C_p$. By Proposition 3.3.3, if $xZ(G), yZ(G)$ are rationally conjugate in $G/Z(G)$, then $x \sim_z y$. By hypothesis

$(\ast) \quad x \sim_z y \implies xZ(G)$ and $yZ(G)$ are rationally conjugate.

If possible, let $A$ be an abelian subgroup of index $p$. Then, $Z(G) < A < G$. Also, since $G/Z(G) \not\cong C_p \times C_p$ (equivalently $[G: Z(G)] \neq p^2$) and $G$ is non-abelian, we have

$$[A: Z(G)] \geq p^2.$$

**Case 1.** $G/Z(G)$ is abelian:

(i) If $A/Z(G)$ is cyclic (which has order $\geq p^2$), let $aZ(G)$ be a generator of $A/Z(G)$. Then $Z_G(a) = Z_G(a^p) = A$, hence $a \sim_z a^p$; but $aZ(G)$ and $a^pZ(G)$ are not rationally conjugate in $G/Z(G)$, since they have different orders, a contradiction to $(\ast)$.

(ii) If $A/Z(G)$ is non-cyclic, there exist $a, b \in A \setminus Z(G)$, such that $\langle aZ(G) \rangle \neq \langle bZ(G) \rangle$. Then $Z_G(a) = Z_G(b) = A$, hence $a \sim_z b$; but $aZ(G)$ and $bZ(G)$ are not rationally conjugate in $G/Z(G)$, a contradiction to $(\ast)$.

**Case 2.** $G/Z(G)$ is non-abelian:

Then $Z(G) < Z_2(G)$. Since $A/Z(G)$ is an abelian subgroup of index $p$ in $G/Z(G)$,

$$Z_2(G)/Z(G) = Z(G/Z(G)) < A/Z(G),$$

hence $Z_2(G) < A$. 

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For any \( a \in A \setminus Z_2(G) \) and \( b \in Z_2(G) \setminus Z(G) \), \( Z_G(a) = A = Z_G(b) \), hence \( a, b \) are \( z \)-equivalent in \( G \), but \( aZ(G), bZ(G) \) are not rationally conjugate, since only \( bZ(G) \) is central, a contradiction to (*)).

By Proposition 4.1.1, if \([G: Z(G)] = p^k\), then \( G \) has at most \( \frac{p^k - 1}{p - 1} + 1 \) \( z \)-classes.

**Theorem 4.2.2.** Let \( G \) be a non-abelian finite \( p \)-group with \([G: Z(G)] = p^k\). If the number of \( z \)-classes in \( G \) is \( \frac{p^k - 1}{p - 1} + 1 \) then either \( G/Z(G) \cong C_p \times C_p \) or the following holds:

1. \( G \) has no abelian subgroup of index \( p \).
2. \( G/Z(G) \) is elementary abelian. Equivalently, \( G \) is isoclinic to a special \( p \)-group.

**Proof.** Suppose that \( G/Z(G) \not\cong C_p \times C_p \). We prove (1) and (2). By hypothesis and surjectivity of \( h \) in Proposition 3.3.3,

\[
\frac{p^k - 1}{p - 1} + 1 = \text{the number of } z \text{-classes in } G \\
\leq \text{the number of rational conjugacy classes in } G/Z(G) \quad (**) \\
\leq \frac{p^k - 1}{p - 1} + 1. \quad (***)
\]

Thus the inequalities (***) are indeed equalities. Now, by equality in (**), \( G \) has no abelian subgroup of index \( p \) (Theorem 4.2.1), which proves (1). Since \( |G/Z(G)| = p^k \), the equality in (***) holds only if \( G/Z(G) \) is elementary abelian. By Theorem 2.7.3, there is a group \( G_1 \) isoclinic to \( G \) such that \( Z(G_1) \subseteq [G_1, G_1] \). Since \( G_1/Z(G_1) \cong G/Z(G) \) is (elementary) abelian, it follows that \( Z(G_1) = [G_1, G_1] \). By Proposition 2.2.1, it is easy to prove that \( [G_1, G_1] \) is also elementary abelian. Thus \( G_1 \) is a special \( p \)-group.

By Proposition 4.1.3, a non-abelian finite \( p \)-group has at least \( p + 2 \) \( z \)-classes. We give a necessary and sufficient condition on a finite \( p \)-group \( G \), which has exactly \( p + 2 \) \( z \)-classes.

**Theorem 4.2.3.** Let \( G \) be a non-abelian finite \( p \)-group. Then \( G \) has exactly \( p + 2 \) \( z \)-classes if and only if either \( G/Z(G) \cong C_p \times C_p \) or the following holds:
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(1) $G$ has a unique abelian subgroup of index $p$.

(2) The center of $G/Z(G)$ has order $p$.

Proof. Suppose that $G$ has exactly $p + 2$ $z$-classes. Then we can assume that

\[(*) \quad [G: Z(G)] \geq p^3,\]

and we prove (1) and (2). Let \{1, $x_1, x_2, \ldots, x_{p+1}\}$ be a set of representatives of the $z$-classes of $G$. As in the proof of Proposition 4.1.3, we choose maximal subgroups $H_i$ of $G$ containing $Z(G)$ such that $R(x_i) \subseteq H_i \ (1 \leq i \leq p + 1)$. Then \(R(1) = Z(G) \subseteq H_i\), and

\[G = R(1) \cup (\cup_{i=1}^{p+1} R(x_i)) \subseteq \cup_{i=1}^{p+1} H_i.\]

By Theorem 2.4.5, every $H_i$ is maximal subgroup and \([G: \cap_{i=1}^{p+1} H_i] = p^2.\)

Claim 1. $H_i$ is abelian for some $i$

Let $L = \cap_{i=1}^{p+1} H_i$. Then \((H_i \setminus L) \cap (H_j \setminus L) = \phi \text{ for } i \neq j.\) For every $i$, consider $x'_i \in H_i \setminus L$. Then, $x'_i$ is $z$-equivalent to $x_j$ for some $j$, hence $x'_i \in R(x_j) \subseteq H_j$. If $j \neq i$, then $x'_i \in H_j \cap H_i = L$, a contradiction. Hence, $x'_i \in R(x_i)$. Thus \(\{1, x'_1, x'_2, \ldots, x'_{p+1}\}\) is a set of representatives of the $z$-classes of $G$, such that $x'_i \notin L$. Thus, without loss of generality, we can assume that the representatives $x_1, \ldots, x_{p+1}$ are not in $L$. This also shows that, all the elements of $H_i \setminus L$ are $z$-equivalent to $x_i$:

\[H_i \setminus L \subseteq R(x_i) \quad (1 \leq i \leq p + 1).\]

Let $A$ be a maximal abelian normal subgroup of $G$. Then \(Z(G) \subseteq A\) (otherwise, $AZ(G)$ will be an abelian normal subgroup, properly containing $A$). If $Z(G) = A$, then, consider a normal subgroup of order $p$ in $G/Z(G)$, say $A_1/Z(G)$ (such a subgroup exists since $G/Z(G)$ is a $p$-group). Since $A_1/Z(G)$ is cyclic, $A_1$ is an abelian normal subgroup of $G$ with $A < A_1$, a contradiction to the choice of $A$. Thus,

\[Z(G) < A\]

Consider $a \in A \setminus Z(G)$. By Theorem 3.2.4, $R(a) \subseteq A \setminus Z(G)$. Since $a \notin Z(G)$, we have $a \sim_z x_i$ for some $i \ (1 \leq i \leq p + 1)$. Therefore, \(H_i \setminus L \subseteq R(x_i) = R(a) \subseteq A \setminus Z(G) \subseteq A.\)
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Then $|A| > |H_i \setminus L| = p^{n-1} - p^{n-2} \geq p^{n-2}$. It follows that $A$ is an abelian subgroup of index $p$, and its uniqueness follows by Theorem 2.4.2, proving (1). Further, since $a \in R(x_i) \cap A$, and all elements of $A \setminus Z(G)$ are $z$-equivalent to $a$, we have

$$A \setminus Z(G) \subseteq R(a) = R(x_i) \subseteq H_i.$$ 

Since $Z(G) \subseteq H_i$, we have $A \subseteq H_i$. Hence $A = H_i$, and $H_i$ is abelian.

Without loss of generality, let $H_1 = A$, which is abelian. By Theorem 3.2.5, $A \setminus Z(G)$ is a $z$-class, and we have two partitions of $G$:

$$G = Z(G) \cup (A \setminus Z(G)) \cup R(x_2) \cup \cdots \cup R(x_{p+1}),$$

$$G = Z(G) \cup (A \setminus Z(G)) \cup (H_2 \setminus L) \cup \cdots \cup (H_{p+1} \setminus L).$$

Since $H_i \setminus L \subseteq R(x_i)$ ($2 \leq i \leq p + 1$), it follows that $R(x_i) = H_i \setminus L$.

**Claim 2:** $G/Z(G)$ is non-abelian.

Suppose that $G/Z(G)$ is abelian. Then all the subgroups of $G$ containing $Z(G)$ are normal. In particular, $Z_G(x_i) \leq G$ for $1 \leq i \leq p + 1$. By hypothesis, there are exactly $p + 1$ $z$-classes other than $Z(G)$. In other words, there are exactly $p + 1$ conjugacy classes of proper centralizers. Hence $\{Z_G(x_i)\}_{1 \leq i \leq p+1}$ are the only proper centralizers in $G$. Therefore, $G = \bigcup_{i=1}^{p+1} Z_G(x_i)$. By Theorem 2.4.5,

$$[G: Z(G)] = [G: \bigcap_{i=1}^{p+1} Z_G(x_i)] = p^2,$$  

a contradiction to the assumption ($\ast$).

**Claim 3.** $[N_G(Z_G(x_i)): Z_G(x_i)] = p$ for $2 \leq i \leq p + 1$.

Fix $i$, with $2 \leq i \leq p + 1$. Let $[G: Z(G)] = p^k$. By Claim 2, $k \geq 3$. Since $x_i \notin A$, by Theorem 2.3.5(i) and (ii), $[Z_G(x_i): Z(G)] = p$ and

$$Z_G(x_i) = Z(G) \cup x_i Z(G) \cup \cdots \cup x_i^{p-1} Z(G).$$

It is easy to see that

$$F'_{x_i} = \{ g \in G: Z_G(g) = Z_G(x_i) \} = Z_G(x_i) \setminus Z(G).$$
If \( Z_G(x_i) \leq G \), then \( Z_G(x_i)/Z(G) \) will be a normal subgroup of order \( p \), it must be central and since \( A/Z(G) \) is an abelian subgroup of index \( p \) in the non-abelian group \( G/Z(G) \), we have \( Z_G(x_i)/Z(G) \leq A/Z(G) \), and hence \( x_i \in A \), a contradiction. Since \( G \) is a \( p \)-group, \( Z_G(x_i) < N_G(Z_G(x_i)) \), and

\[
[G: N_G(Z_G(x_i))] < [G: Z_G(x_i)] = p^{k-1}.
\]

Since \([G: N_G(Z_G(x_i))]\) is the number of distinct conjugates of \( Z_G(x_i) \) in \( G \), if \( Z_G(x_i) \) has \( p^{k_i} \) conjugates in \( G \), then \( p^{k_i} < p^{k-1} \), i.e. \( p^{k_i} \leq p^{k-2} \), and

\[
p^{n-1} - p^{n-2} = |H_i \setminus L| = |R(x_i)| = [G: N_G(Z_G(x_i))] |F'_x| \\
\leq p^{k-2} (p - 1) |Z(G)| = p^{k-2} (p - 1) p^{n-k} = p^{n-1} - p^{n-2}
\]

Thus \( k_i = k - 2 \) and \([G: N_G(Z_G(x_i))] = p^{k-2} \). Since

\[
[G: Z(G)] = p^k \quad \text{and} \quad [Z_G(x_i): Z(G)] = p
\]

we have \([G: Z_G(x_i)] = p^{k-1} \) and the claim follows.

By Theorem 2.3.5(iii),

\[
Z(G/Z(G)) = N_G(Z_G(x_i))/Z_G(x_i) \cong C_p.
\]

This proves (2).

Conversely, if \( G/Z(G) \cong C_p \times C_p \), then \( G \) has \( p + 2 \) \( z \)-classes (as illustrated in the Example 4.1.4). Therefore, suppose that \([G: Z(G)] \geq p^3 \), and (1) and (2) holds. Let \( A \) be the abelian subgroup of index \( p \) in \( G \). By Theorem 3.2.5, \( A \setminus Z(G) \) is a \( z \)-class. Let \( x \in G \setminus A \) be arbitrary. By Theorem 2.3.5(i), \([Z_G(x): Z(G)] = p\), and

\[
N_G(Z_G(x))/Z_G(x) = Z(G/Z(G)) \cong C_p.
\]

Hence \([N_G(Z_G(x)): Z(G)] = p^2 \) and \([G: N_G(Z_G(x_i))] = p^{k-2} \). In other words, \( Z_G(x) \) has \( p^{k-2} \) conjugates in \( G \). Since \([Z_G(x): Z(G)] = p\),

\[
F'_x = Z_G(x) \setminus Z(G) = xZ(G) \cup x^2Z(G) \cup \cdots x^{p-1}Z(G).
\]
Thus,
\[ |R(x)| = |G: N_G(Z_G(x))|, |F'_x| = p^{k-2}(p-1)|Z(G)| = p^{n-1} - p^{n-2}. \]

If \( G \setminus A \) is union of \( m \) \( z \)-classes, then
\[ p^n - p^{n-1} = |G \setminus A| = m(p^{n-1} - p^{n-2}), \]
hence \( m = p \). Since \( A \) is union of two \( z \)-classes, \( G \) has exactly \( p + 2 \) \( z \)-classes.

\[ \square \]

**Alternate Proof:** In the proof of the necessary part of Theorem 4.2.3, the existence of abelian subgroup of index \( p \) can be shown in the following way also:

Let \( A \) be a maximal abelian normal subgroup of \( G \). Clearly, \( Z(G) \subseteq A \). If \( Z(G) = A \), then, in \( G/Z(G) \), consider a normal subgroup of order \( p \), say \( A_1/Z(G) \). Then \( A_1 \) is an abelian normal subgroup of \( G \) with \( Z(G) = A < A_1 \), contradicting the choice of \( A \).

Thus, for any maximal abelian normal subgroup \( A \) of \( G \), \( Z(G) < A \). Note that, by hypothesis, \( G \) is non-abelian, hence \( Z(G) < A < G \).

By Theorem 3.2.4, \( A \) is a union of \( z \)-classes. Consider \( x_1 \in A \setminus Z(G) \). Let \( \{1, x_1, \cdots, x_{p+1}\} \) be a set of representatives of the \( z \)-classes of \( G \). As in the proof of Proposition 4.1.3, we can choose maximal subgroups \( H_2, \cdots, H_{p+1} \) containing \( Z(G) \) such that the \( z \)-class of \( x_i \) is contained in \( H_i \), \( 2 \leq i \leq p + 1 \). Then, it follows that
\[ G = A \cup H_2 \cup \cdots \cup H_{p+1}. \]

By Theorem 2.4.5, \( A \) (and \( H_i \)'s) must be maximal, which proves the existence of abelian subgroup of index \( p \).

**Reformulation of Theorem 4.2.3:**

We give a reformulation of the Theorem 4.2.3. Let \( G \) be a non-abelian finite \( p \)-group, such that \( G \) has an abelian subgroup \( A \) of index \( p \), and \( |Z(G)| = p \). By Theorem 2.2.2, \([G: \gamma_2(G)] = p^2\), and
\[ Z(G) \leq \gamma_2(G) \leq A. \]

If \( Z(G) = \gamma_2(G) \), then \( G \) will be a non-abelian group of order \( p^3 \), hence will be of maximal class. If \( Z(G) < \gamma_2(G) \), then \( \gamma_2(G/Z(G)) = \gamma_2(G)/Z(G) \), and its index in \( G/Z(G) \) will be \( p^2 \). By Theorem 2.2.2, \( G/Z(G) \) has center of order \( p \), i.e. \( Z_2(G)/Z(G) \cong C_p \).

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Continuing this way, we see that the upper central series of $G$ has maximum length, hence $G$ is of maximal class. Then, Theorem 4.2.3 can be reformulated as:

**Theorem 4.2.4.** Let $G$ be a non-abelian finite $p$-group. Then $G$ has exactly $p + 2$ $z$-classes if and only if either $G/Z(G) \cong C_p \times C_p$ or the following holds:

1. $G$ has a unique abelian subgroup of index $p$.
2. $G/Z(G)$ is of maximal class.

**Remarks 4.2.5.** (i) In any non-abelian group $G$ of order $p^3$, we have $G/Z(G) \cong C_p \times C_p$, hence $G$ has exactly $p + 2$ $z$-classes.

(ii) It is interesting to note that any non-abelian group $G$ of order $p^4$ has exactly $p + 2$ $z$-classes: $G$ always has an abelian subgroup of index $p$ (see Theorem 2.4.1), and $[G: Z(G)] \in \{p^2, p^3\}$. If $[G: Z(G)] = p^2$, then $G$ has $p + 2$ $z$-classes (by Theorem 4.2.3).

If $[G: Z(G)] = p^3$, i.e. $|Z(G)| = p$, then $|G'| = p^2$ (by Theorem 2.2.2). Therefore $Z(G) < G'$ and $G/Z(G)$ is non-abelian group of order $p^3$; its center has order $p$. By Theorem 4.2.3, $G$ has $p + 2$ $z$-classes.

**Example 4.2.6.** The simplest examples of groups with $p + 2$ $z$-classes will be groups $G$ with $G/Z(G) \cong C_p \times C_p$. Such a well known group is

$$G_n = C_{p^n} \rtimes C_p = \langle x, y : x^{p^n} = y^p = 1, y^{-1}xy = x^{1+p^{n-1}} \rangle, \quad n \geq 2.$$ 

Here $|G_n| = p^{n+1}$. We give an infinite family of $p$-groups in which $[G: Z(G)]$ takes every possible value, and $G$ has $p + 2$ $z$-classes (cf. Fernandez-Alcober [1], Ex. 3.4, p. 202).

Let $H$ be the abelian group defined by the generators $\{s_i : i \geq 1\}$ subject to the relations

$$s_i^{p^i} s_i \cdots s_{i+p-1} = 1, \text{ for } 1 \leq i \leq m - 1$$

$$s_i = 1 \text{ for } i \geq m.$$ 

By induction on $m$, we can show that $|H| = p^{m-1}$. The map $s : s_i \mapsto s_is_{i+1}$ extends to an automorphism of $H$ of order $p$. The group $G = H \rtimes \langle s \rangle$ is a $p$-group of maximal
class, with an abelian subgroup $H$ of index $p$. By Theorem 4.2.4, $G$ has exactly $p + 2$ $z$-classes.

Note that, for $p = 2$, $G$ is isomorphic to the dihedral group of order $2^n$.

**Remark 4.2.7.** By above example, we see that, there are infinitely many non-isoclinic finite $p$-groups with exactly $p + 2$ $z$-classes.